1 Method of images

The method of images is a method that allows us to solve certain potential problems as well as obtaining a Green's function for certain spaces. Recall that the Green's function satisfies the equation ("formal" notes eqn 3):

$$\nabla'^2 G\left(\vec{x}, \vec{x}'\right) = -4\pi\delta\left(\vec{x} - \vec{x}'\right) \tag{1}$$

subject to the boundary conditions

$$G_D = 0 \text{ on S} \tag{2}$$

or

$$\frac{\partial G}{\partial n} = C \text{ on S} \tag{3}$$

and thus the solution is of the form

$$G\left(\vec{x}, \vec{x}'\right) = \frac{1}{|\vec{x} - \vec{x}'|} + \psi\left(\vec{x}, \vec{x}'\right) \tag{4}$$

where

$$\nabla'^2 \psi\left(\vec{x}, \vec{x}'\right) = 0 \text{ in } V \tag{5}$$

Thus the Green's function is $4\pi\varepsilon_0$ times the potential at $\vec{x}$ due to a unit point charge at $\vec{x}'$ in the volume V plus an additional term, with no sources in V, that fixes up the boundary conditions, that is, a term due to sources outside V. Thus we can make progress in finding the Dirichlet Green's function by finding the potential due to a point charge in V with grounded boundaries.

1.1 Plane boundary

Suppose the volume of interest is the half-space $z > 0$. A point charge q is placed at a distance d from the $x - y-$plane, which is a conducting boundary. What is the potential for $z > 0$?

First, the conducting plane must be at a constant potential, which we may take to be zero. (If the potential is V_0, we can just add V_0 to our solution at the end.) Then we may place an *image charge* $-q$ at distance d from the boundary, but on the opposite side. Then the potential due to these two charges everywhere on the boundary $z = 0$ is zero, since every point on the boundary is equidistant from the two charges. Since the image charge we added is *outside* our volume, it does not contribute to the value of $\nabla^2\Phi$ in V. Thus, putting the $z-$axis through q, the solution is

$$\Phi\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{|\vec{x} - d\hat{z}|} - \frac{q}{|\vec{x} + d\hat{z}|}\right\}$$

Properties of this solution are explored in Jackson problem 2.1. Note that the image charge represents the charge density drawn onto the plane through the ground wire.

To obtain the Green's function for the half-space, we simply set $q = 1$, $d\hat{z} = \vec{x}'$, and

multiply by $4\pi\varepsilon_0$. Then

$$G\left(\vec{x}, \vec{x}'\right) = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}''|} \tag{6}$$

where

$$\vec{x}'' = (x', y', -z')$$

is the position vector of the image point. This expression shows clearly that the physical dimensions of G are [1/length].

This Green's function (6) is explicitly in the form (4), but it is not very convenient to use. For example, suppose the problem of interest has potential V_0 within a circle of radius a on the plane $z = 0$, with the rest of the plane grounded. (See Jackson problem 2.7.) The outward normal from the volume on the plane is $\hat{n} = -\hat{z}$. So we would need first to compute ("formal" notes equation 7)

$$\begin{aligned}
\left.\frac{\partial G}{\partial n'}\right|_{z'=0} &= \left.\hat{n}' \cdot \vec{\nabla}' G\right|_{z'=0} = \left.-\hat{z} \cdot \vec{\nabla}' G\right|_{z'=0} = \left.-\frac{\partial G}{\partial z'}\right|_{z'=0} \\
&= \left. -\left(\frac{(z - z')}{\left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{3/2}} + \frac{z + z'}{\left[(x - x')^2 + (y - y')^2 + (z + z')^2\right]^{3/2}}\right)\right|_{z'=0} \\
&= \left. -\frac{2z}{|\vec{x} - \vec{x}'|^3}\right|_{z'=0}
\end{aligned}$$

and then evaluate

$$\begin{aligned}
\Phi\left(\tilde{\mathbf{x}}\right) &= \frac{V_0}{4\pi} \int_{\text{circle}} \left.\frac{2z}{|\vec{x} - \vec{x}'|^3}\right|_{z'=0} dx'\, dy' \\
&= \frac{V_0 z}{2\pi} \int_{\text{circle}} \frac{1}{\left(r^2 + r'^2 - 2rr' \cos\left(\phi - \phi'\right) + z^2\right)^{3/2}} r'\, dr'\, d\phi'
\end{aligned}$$

where r, ϕ are polar coordinates in the $x - y$ plane. This is ugly. (We encountered a similar integral in the bar magnet example, but that was only a 1/2 power. Later on we will identify some methods for doing this integral, but we will also find more convenient expressions for G.)

1.2   Images in a sphere

Now let the conducting boundary be a sphere of radius a and suppose we have a point charge q at a distance d from the center of the sphere, where $d > a$. We want to find the potential outside the sphere.

Learning from our experience above, we conjecture that we can place an image charge *inside* the sphere (and thus outside our volume V) and form the potential in V as the sum of the potential due to the two charges. The boundary condition is that $\Phi\left(\vec{x}\right) = 0$ everywhere on the surface of the sphere ($r = a$). The system has azimuthal symmetry about the line from the center of the sphere to the charge q. Rotate the system about this line and nothing changes. Thus the image charge q' must lie on this line at a distance d' from the center. Then we have two unknowns in our potential: q' and d', and we need pick only two points on the sphere to solve for the two unknowns. The most convenient two points lie on the

ends of the diameter through the image charge, as shown (P and Q in the diagram).
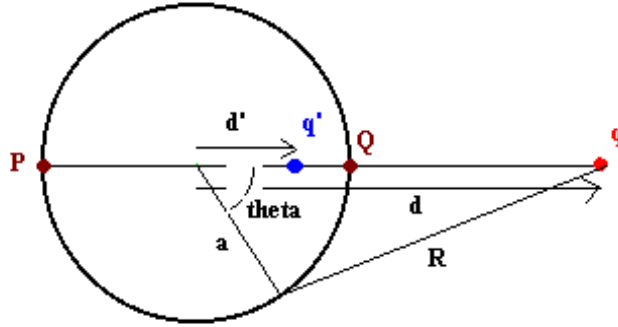
The potentials are

$$\Phi_P = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{d+a} + \frac{q'}{d'+a}\right\} = 0 \tag{7}$$

and

$$\Phi_Q = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{d-a} + \frac{q'}{a-d'}\right\} = 0 \tag{8}$$

Then from equation (7)

$$q(d'+a) + q'(d+a) = 0$$

and from (8)

$$q(a-d') + q'(d-a) = 0$$

Adding these two relations, we can eliminate d' to get

$$2qa + 2q'd = 0 \Rightarrow q' = -q\frac{a}{d} \tag{9}$$

This result has the nice property that the image charge is negative if q is positive, as we found in the planar case. Once again the image charge represents the charge drawn onto the surface of the sphere through the ground wire.

Now we subtract the two relations to obtain an expression for d' :

$$\begin{aligned}
2qd' + 2q'a &= 0 \\
d' &= -a\frac{q'}{q} = -a\left(-\frac{a}{d}\right) = \frac{a^2}{d} \tag{10}
\end{aligned}$$

The image charge is inside the sphere if $d > a$, as we need. Conversely, if $d < a$, the image charge is outside the sphere. (You are asked to confirm this result in Problem 2.2.)

We can perform two checks on this result. First let's find the potential at an arbitrary point on the sphere. We put the polar axis through q, so that the potential is independent of

3

ϕ. Then

$$
\begin{aligned}
4\pi\varepsilon_0 \Phi\left(a, \theta\right) &= \frac{q}{R} + \frac{q'}{R'} \\
&= \frac{q}{\sqrt{d^2 + a^2 - 2ad\cos\theta}} - \frac{qa/d}{\sqrt{(d')^2 + a^2 - 2ad'\cos\theta}} \\
&= \frac{q}{d}\left\{\frac{1}{\sqrt{1 + \frac{a^2}{d^2} - 2\frac{a}{d}\cos\theta}} - \frac{a}{\sqrt{(a^2/d)^2 + a^2 - 2\left(a^3/d\right)\cos\theta}}\right\} = 0
\end{aligned}
$$

for all θ, as expected.

Second, let's check that we get back the result from §1.1 as $a \to \infty$ (plane boundary). We have to be a bit careful here, because if we immediately let $a \to \infty$, the point from which we are measuring our distances moves off infinitely far to the left, and we will learn nothing. So first we write our results in terms of distance from the surface of the sphere. The charge q is a distance $d - a = h$ from the surface, and then

$$
q' = -q\frac{a}{a+h} = -\frac{q}{1 + h/a} \to -q \text{ as } a \to \infty
$$

as required. The distance of the image from the surface is

$$
h' = a - d' = a - \frac{a^2}{a+h} = \frac{ah}{a+h} = \frac{h}{1 + h/a} \to h \text{ as } a \to \infty
$$

and this is the second required result.

Finally we can write the Dirichlet Green's function for the region outside a sphere of radius a by setting $q = 1$ and multiplying by $4\pi\varepsilon_0$:

$$
G_D\left(\vec{x}, \vec{x}'\right) = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{a}{r'}\frac{1}{|\vec{x} - \vec{x}''|} \tag{11}
$$

where $\vec{x}''$ is the position vector of the image point. (This is Jackson eqn 2.17.) Again, while correct, this is pretty ugly and will be difficult to use.

1.3 Images in a cylinder

The basic ideas and methods are the same as we have used in the plane and sphere cases. See Jackson problem 2.11.

1.4 Use of images to solve problems

Jackson P 2.10 asks us to compute the potential inside a parallel-plate capacitor with a small hemispherical boss on one plate. We *model* the system by putting a point charge q at a very large distance d from the plane. A field line diagram shows us that the field will be close to uniform in a region $d \gg r \gg a$. Then we can use the image system shown to model the capacitor, since the image system puts potential zero on the lower plate. (q and $-q$, q' and $-q'$ form pairs that make the potential on the plane zero; q and q', $-q$ and $-q'$ form pairs that make the potential on the sphere zero.) Then we can show that as $d \to \infty$ we obtain a uniform field at large distance from the lower plate, and we set that uniform field equal to the given value of E_0, thus determining the necessary charge q.
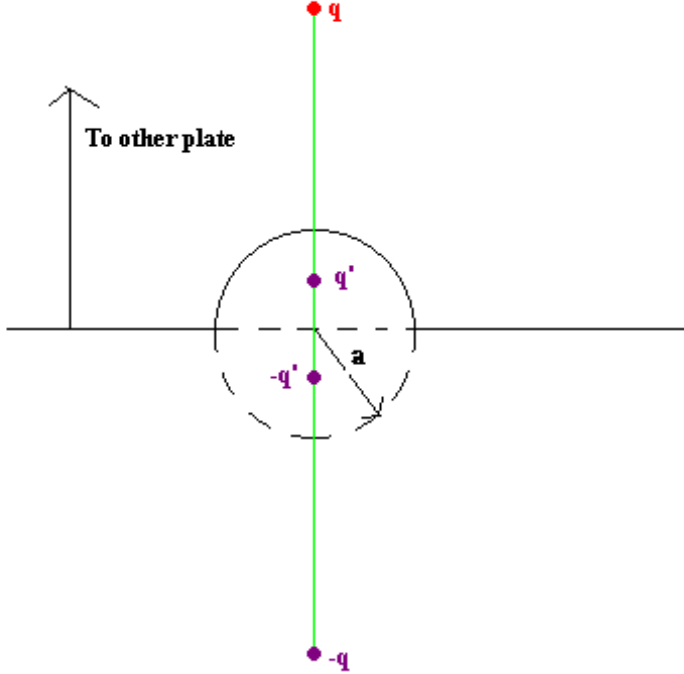
Setup: From §1.2 eqns (9) and (10), the image charge $q' = -\frac{a}{d}q$ and its distance from the plane is $d' = \frac{a^2}{d}$. The potential due to the four charges (one real charge and three images) is

$$\Phi\left(r,\theta\right) = kq \left\{ \begin{array}{c} 1/\sqrt{r^2 + d^2 - 2rd\cos\theta} - 1/\sqrt{r^2 + d^2 + 2rd\cos\theta} \\ -\left(\frac{a}{d}\right)/\sqrt{r^2 + \frac{a^4}{d^2} - 2r\frac{a^2}{d}\cos\theta} + \left(\frac{a}{d}\right)/\sqrt{r^2 + \frac{a^4}{d^2} + 2r\frac{a^2}{d}\cos\theta} \end{array} \right\} \quad (12)$$

To see why this works, look at the potential for $d \gg r \gg a$. We drop terms in $(a/d)^2$ to get

$$\Phi\left(r,\theta\right) \simeq \frac{kq}{d} \left\{ \begin{array}{c} 1/\sqrt{\frac{r^2}{d^2} + 1 - 2\frac{r}{d}\cos\theta} - 1/\sqrt{\frac{r^2}{d^2} + 1 + 2\frac{r}{d}\cos\theta} \\ -\left(\frac{a}{r}\right)/\sqrt{1 - 2\frac{a^2}{dr}\cos\theta} + \left(\frac{a}{r}\right)/\sqrt{1 + 2\frac{a^2}{rd}\cos\theta} \end{array} \right\}$$

Next expand the square roots, dropping terms in $(r/d)^2$ and $(a/r) \times (a/d)$.

$$\begin{aligned} \Phi\left(r,\theta\right) &\simeq \frac{kq}{d}\left\{ 1 + \frac{r}{d}\cos\theta - \left(1 - \frac{r}{d}\cos\theta\right) - \frac{a}{r} + \frac{a}{r} \right\} \\ &= \frac{kq}{d}2\frac{r}{d}\cos\theta = \frac{2kq}{d^2}z \;\; \text{to first order in small quantities} \end{aligned}$$

This corresponds to the given uniform field provided that $E_0 = 2kq/d^2$, or, if we choose

$$q = E_0 d^2/2k. \quad (13)$$

Solve: **(a)** To find the surface charge densities, begin with the field components. From

(12), we have

$$\frac{\partial \Phi}{\partial r} = kq \left\{ -\frac{r - d\cos\theta}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}} + \frac{r + d\cos\theta}{(r^2 + d^2 + 2rd\cos\theta)^{3/2}} \right.$$

$$\left. + \frac{a\left(r - \frac{a^2}{d}\cos\theta\right)}{d\left(r^2 + \frac{a^4}{d^2} - 2r\frac{a^2}{d}\cos\theta\right)^{3/2}} - \frac{a\left(r + \frac{a^2}{d}\cos\theta\right)}{d\left(r^2 + \frac{a^4}{d^2} + 2r\frac{a^2}{d}\cos\theta\right)^{3/2}} \right\}$$

and at $r = a$

$$E_r = kq \left\{ \frac{a - d\cos\theta}{(a^2 + d^2 - 2ad\cos\theta)^{3/2}} - \frac{a + d\cos\theta}{(a^2 + d^2 + 2ad\cos\theta)^{3/2}} \right.$$

$$\left. - \frac{a\left(a - \frac{a^2}{d}\cos\theta\right)}{d\left(a^2 + \frac{a^4}{d^2} - 2a\frac{a^2}{d}\cos\theta\right)^{3/2}} + \frac{a\left(a + \frac{a^2}{d}\cos\theta\right)}{d\left(a^2 + \frac{a^4}{d^2} + 2a\frac{a^2}{d}\cos\theta\right)^{3/2}} \right\}$$

$$= kq \left\{ \frac{a - d\cos\theta}{(a^2 + d^2 - 2ad\cos\theta)^{3/2}} - \frac{a + d\cos\theta}{(a^2 + d^2 + 2ad\cos\theta)^{3/2}} \right.$$

$$\left. - \frac{da^2(d - a\cos\theta)}{a^3(d^2 + a^2 - 2ad\cos\theta)^{3/2}} + \frac{da^2(d + a\cos\theta)}{a^3(d^2 + a^2 + 2ad\cos\theta)^{3/2}} \right\}$$

$$= kq \left[\frac{a - d\cos\theta - \frac{d}{a}(d - a\cos\theta)}{(a^2 + d^2 - 2ad\cos\theta)^{3/2}} - \frac{a + d\cos\theta - \frac{d}{a}(d + a\cos\theta)}{(a^2 + d^2 + 2ad\cos\theta)^{3/2}} \right]$$

$$= \frac{qa}{4\pi\varepsilon_0}\left(1 - \frac{d^2}{a^2}\right)\left[\frac{1}{(a^2 + d^2 - 2ad\cos\theta)^{3/2}} - \frac{1}{(a^2 + d^2 + 2ad\cos\theta)^{3/2}} \right]$$
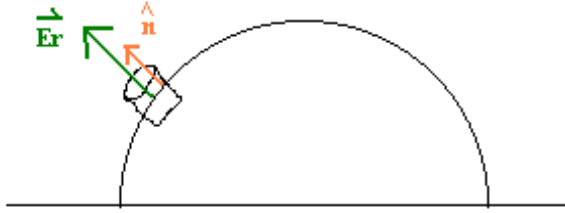
Then the charge density is (notes 1 eqn 11 with $\vec{E}_2 = 0$ inside the boss and $\hat{n} = \hat{r}$, see diagram):

$$\sigma(\theta) = \varepsilon_0 E_r = \frac{qa}{4\pi}\left(1 - \frac{d^2}{a^2}\right)\left[\frac{1}{(a^2 + d^2 - 2ad\cos\theta)^{3/2}} - \frac{1}{(a^2 + d^2 + 2ad\cos\theta)^{3/2}} \right]$$

Analysis: Notice that σ is zero in the corners at $\theta = \pi/2$, as expected. Also since $1 \geq \cos\theta \geq 0$ on the boss and $d > a$, the charge density is negative on the boss if q is positive, as expected.

Solve: The total charge on the boss is (with $\mu = \cos\theta$)

$$
\begin{aligned}
Q_{\text{boss}} &= \int \sigma\, dA = 2\pi \int_0^{\pi/2} \sigma(\theta)\, a^2 \sin\theta\, d\theta \\[2mm]
&= \frac{2\pi}{4\pi} qa^3 \left(1 - \frac{d^2}{a^2}\right) \int_0^1 \left(\frac{1}{(a^2 + d^2 - 2a\mu d)^{3/2}} - \frac{1}{(a^2 + d^2 + 2a\mu d)^{3/2}} \right) d\mu \\[2mm]
&= \frac{qa^3}{2}\left(1 - \frac{d^2}{a^2}\right) \frac{-1}{2ad} \left(\frac{-2}{(a^2 + d^2 - 2da\mu)^{1/2}} - \frac{2}{(a^2 + d^2 + 2da\mu)^{1/2}} \right)\Bigg|_0^1 \\[2mm]
Q_{\text{boss}} &= \frac{q}{2}\left(1 - \frac{d^2}{a^2}\right)\frac{a^2}{d}\left(\frac{1}{(a^2+d^2-2ad)^{1/2}} + \frac{1}{(a^2+d^2+2ad)^{1/2}} - \frac{2}{\sqrt{a^2+d^2}} \right) \\[2mm]
&= -\frac{q}{2}\frac{(d^2-a^2)}{d}\left(\frac{1}{d-a} + \frac{1}{a+d} - \frac{2}{\sqrt{a^2+d^2}} \right) \\[2mm]
Q_{\text{boss}} &= -q\left(1 - \frac{d^2 - a^2}{d\sqrt{a^2+d^2}} \right)
\end{aligned}
$$

which is Jackson's result.

Analysis: As $d \to \infty$ for fixed q, $Q_{\text{boss}} \to 0$. Is this what you would have expected? The induced charge also goes to zero as $a \to 0$, as the boss disappears in this case. $|Q_{\text{boss}}| < q$. A field line diagram shows why this must be the case: not every field line leaving the point charge reaches the boss.

Solve: The charge density on the plane is
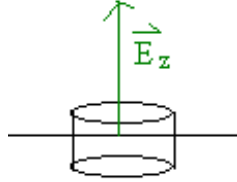
$$
\sigma(\rho) = \varepsilon_0 E_z = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial z}\right|_{z=0}
$$

Here it is more convenient to express the potential in cylindrical coordinates with $z = r\cos\theta$:

$$
\begin{aligned}
\Phi(\rho, z) = kq \Bigg\{ & \frac{1}{\sqrt{z^2 + \rho^2 + d^2 - 2zd}} - \frac{1}{\sqrt{z^2 + \rho^2 + d^2 + 2zd}} \\[2mm]
& - \frac{a}{d\sqrt{z^2 + \rho^2 + \frac{a^4}{d^2} - 2z\frac{a^2}{d}}} + \frac{a}{d\sqrt{z^2 + \rho^2 + \frac{a^4}{d^2} + 2z\frac{a^2}{d}}} \Bigg\} \quad (14)
\end{aligned}
$$

Thus

$$\sigma(\rho) = -\varepsilon_0 kq \left[\frac{-(z-d)}{(z^2 + \rho^2 + d^2 - 2zd)^{3/2}} - \frac{-(z+d)}{(z^2 + \rho^2 + d^2 + 2zd)^{3/2}} \right.$$

$$\left. \left. + \frac{a(z - a^2/d)}{d\left(z^2 + \rho^2 + \frac{a^4}{d^2} - 2z\frac{a^2}{d}\right)^{3/2}} - \frac{a(z + a^2/d)}{d\left(z^2 + \rho^2 + \frac{a^4}{d^2} + 2z\frac{a^2}{d}\right)^{3/2}} \right] \right|_{z=0}$$

$$= -\frac{qd}{2\pi} \left(\frac{1}{(\rho^2 + d^2)^{3/2}} - \frac{a^3}{d^3 \left(\rho^2 + \frac{a^4}{d^2}\right)^{3/2}} \right)$$

where the second term is negligible if $d \gg a$. In this case we get back the result of problem 2.1(a).

The total charge on the plane is:

$$Q_{\text{plane}} = -\frac{qd}{2\pi} 2\pi \int_a^\infty \left(\frac{1}{(\rho^2 + d^2)^{3/2}} - \frac{a^3}{d^3 \left(\rho^2 + \frac{a^4}{d^2}\right)^{3/2}} \right) \rho \, d\rho$$

$$= -\frac{qd}{2} \left. \left(\frac{-2}{(\rho^2 + d^2)^{1/2}} - \frac{a^3(-2)}{d^3 \left(\rho^2 + \frac{a^4}{d^2}\right)^{1/2}} \right) \right|_a^\infty$$

$$= -qd \left(\frac{1}{(a^2 + d^2)^{1/2}} - \frac{a^3}{d^3 \left(a^2 + \frac{a^4}{d^2}\right)^{1/2}} \right)$$

$$= -\frac{q}{d} \frac{d^2 - a^2}{\sqrt{d^2 + a^2}}$$

Analysis: $Q_{\text{plane}} \to -q$ as $a \to 0$ (flat plate) or $d \to \infty$. The total induced charge on the conducting surface is:

$$Q_{\text{boss}} + Q_{\text{plane}} = -q \left(1 - \frac{d^2 - a^2}{d\sqrt{a^2 + d^2}} \right) - q\frac{1}{d} \frac{d^2 - a^2}{\sqrt{d^2 + a^2}}$$

$$= -q$$

which is the sum of the image charges, as expected.

Solve: Now let's put in the value for q that gets us to the capacitor-plus-boss system (eqn 13): $q = E_0 d^2/2k$. Then, for $d \gg a$, the charge on the boss is:

$$Q_{\text{boss}} = -q \left(1 - \frac{1 - a^2/d^2}{\sqrt{a^2/d^2 + 1}} \right) \simeq -\frac{E_0}{2k} d^2 \left[1 - \left(1 - \frac{a^2}{d^2} \right) \left(1 - \frac{1}{2}\frac{a^2}{d^2} \right) + \mathcal{O}\left(\frac{a}{d}\right)^4 \right]$$

$$= -\frac{E_0}{2k} d^2 \left(\frac{3}{2}\frac{a^2}{d^2} \right) = -\frac{3}{4}\frac{E_0}{k} a^2 = -3\pi\varepsilon_0 E_0 a^2$$

Analysis: This is Jackson's answer in (b). Notice that d disappears in the limit $d \to \infty$, as required.

Solve: The charge densities in the capacitor system are:

$$
\begin{aligned}
\sigma(\theta) &= \varepsilon_0 E_0 \frac{a}{2}\left(1-\frac{d^2}{a^2}\right)\left(\frac{d^2}{(a^2+d^2-2ad\cos\theta)^{3/2}} - \frac{d^2}{(a^2+d^2+2ad\cos\theta)^{3/2}}\right) \\
&= \varepsilon_0 E_0 \frac{a}{2d}\left(1-\frac{d^2}{a^2}\right)\left(\frac{1}{(a^2/d^2+1-2a/d\cos\theta)^{3/2}} - \frac{1}{(a^2/d^2+1+2a/d\cos\theta)^{3/2}}\right) \\
&\simeq \varepsilon_0 E_0 \frac{a}{2d}\left(-\frac{d^2}{a^2}\right)\left(1+3\frac{a}{d}\cos\theta - \left(1-3\frac{a}{d}\cos\theta\right) + \cdots\right) \\
&= \varepsilon_0 E_0 \frac{a}{2d}\left(-\frac{d^2}{a^2}\right)\left(6\frac{a}{d}\cos\theta + \mathcal{O}\left(\frac{a}{d}\right)^3\right) \to -3\varepsilon_0 E_0 \cos\theta \text{ as } d\to\infty
\end{aligned}
$$

on the boss, and on the plane

$$
\begin{aligned}
\sigma(\rho) &= -\frac{E_0 d^3}{4\pi k}\left(\frac{1}{(\rho^2+d^2)^{3/2}} - \frac{a^3}{d^3\left(\rho^2+\frac{a^4}{d^2}\right)^{3/2}}\right) \\
&= -\frac{E_0}{4\pi k}\left(1-\frac{3\rho^2}{2d^2} - \frac{a^3}{\rho^3\left(1+\frac{a^4}{d^2\rho^2}\right)^{3/2}}\right) \\
&= -\frac{E_0}{4\pi k}\left(1-\frac{3\rho^2}{2d^2} - \frac{a^3}{\rho^3}\left(1-\frac{3}{2}\frac{a^4}{d^2\rho^2}\right)\right) \\
&\to -\varepsilon_0 E_0\left(1-\frac{a^3}{\rho^3}\right) \text{ as } d\to\infty
\end{aligned}
$$

Analysis: Note that the charge density is zero where the boss meets the plate ($\rho = a$, $\theta = \pi/2$), as expected near a sharp "hole" in the conductor. Also $\sigma \to -\varepsilon_0 E_0$ as $\rho \to \infty$, the expected result for a flat plate with a uniform field above.
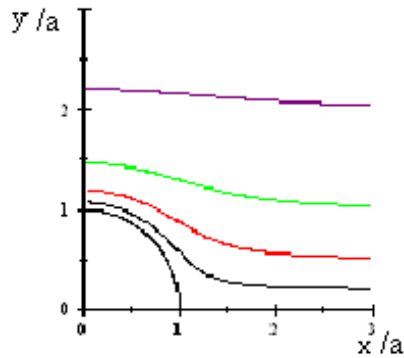
The plot shows $\sigma/\varepsilon_0 E_0$ versus ρ/a for $\rho > a$ and versus $2\theta/\pi$ for $0 < \theta < \pi/2$

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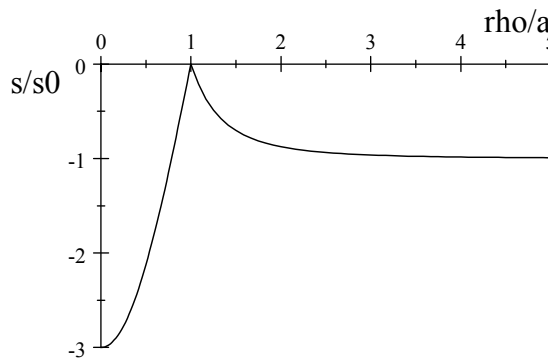
Solve: The potential (14) is

$$\Phi\left(\rho, z\right) \;\; = \;\; \frac{kq}{d}\left\{\frac{1}{\sqrt{1+\left(\frac{z}{d}\right)^2+\left(\frac{\rho}{d}\right)^2-2\frac{z}{d}}} - \frac{1}{\sqrt{1+\left(\frac{z}{d}\right)^2+\left(\frac{\rho}{d}\right)^2+2\frac{z}{d}}}\right.$$

$$\left.- \frac{a}{d\sqrt{\left(\frac{z}{d}\right)^2+\left(\frac{\rho}{d}\right)^2+\frac{a^4}{d^4}-2\frac{z}{d}\frac{a^2}{d^2}}} + \frac{a}{d\sqrt{\left(\frac{z}{d}\right)^2+\left(\frac{\rho}{d}\right)^2+\frac{a^4}{d^4}+2\frac{z}{d}\frac{a^2}{d^2}}}\right\}$$

$$\simeq \;\; \frac{E_0 d}{2}\left\{1+\frac{z}{d}-\left[1-\frac{z}{d}\right]-\frac{a}{\sqrt{z^2+\rho^2}}2\frac{z}{d}\frac{a^2}{z^2+\rho^2}\right\}$$

$$\simeq \;\; E_0 z\left(1-\frac{a^3}{\left(z^2+\rho^2\right)^{3/2}}\right) \quad \text{as } d\to\infty$$

Analysis: This function is plotted below. All distances are scaled by a. (Oops- the vertical axis should be labelled z/a.)

Values of $\Phi/E_0 a$ are: black 0.2, red 0.5, green 1, purple 2. The surfaces flatten out as distance from the boss increases. Aagin this is the expected result.

$$\Phi \;\; \to \;\; E_0 z \quad \text{as } \rho/a \to \infty$$
$$\vec{E} \;\; \to \;\; -E_0 \hat{z}$$

as expected.

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