1 Fundamentals

1.1 Overview

The objective of this course is: to determine $\vec{E}$ and $\vec{B}$ fields in various physical systems and the forces and/or torques resulting from them.

The domain of applicability is the classical regime in which the field picture is appropriate. In QM the photon is the particle that mediates the E-M force. The field picture is appropriate when interactions involve lots of photons. This regime has widespread applicability. Mostly we'll be working with continuous charge distributions in which the discrete nature of electrons is ignored.

Idealizations we'll make include:

1. The photon has zero mass: the experimental limit is $m_\gamma < 10^{-51}$ g. (See link on web site for details.) The mass of the photon is linked to the form of the point charge potential $\Phi \propto q/r$ and the corresponding inverse square force law. (Most experiments test for the inverse square law rather than trying to measure mass directly.) If the photon had mass, then the potential would be of the form

$$\Phi \propto \frac{1}{r} \exp\left(-m_\gamma \frac{c}{\hbar} r\right)$$

Think about what the force law looks like in this case, and how it differs from a pure inverse-square law.

2. Perfect conductors exist.

3. An extended surface can be maintained at a constant potential, and two different surfaces can be maintained at the same potential ("common ground"). This idealization is closely related to #2.

4. Surface charge and current layers have zero thickness. The actual thickness is of order a few atomic diameters, so this idealization is equivalent to

$$\text{size of system} \gg \text{few atomic diameters}$$

1.2 Units:

The first half of Jackson's book uses SI units, the second half uses Gaussian. Read the appendices on units! SI is an engineering system and is theoretically corrupt. In the Gaussian unit system, E and B are dimensionally equivalent; in SI they are not. This creates difficulties when E and B are different components of the same tensor! (See Chapter 11.) In the Gaussian system all units are derived from the three fundamental units of mass (gm), length (cm), and time (s). For example, the Coulomb force law is used to define the charge unit.

$$F = \frac{q^2}{r^2} \Rightarrow q = r\sqrt{F}$$

and thus the esu (**e**lectro-**s**tatic **u**nit of charge or stat-coulomb) is

$$1 \text{ esu} \;=\; \text{cm}\sqrt{\frac{\text{gm}\cdot\text{cm}}{\text{s}^2}} \;=\; \text{cm}^{3/2}\text{gm}^{1/2}\text{s}^{-1} \tag{1}$$

In SI a pseudo-fundamental unit of current is also included. (This is the corruption.) In cgs-Gaussian units there is no need for constants such as ε_0 and μ_0. The only constants that appear in Maxwell's equations are π and c (the speed of light).

1.3 Fundamental relations:

The fundamental relations of electromagnetic theory are Maxwell's equations. In SI units, we have

Gauss' law:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad\qquad \vec{\nabla}\cdot\vec{D} = \rho_f \tag{2}$$

where ρ_f is the free charge density.

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \text{(no magnetic monopoles)} \tag{3}$$

Faraday's law

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{4}$$

Ampere's law

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad\qquad \vec{\nabla}\times\vec{H} = \vec{j}_f + \frac{\partial\vec{D}}{\partial t} \tag{5}$$

where j_f is the free curent density. We also have the Lorentz force law

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{6}$$

We will spend the entire semester learning how to solve these five equations, and understand the solutions.

Charge conservation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \tag{7}$$

is a useful additional relation, but it is not independent of Maxwell's equations. To see this, start with Gauss' law

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= \frac{\partial}{\partial t}\left(\varepsilon_0\vec{\nabla}\cdot\vec{E}\right) \\
&= \vec{\nabla}\cdot\left(\varepsilon_0\frac{\partial}{\partial t}\vec{E}\right) && t \text{ and } \vec{x} \text{ are independent variables, } \varepsilon_0 \text{ is constant} \\
&= \vec{\nabla}\cdot\left(\frac{\vec{\nabla}\times\vec{B}}{\mu_0} - \vec{j}\right) && \text{Ampere's law} \\
&= -\vec{\nabla}\cdot\vec{j} && \text{divergence of a curl is zero}
\end{aligned}$$

You can also derive this equation from the principle of charge conservation, as in Lea Problem 1.6.

1.4 Nature of the mathematical problem

Maxwell's equations are *linear, 2nd order,* and in the static case, *elliptic PDEs.* Classes of problems we'll be interested in include:

1. Given sources $(\rho, \vec{j})$ everywhere, find the fields.

2. Given boundary conditions on the bounding surface of a finite volume, find the fields in the volume. (Boundary conditions show the effect of charges and currents outside the volume.)

3. Given the fields, find their effect on ρ, $\vec{j}$.

These are well-posed problems with unique solutions. Difficulties can arise because the fields due to sources act back on the sources, changing the position and velocity of charges and thus changing the sources that produce the fields. For the most part we will not get involved with these complications. (But see plasma physics course (Phys 712) , and also J Chapter 16).

Linearity is nice. It gives us the principle of superposition. We can compute the fields due to individual sources and simply add the results. The technique of using Green's functions depends on the linearity of Maxwell's equations.

Structure of the equations.

The equations may be divided into two groups in two ways:

1. Divergence equations and curl equations. This is useful when finding boundary conditions.

2. Equations with sources and equations without sources. This is actually more fundamental– See Ch 11. If we write the equations in 4-vectors, there are only two equations.

The lack of symmetry is due to non-existence of magnetic monopoles.

1.5 Fields in media.

In a material medium, an applied $\vec{E}$ polarizes the material, reducing the net $\vec{E}$ in the interior. It is useful to define a new field $\vec{D}$, the electric displacement,

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

where $\vec{P}$ is the polarization. Since $\vec{P}$ is generally proportional to the net $\vec{E}$, (in which case the material is said to be linear and isotropic)

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

where χ is the susceptibilty, then

$$\vec{D} = \varepsilon_0 (1 + \chi) \vec{E} = \varepsilon \vec{E} \tag{8}$$

Then

$$\begin{aligned}\vec{\nabla}\cdot\vec{D} &= \varepsilon_0\vec{\nabla}\cdot\vec{E}+\vec{\nabla}\cdot\vec{P}\\ &= \rho-\rho_b=\rho_f\end{aligned}$$

where

$$\rho_b=-\vec{\nabla}\cdot\vec{P}$$

is the bound charge density and ρ_f is the free charge density. See Ch 4 for more details. (We'll get to this in April.) This description is most useful in the so-called LIH materials (linear, isotropic, homogeneous) in which ε is a constant scalar. However, ε is always a function of the frequency of the applied fields, and strictly equation (8) is true in the Fourier transform domain.

$$\vec{D}(\omega)=\varepsilon(\omega)\vec{E}(\omega) \tag{9}$$

and this leads to an interesting relation between the time-dependent fields. (See Lea Ch 7 P 12 and Jackson Ch 7.) In static situations we use

$$\varepsilon=\lim_{\omega\to 0}\varepsilon(\omega)$$

For non-isotropic materials ε is a tensor, and the relation is

$$D_i=\varepsilon_{ij}E_j$$

indicating that one component of $\vec{D}$ depends on more than one component of $\vec{E}$. See http://www.physics.sfsu.edu/~lea/courses/grad/plasmawavesi.pdf for an example.

Similarly, atoms have magnetic moments which can be modified and/or aligned by applied fields. (see *e.g.* Ch 29 in Lea and Burke and also J p 191-194) In these cases we have a magnetization $\vec{M}=\chi_m\vec{H}$ and

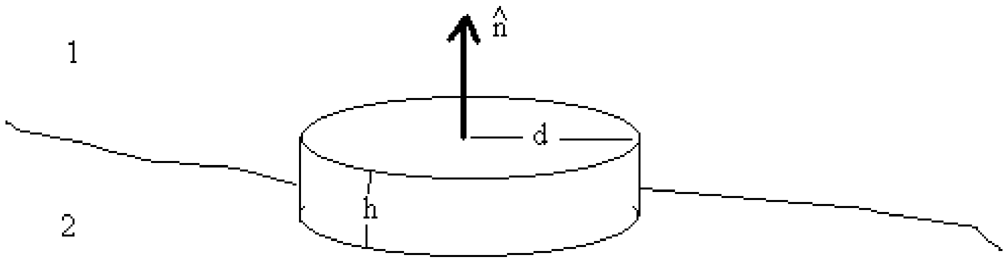
$$\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)=\mu\vec{H}$$

In situations where this description is useful (LIH) we find that $\mu/\mu_0\simeq 1$. The interesting cases with μ/μ_0 substantially different from one are not LIH, and the non-linearity gives rise to some very interesting phenomena (hysteresis etc). We won't get into these much in this course.

2 Boundary conditions

Here we investigate the relations between the field components on either side of a boundary between different materials. We start with the divergence equations, and integrate over a pillbox placed across the boundary. The dimensions of the pillbox are chosen as follows:

$$\begin{aligned} A &\simeq \pi d^2, \qquad d\ll L= \text{ length scale over which any property changes}\\ h &\ll d\end{aligned}$$



Then, from Gauss's law, we have

$$\int_{\text{pillbox}} \vec{\nabla}\cdot\vec{D}\ dV = \int_{\text{pillbox}} \rho_f\ dV = \int \rho_f\ dA dh$$

The free charge is usually confined to the surface (remember: this means within a few atomic diameters of the surface), so defining the surface charge density as $\sigma_f = \int \rho_f\ \ dh$, and applying the divergence theorem on the LHS, we have

$$\oint \vec{D}\cdot\hat{n}\ dA = \int dA \int \rho_f\ \ dh = \int \sigma_f dA$$

With $d \ll L$, σ_f is constant over the area, and we may approximate

$$\oint \vec{D}\cdot\hat{n}\ dA = A\sigma_f$$

With $h \ll d$, the sides of the pillbox give a negligible contribution to the integral on the LHS. Then, with $\hat{n}$ pointing from medium 2 to medium 1, as shown in the diagram,

$$A\left(\vec{D}_1 - \vec{D}_2\right)\cdot\hat{n} = A\sigma_f$$

Thus

$$\left(\vec{D}_1 - \vec{D}_2\right)\cdot\hat{n} = \sigma_f \tag{10}$$

For the electric field

$$\left(\vec{E}_1 - \vec{E}_2\right)\cdot\hat{n} = \frac{\sigma}{\varepsilon_0} \tag{11}$$

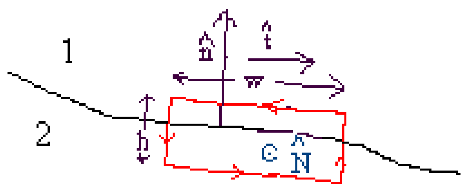
and since $\vec{B}$ satisfies the equation $\vec{\nabla}\cdot\vec{B} = 0$, then

$$\left(\vec{B}_1 - \vec{B}_2\right)\cdot\hat{n} = 0 \tag{12}$$

The normal component of $\vec{B}$ is continuous across any surface.

To avoid sign errors, always draw the pillbox and define the normal $\hat{\mathbf{n}}$.

Next we look at the curl equations and integrate over the area of a rectangle that crosses the boundary. We choose the dimensions such that the rectangle measures $w \times h$ where, as before, $h \ll w \ll L$. We start with Ampere's law.



$$\int \left(\vec{\nabla} \times \vec{H}\right) \cdot \hat{N}\ dA = \int \vec{j}_f \cdot \hat{N}\ dA + \int \frac{\partial \vec{D}}{\partial t} \cdot \hat{N}\ dA$$

Here the unit vector $\hat{t}$ is tangent to the surface, and the normal $\hat{N}$ is perpendicular to the plane of the rectangle, according to the right-hand rule. Then using Stokes' theorem, and for $w \ll L$,

$$\oint \vec{H} \cdot d\vec{\ell} = w \int \vec{j}_f \cdot \hat{N}\ dh + w \int \frac{\partial \vec{D}}{\partial t} \cdot \hat{N}\ dh$$

Then with $h \ll w$, the short sides of the rectangle make a negligible contribution to the line integral on the left hand side. Note that as $h \to 0$, we trap all the surface current inside our rectangle. (Effectively, $\vec{j}$ is a delta function and thus $\to \infty$ right at the surface.) However, the time derivative of $\vec{D}$ remains finite everywhere, and so the 2nd integral on the right hand side always goes to zero as $h \to 0$. Thus

$$\left[\vec{H}_1 \cdot (-\hat{t}) + \vec{H}_2 \cdot \hat{t}\right] w = w\vec{K}_f \cdot \hat{N}$$

where

$$\vec{K}_f = \int \vec{j}_f\ dh$$

is the free surface current density.

We may rewrite the LHS as follows

$$\left(\vec{H}_2 - \vec{H}_1\right) \cdot \left(\hat{n} \times \hat{N}\right) = \left[\left(\vec{H}_2 - \vec{H}_1\right) \times \hat{n}\right] \cdot \hat{N} = \vec{K}_f \cdot \hat{N}$$

Now we may orient our rectangle arbitrarily in the sense that $\hat{N}$ can be any vector in the plane perpendicular to $\hat{n}$. For example, if $\hat{n} = \hat{z}$, we can choose $\hat{N} = \hat{x}$ or $\hat{N} = \hat{y}$. Thus we get all components of the equation

$$\left(\vec{H}_2 - \vec{H}_1\right) \times \hat{n} = \vec{K}_f = \hat{n} \times \left(\vec{H}_1 - \vec{H}_2\right) \tag{13}$$

or, for $\vec{B}$,

$$\hat{n} \times \left(\vec{B}_1 - \vec{B}_2\right) = \mu_0 \vec{K} \tag{14}$$

Similarly for $\vec{E}$

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2\right) = 0 \tag{15}$$

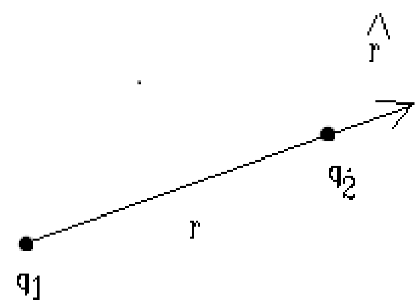
The tangential component of $\vec{E}$ is continuous across any surface.

3 Derivation of equations from experimental results

In this section we shall assume that all fields and charges change slowly, and all speeds $v \ll c$.

3.1 Coulomb's Law

Physics is an experimental science, so Maxwell's equations have their roots in experiment. Cavendish (in England) and Coulomb (in France, 1785) independently determined the electostatic force law. The force exerted on one point charge by another is



$$\vec{F}_{\text{on 2 due to 1}} \propto q_1 q_2 \frac{\hat{r}}{r^2}$$

The constant of proportionality depends on the unit system. In Gaussian units it is 1 (because this equation defines the charge unit, see eq. 1). In SI it is

$$k = 9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 = \frac{1}{4\pi\varepsilon_0}$$

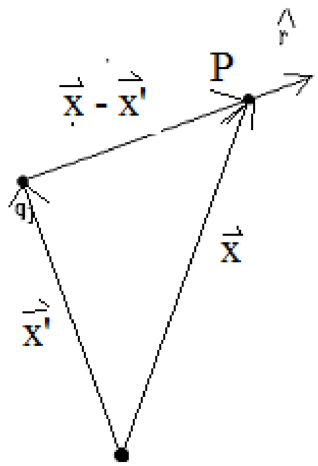
We *define* the electric field by placing a point test charge q at a point P and measuring the force on it. Then

$$\vec{E}(P) = \lim_{q \to 0} \frac{\vec{F}}{q}$$

Thus the field due to the point charge q_1 at the position of q_2 is

$$\vec{E}(P) = \frac{kq_1}{r^2}\hat{r}$$

Next we invoke the observed linearity of the forces to obtain the principle of superposition for $\vec{E}$



$$\vec{E}(\vec{x}) = \sum_{\text{all charges}} k\frac{q_i}{r_i^2}\hat{r}_i = k\sum_i q_i \frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|^3}$$

and then for a continuous distribution with $dq = \rho(\vec{x}')\,dV'$,

$$\vec{E}(\vec{x}) = k\int \rho(\vec{x}')\,\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}d^3\vec{x}' \tag{16}$$

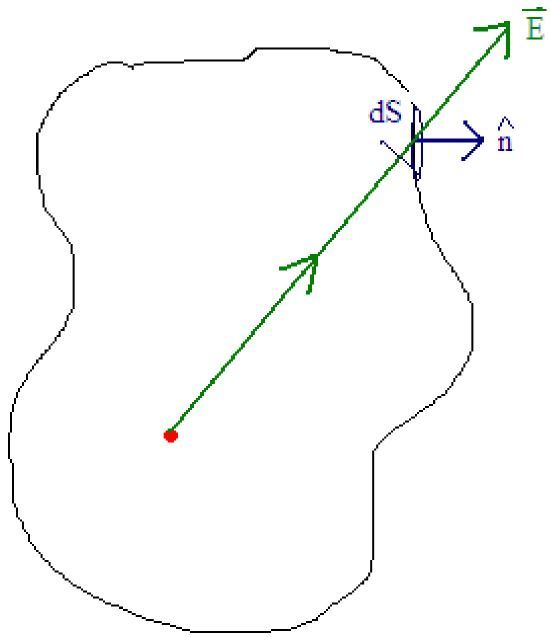
The standard convention is to use the primed coordinates to describe the position of the source and the unprimed coordinates for the field point.

3.2 Gauss' Law

Consider the electric field $\vec{E}_q$ due to a point charge q at a point P on a surface S that surrounds the charge. Let dS be an element of the surface with outward normal $\hat{n}$. Then

$$\vec{E}_q \cdot \hat{n}\; dS = k\frac{q}{r^2}\hat{r}\cdot\hat{n}\; dS$$

The dot product $\hat{r} \cdot \hat{n}\ dS$ projects the area dS onto the plane perpendicular to $\hat{r}$. By definition of solid angle, this projected area is $r^2 d\Omega$, and the flux is



$$\oint \vec{E}_q \cdot \hat{n}\ dS = \oint \frac{kq}{r^2} r^2 d\Omega = \frac{q}{4\pi\varepsilon_0} \oint d\Omega = \frac{q}{\varepsilon_0}$$

If q is outside S, then there is an area element on the "near" side whose contribution exactly cancels the element on the "far" side, and the result is zero. Now we can use the principle of superposition to sum the contributions from all the charges inside, to obtain Gauss' law:

$$\oint \vec{E} \cdot \hat{n}\ dS = \frac{\text{all charge inside}}{\varepsilon_0} = \frac{\int_V \rho\ dV}{\varepsilon_0} \tag{17}$$

The final step is to use the divergence theorem on the LHS

$$\int_V \vec{\nabla} \cdot \vec{E}\ dV = \int_V \frac{\rho}{\varepsilon_0} dV$$

Since this result must be true for any and every volume V, then

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

which is our first Maxwell equation (2).

3.3 Ampere's Law

We start with the Biot-Savart Law (discovered in 1820 by Jean-Baptiste Biot and Felix Savart). The magnetic field element $d\vec{B}$ produced by current I in a wire segment $d\vec{l}$ is

$$d\vec{B} = k_m I d\vec{\ell} \times \frac{\hat{r}}{r^2}$$

As in Coulomb's law, $\hat{r}$ points from the source to the field point. More generally, for a current density $\vec{j}$, $(Id\vec{\ell} \to \vec{j}dV')$

$$\vec{B}(\vec{x}) = \int d\vec{B} = \frac{\mu_0}{4\pi}\int \vec{j}(\vec{x}') \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} d^3\vec{x}' \tag{18}$$

Note the following **useful relation:**

$$\begin{aligned}
\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|} &= \vec{\nabla}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\
&= -\frac{x-x'}{|\vec{x}-\vec{x}'|^3}\hat{x} - \frac{y-y'}{|\vec{x}-\vec{x}'|^3}\hat{y} - \frac{z-z'}{|\vec{x}-\vec{x}'|^3}\hat{z} \\
\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|} &= -\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|} \qquad (19)
\end{aligned}$$

So we may write

$$\vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi}\int \vec{j}(\vec{x}') \times \vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$$

From the front cover

$$\vec{\nabla} \times (\psi\vec{a}) = \vec{\nabla}\psi \times \vec{a} + \psi\vec{\nabla} \times \vec{a} = -\vec{a} \times \vec{\nabla}\psi + \psi\vec{\nabla} \times \vec{a}$$

We may apply this result to our expression for $\vec{B}$, with $\vec{a} = \vec{j}(\vec{x}')$ and $\psi = 1/|\vec{x}-\vec{x}'|$, but since $\vec{j}(\vec{x}')$ is not a function of $\vec{x}$, then $\vec{\nabla} \times \vec{a} = 0$ and so (note the sign)

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{\nabla} \times \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$$

We pay pull the differential operator in unprimed coordnates out of the integral over primed coortdinates beause these are independent variables.

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\vec{\nabla} \times \int \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' \tag{20}$$

Since $\mu_0/4\pi$ is constant, we may write this relation as

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

with

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}' \tag{21}$$

so we immediately have

$$\vec{\nabla}\cdot\vec{B} = 0,$$

our second Maxwell equation (3).

To obtain Ampere's law, we take the curl of (20):

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\vec{\nabla}\times\int\frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}'\right)$$

Expand the double derivative:

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\left[\vec{\nabla}\left(\vec{\nabla}\cdot\int\frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}'\right) - \vec{\nabla}^2\int\frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}'\right]$$

Move the differential operator inside the integral, where it operates only on the $1/|\vec{x}-\vec{x}'|$ factor. In the first integral we also use the results (19) and

$$\vec{\nabla}\cdot(\psi\vec{a}) = \left(\vec{\nabla}\psi\right)\cdot\vec{a} + \psi\vec{\nabla}\cdot\vec{a} \tag{22}$$

I leave it to you to verify that the last term also simplifies as follows[1]:

$$\begin{aligned}\vec{\nabla}\times\vec{B} &= \frac{\mu_0}{4\pi}\left[\vec{\nabla}\left(\int\vec{j}(\vec{x}')\cdot\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}d^3\vec{x}'\right) - \int\vec{j}(\vec{x}')\vec{\nabla}^2\frac{1}{|\vec{x}-\vec{x}'|}d^3\vec{x}'\right] \\ &= \frac{\mu_0}{4\pi}\left[\vec{\nabla}\left(-\int\vec{j}(\vec{x}')\cdot\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|}d^3\vec{x}'\right) + 4\pi\int\vec{j}(\vec{x}')\,\delta(\vec{x}-\vec{x}')\,d^3\vec{x}'\right]\end{aligned}$$

where we have used Lea eqn 6.26. In the first term, use relation (22) again, and in the second use the sifting property:

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\left\{\vec{\nabla}\left(-\int\left[\vec{\nabla}'\cdot\frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}'|}\vec{\nabla}'\cdot\vec{j}(\vec{x}')\right]d^3\vec{x}'\right) + 4\pi\vec{j}(\vec{x})\right\}$$

Now use the divergence theorem and charge conservation.

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\left[-\int_{S_\infty}\frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}\cdot\hat{n}'dA' + \int\frac{1}{|\vec{x}-\vec{x}'|}\left(-\frac{\partial\rho(\vec{x}',t)}{\partial t}\right)d^3\vec{x}'\right] + \mu_0\vec{j}(\vec{x})$$

If the current distribution is confined to a finite region of space, the surface integral is zero. (Beware infinite wires!)

$$\vec{\nabla}\times\vec{B} = \frac{\partial}{\partial t}\frac{\mu_0}{4\pi}\left[-\vec{\nabla}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}'\right] + \mu_0\vec{j}(\vec{x}) \tag{23}$$

[1] Hint: ∇^2 is a scalar operator. Use index notation.

and evaluating the gradient and comparing with equation (16)[2] we finally have Ampere's law.

$$\begin{aligned}\vec{\nabla}\times\vec{B} &= \frac{\partial}{\partial t}\frac{\mu_0}{4\pi}\int\frac{\rho\left(\vec{x}'\right)\left(\vec{x}-\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|^3}d^3\vec{x}'+\mu_0\vec{j}\left(\vec{x}\right)\\ &= \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}+\mu_0\vec{j}\left(\vec{x}\right)\end{aligned}$$

3.4 Faraday's Law

In 1831 Faraday published his experimental result:

$$\text{Emf}\propto-\frac{d}{dt}\,(\text{magnetic flux through circuit})$$

or, in symbol form

$$\oint_C\vec{E}\cdot d\vec{\ell}=-\frac{d}{dt}\int_S\vec{B}\cdot\hat{n}\;dA \tag{24}$$

where S is a surface spanning the curve C. Note that the time derivative can give a non-zero result for any of the following:

1. $\vec{B}$ is a function of t.
2. The angle between $\vec{B}$ and $\hat{n}$ is a function of t.
3. The shape or area of the surface is a function of t.

Items (2) and (3) give rise to "motional EMF".
Comments:

1. The electric field is the field that would make current flow in the circuit if the circuit were a conducting wire. That means, $\vec{E}$ is measured in the *rest frame* of the circuit element $d\vec{\ell}$.
2. The constant of proportionality (here unity) depends on the unit system chosen. In Gaussian units it is $1/c$.
3. The minus sign expresses energy conservation. It makes sense here because the direction of $\hat{n}$ is related to the direction of $d\vec{l}$ through the usual right hand rule. It is sometimes given a separate name as Lenz's Law.

First let's choose our curve C to be stationary. Then the rate of change of flux is due entirely to the time rate of change of $\vec{B}$ itself, and the derivative may be moved inside the integral, where it becomes $\partial/\partial t$. (All spatial information disappears in the integral over S). We may also convert the LHS to a surface integral using Stokes' theorem:

$$\int_S\left(\vec{\nabla}\times\vec{E}\right)\cdot\hat{n}\;dA=-\int_S\frac{\partial}{\partial t}\vec{B}\cdot\hat{n}\;dA \tag{25}$$

[2] This step is valid only when the charge density changes slowly.

Now this relation must hold true for *any* stationary curve C and *any* corresponding spanning surface S, and so we may conclude:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

which is Faraday's law (4).

But there is more to learn from this experiment. What happens when we allow C to have a uniform velocity $\vec{v}$**?** There is an additional contribution to the change in flux as the curve moves to a region with a differing value of $\vec{B}$. Using a Taylor series expansion, we have:

$$B\left(\vec{r} + \delta\vec{r}\right) = B\left(\vec{r} + \vec{v}\delta t\right) = \vec{B}\left(\vec{r}\right) + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}\delta t + \cdots$$

Then

$$B\left(\vec{r} + \delta\vec{r}\right) - \vec{B}\left(\vec{r}\right) = \delta\vec{B} = \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}\delta t$$

to first order in δt. Thus the change in flux due to motion of the curve is:

$$\delta\Phi_m = \int_S \delta\vec{B} \cdot \hat{n}\ dA = \int_S \left[\left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}\right] \cdot \hat{n}\ dA\delta t$$

and hence, in the limit $\delta t \to 0$,

$$\left.\frac{d}{dt}\Phi_m\right|_{\text{motion}} = \int_S \left[\left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}\right] \cdot \hat{n}\ dA$$

due to motion of the curve. (This is one kind of "motional emf".) Adding the contribution from $\partial\vec{B}/\partial t$ that we have already calculated, we have

$$\left.\frac{d}{dt}\Phi_m\right|_{\text{total}} = \int_S \frac{\partial}{\partial t}\vec{B} \cdot \hat{n}\ dA + \int_S \left[\left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}\right] \cdot \hat{n}\ dA$$

and applying Faraday's law:

$$\oint_C \vec{E}' \cdot d\vec{\ell} = -\left\{\int_S \frac{\partial}{\partial t}\vec{B} \cdot \hat{n}\ dA + \int_S \left[\left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}\right] \cdot \hat{n}\ dA\right\}$$

Remember: the electric field $\vec{E}'$ is measured in the rest frame of $d\vec{\ell}$, *i.e.* the frame moving with velocity $\vec{v}$ with respect to the lab, while $\vec{B}$ is measured in the lab frame.

Now we'd like to convert the last term on the right to a line integral, so we use a result from the cover of Jackson:

$$\vec{\nabla} \times \left(\vec{B} \times \vec{v}\right) = \vec{B}\left(\vec{\nabla} \cdot v\right) - \vec{v}\left(\vec{\nabla} \cdot \vec{B}\right) - \left(\vec{B} \cdot \vec{\nabla}\right)\vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}$$

But here we have chosen $\vec{v}$ to be uniform, so all its spatial derivatives are zero, and from the second Maxwell equation, $\vec{\nabla} \cdot \vec{B} = 0$, so

$$\vec{\nabla} \times \left(\vec{B} \times \vec{v}\right) = \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}$$

Thus

$$\begin{aligned}
\oint_C \vec{E}' \cdot d\vec{\ell}' &= -\left\{ \int_S \frac{\partial}{\partial t}\vec{B} \cdot \hat{n}\ dA + \int_S \left[\vec{\nabla} \times \left(\vec{B} \times \vec{v} \right) \right] \cdot \hat{n}\ dA \right\} \\
&= -\int_S \frac{\partial}{\partial t}\vec{B} \cdot \hat{n}\ dA - \oint_C \left(\vec{B} \times \vec{v} \right) \cdot d\vec{\ell} \qquad (26)
\end{aligned}$$

Comparing equation (26) with equations (24) and (25) we may replace the surface integral of $\frac{\partial}{\partial t}\vec{B} \cdot \hat{n}$ with a line integral involving $\vec{E}$ in the lab frame[3]:

$$\oint_C \vec{E}' \cdot d\vec{\ell}' = \oint_C \vec{E} \cdot d\vec{\ell} + \oint_C \left(\vec{v} \times \vec{B} \right) \cdot d\vec{\ell}$$

where again the result is true for *any* curve C moving at uniform velocity $\vec{v}$. Thus we obtain the (non-relativistic) transformation law:

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B} \qquad (27)$$

The result is consistent with the non-relativistic Lorentz force law (6) in this unit system. It is valid when $v \ll c$. Thus we have established that the constant of proportionality in Faraday's law is linked to the transformation properties of the electric field. (See Chapter 11 for relativistic corrections to this result.)

Strictly, all these derivations are valid when the fields change *slowly*, although the results (except 27) are always valid. We'll investigate further in Chapters 6 and 7.

4 Potentials and gauge choice

4.1 Static case

We start with Faraday's law, which in the static case reduces to

$$\vec{\nabla} \times \vec{E} = 0$$

and thus we may write

$$\vec{E} = -\vec{\nabla}\Phi \qquad (28)$$

The minus sign is convential. Comparing with equation (23), we may write the solution for Φ :

$$\Phi\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho\left(\vec{x}'\right)}{\left|\vec{x} - \vec{x}'\right|} d^3\vec{x}' \qquad (29)$$

provided that the integral exists. (See below.)

We may also evaluate Φ as follows:

$$\int_1^2 \vec{E} \cdot d\vec{l} = \int_1^2 -\vec{\nabla}\Phi \cdot d\vec{l} = -\int_1^2 d\Phi = \Phi_1 - \Phi_2 \qquad (30)$$

[3]Strictly, we must apply Faraday's law to a curve C' at rest in the lab that instantaneously coincides with the moving curve C.

Thus the integral is path independent.

The integral (29) does not exist if $\rho = \lambda\delta(x)\,\delta(y)$ corresponding to an infinite line charge along the $z-$axis. Then the integral becomes

$$\Phi(\vec{x}) = \int_{-\infty}^{+\infty} \frac{\lambda}{\sqrt{x^2 + y^2 + (z - z')^2}} dz'$$

Let $z' - z = \sqrt{x^2 + y^2} \tan\theta$. Then

$$\begin{aligned} \Phi(\vec{x}) &= \int_{-\pi/2}^{+\pi/2} \frac{\lambda}{\sqrt{x^2+y^2}\sqrt{1+\tan^2\theta}} \sec^2\theta d\theta \\ &= \frac{\lambda}{\sqrt{x^2+y^2}} \int_{-\pi/2}^{+\pi/2} \sec\theta d\theta \\ &= \frac{\lambda}{\sqrt{x^2+y^2}} \ln(\sec\theta + \tan\theta)|_{-\pi/2}^{\pi/2} \end{aligned}$$

and now we are in trouble because the logarithm is undefined at both limits. The problem arises because our charge distribution extends to infinity. Of course this is a non-physical idealization. In cases like this there is usually enough symmetry that we can proceed by finding $\vec{E}$ using the integral form of Gauss' law, and integrating $\vec{E}$ to get the potential using eqn (30). For the line charge,

$$\begin{aligned} \Phi_2 - \Phi_1 &= \int_2^1 \vec{E} \cdot d\vec{\ell} = \int_2^1 \frac{\lambda}{2\pi\varepsilon_0 r} dr \\ &= \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_1}{r_2}\right) \end{aligned}$$

We must choose a reference point where $\Phi = 0$. We may not choose the reference point at infinity (or on the line) because there is charge there. So if the reference point is at a distance r_0 from the line, then

$$\Phi(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r_0}{r}\right) \tag{31}$$

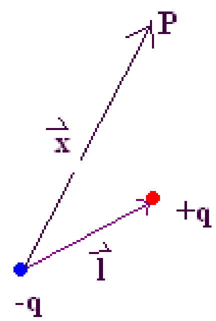
Alternatively, using the expression (28) for $\vec{E}$ in Gauss' law, we have the differential equation (Poisson's equation) for Φ :

$$\nabla^2\Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\varepsilon_0} \tag{32}$$

We'll learn how to solve this differential equation later.

Example: dipole potential.

The standard model for an ideal dipole is two equal but opposite charges $\pm q$ separated by a small displacement $\vec{\ell}$.



The dipole moment is

$$\vec{p} = \lim_{q\to\infty,\ell\to 0} q\vec{\ell}$$

We begin with equation (29) with the charge density $\rho(\vec{x}) = -q\delta(\vec{x}) + q\delta\left(\vec{x} - \vec{\ell}\right)$ (dipole at the origin). Then

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{1}{4\pi\varepsilon_0}\int \frac{-q\delta(\vec{x}') + q\delta\left(\vec{x}' - \vec{\ell}\right)}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \\
&= \frac{1}{4\pi\varepsilon_0}\left[\frac{-q}{|\vec{x}|} + \frac{q}{\left|\vec{x} - \vec{\ell}\right|}\right],
\end{aligned}$$

the expected result for two point charges. Since $\ell \to 0$, we use a binomial expansion in the second term. First factor out $|\vec{x}|$.

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{q}{4\pi\varepsilon_0|\vec{x}|}\left[-1 + \frac{1}{\sqrt{1 - 2\frac{\vec{\ell}\cdot\vec{x}}{|\vec{x}|^2} + \frac{l^2}{|\vec{x}|^2}}}\right] \\
&= \frac{q}{4\pi\varepsilon_0|\vec{x}|}\left[-1 + 1 + \frac{\vec{\ell}\cdot\vec{x}}{|\vec{x}|^2} + \cdots\right] \\
&\to \frac{\vec{p}\cdot\vec{x}}{4\pi\varepsilon_0|\vec{x}|^3} \text{ as } \ell \to 0 \qquad (33)
\end{aligned}$$

or, placing the dipole at a point $\vec{x}'$, we get

$$\Phi_{\text{dipole}}(\vec{x}) = \frac{\vec{p}\cdot(\vec{x} - \vec{x}')}{4\pi\varepsilon_0|\vec{x} - \vec{x}'|^3} = -\frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\vec{\nabla}\left|\frac{1}{\vec{x} - \vec{x}'}\right| \qquad (34)$$

Note that this potential decreases as one over the square of the distance from the source, as compared with 1/distance for a point charge.

4.1.1 Magnetic scalar potential

In regions where $\vec{j}(\vec{x}) = 0$, then Ampere's law becomes $\vec{\nabla} \times \vec{B} = 0$, so we may also write

$$\vec{B} = -\vec{\nabla}\Phi_m$$

where

$$\nabla^2 \Phi_m(\vec{x}) = 0 \qquad (35)$$

Thus the magnetic scalar potential satisfies Laplace's equation. This equation is easier to solve than Poisson's equation, but we have lost all the information about the sources! The result is still useful anywhere that $\vec{j} = 0$ (*ie* outside the source) but we have to use tricks to find the potential. (See J problem 5.1 for example.)

A simple example: Current in a long straight wire. Because of the azimuthal symmetry around the wire and translational symmetry parallel to the wire, we can easily find the field using Ampere's Law:

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{I}{\rho} \hat{\phi}$$

where ρ is the radial coordinate in a cylindrical coordinate system. This suggests that we can use the scalar potential

$$\Phi_m = -\frac{\mu_0 I}{2\pi}\phi$$

This is not ideal, however, because Φ_m is not single-valued. It has the countably infinite number of values $\Phi_m = -\frac{\mu_0 I}{2\pi}(\phi + 2n\pi)$ at any point with azimuthal angle ϕ. We won't get into much trouble, though, because all these values give the same field $\vec{B}$.

4.1.2 Magnetic vector potential

When $\vec{j} \neq 0$, we start with

$$\vec{\nabla} \cdot \vec{B} = 0$$

which tells us we can express $\vec{B}$ as the curl of a vector field $\vec{A}$ (see, *eg*, equation 20). Then Ampere's law reduces to

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{j} \qquad (36)$$

We may add the gradient of any scalar function ψ to $\vec{A}$ without changing the value of $\vec{B}$. We may use this freedom to set

$$\vec{\nabla} \cdot \vec{A} = 0 \qquad (37)$$

This is the *Coulomb Gauge.* (Suppose $\vec{\nabla} \cdot \vec{A}' \neq 0$. We let $\vec{A} = \vec{A}' + \vec{\nabla}\psi$. Then $\vec{\nabla} \cdot \vec{A} = 0 = \vec{\nabla} \cdot \vec{A}' + \nabla^2\psi$ and we can solve this equation to find a suitable ψ.) Then the equation for the potential $\vec{A}$ is

$$\nabla^2 \vec{A} = -\mu_0 \vec{j} \qquad (38)$$

If we work in Cartesian components, each component of this equation has the same form as the equation for Φ, and so the solution also has the same form, as indeed we have already found (equation 21). Do not attempt to use this result in other than Cartesian components! While true as a vector relation, the integration must be handled with extreme care.

Again the solution (21) is valid only when the integral exists. We cannot use this expression to compute the vector potential due to a current in a long straight wire. But for a long wire, it is easy to find $\vec{B}$ using the integral form of Ampere's law. If we need $\vec{A}$ rather than just $\vec{B}$, we have to work from the differential equation (38), which, with the wire along the $z-$axis, takes the form:

$$\nabla^2 \vec{A} = -\mu_0 I \ \hat{z} \ \delta(x) \delta(y)$$

So, in Cartesian components,

$$\nabla^2 A_x = \nabla^2 A_y = 0$$

and

$$\nabla^2 A_z = -\mu_0 I \delta(x) \delta(y) \tag{39}$$

The Coulomb gauge condition is

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

Since the symmetry of the situation (translational symmetry along the direction parallel to the wire and rotational symmetry about the wire) leads us to expect $\vec{A}$ to be independent of z, ($\frac{\partial A_z}{\partial z} = 0$), and $A_x = A_y$, we should be able to find a solution with $A_x = A_y = 0$. Then equation (39) has the solution:

$$A_z = -\frac{\mu_0}{2\pi} I \ln \frac{\rho}{\rho_0}$$

where $\rho = \sqrt{x^2 + y^2}$. (Recall that $\nabla^2 \ln(\rho/\rho_0) = 2\pi\delta(\vec{\rho})$. Lea Chapter 6 eqn. 6.27) The reference point where $\vec{A} = 0$ is at distance ρ_0 from the line.

We can check this result by calculating $\vec{B}$:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \hat{\phi}\left(-\frac{\partial A_z}{\partial \rho}\right) = \hat{\phi}\left(\frac{\mu_0}{2\pi\rho} I\right)$$

which agrees with the result from Ampère's law.

4.2 Time dependence

When the fields are time dependent, $\vec{\nabla} \times \vec{E}$ is no longer zero. But $\vec{\nabla} \cdot \vec{B}$ is always zero, and so it makes sense to start with the vector potential $\vec{A}$. Then Faraday's law becomes:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times A = -\vec{\nabla} \times \frac{\partial}{\partial t} \vec{A}$$

or

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

So now we may write

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\Phi$$

leading to

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \tag{40}$$

The first term represents the field due to charges, or the *Coulomb field,* while the second represents the induced field. To see how this decomposition arises from Maxwell's equations, we put these potentials into Ampere's law:

$$\begin{aligned}
\vec{\nabla} \times \vec{B} &= \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \\
\vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \nabla^2 \vec{A} &= \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \right)
\end{aligned}$$

Rearranging, and reordering the partial derivatives of Φ, we find

$$\begin{aligned}
\vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) + \mu_0 \varepsilon_0 \vec{\nabla} \frac{\partial}{\partial t} \Phi + \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{A} - \nabla^2 \vec{A} &= \mu_0 \vec{j} \\
\vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial}{\partial t} \Phi \right) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} - \nabla^2 \vec{A} &= \mu_0 \vec{j} \qquad (41)
\end{aligned}$$

Here the convenient gauge choice is

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial}{\partial t} \Phi = 0 \tag{42}$$

– the *Lorentz Gauge.* With this gauge, the equation (41) for $\vec{A}$ becomes

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} - \nabla^2 \vec{A} = \mu_0 \vec{j} \tag{43}$$

Thus $\vec{A}$ satisfies a wave equation with wave speed c and source $\mu_0 \vec{j}$.

Next we look at Gauss' law, using (40) for $\vec{E}$:

$$\vec{\nabla} \cdot \left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\varepsilon_0}$$

Expand out the divergence, and use the gauge condition (42) to eliminate $\vec{\nabla} \cdot \vec{A}$.

$$\begin{aligned}
-\nabla^2 \Phi - \frac{\partial}{\partial t} \left(-\frac{1}{c^2} \frac{\partial}{\partial t} \Phi \right) &= \frac{\rho}{\varepsilon_0} \\
-\nabla^2 \Phi + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi &= \frac{\rho}{\varepsilon_0} \qquad (44)
\end{aligned}$$

So here, too, Φ satisfies a wave equation, with wave speed c, and this time the source is ρ/ε_0.

In the special case that the fields are time independent, the Coulomb and Lorentz Gauges are the same. We'll have more to say about these gauges later.

Even with the gauge condition imposed, there is still some freedom in our potentials. Suppose we have a vector potential $\vec{A}$ and a scalar potential Φ. We want to impose the Lorentz gauge condition, so we look for a vector $\vec{A}'$ where $\vec{A}' = \vec{A} + \vec{\nabla}\psi$ and Φ' where $\Phi' = \Phi + \chi$. We want the same values of the fields with both sets of potentials, so:

$$\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \left(\vec{A} + \vec{\nabla}\psi\right) = \vec{\nabla} \times \vec{A} = \vec{B}$$

and

$$\begin{aligned} \vec{E}' &= -\vec{\nabla}\Phi' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}(\Phi + \chi) - \frac{\partial\left(\vec{A} + \vec{\nabla}\psi\right)}{\partial t} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\chi - \frac{\partial \tilde{\boldsymbol{\nabla}}\psi}{\partial t} \\ &= \vec{E} - \vec{\nabla}\left(\chi + \frac{\partial \psi}{\partial t}\right) = \vec{E} \end{aligned}$$

Thus we must have

$$\chi = -\frac{\partial \psi}{\partial t} + C \tag{45}$$

where C is a constant. Now imposing the Lorentz gauge condition,

$$\begin{aligned} \vec{\nabla} \cdot \vec{A}' + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi' &= 0 \\ \vec{\nabla} \cdot \left(\vec{A} + \vec{\nabla}\psi\right) + \frac{1}{c^2}\frac{\partial}{\partial t}(\Phi + \chi) &= 0 \\ \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi + \frac{1}{c^2}\frac{\partial}{\partial t}\chi + \nabla^2\psi &= 0 \end{aligned}$$

If $\vec{A}$ and Φ satisfy the Lorentz gauge condition, $\vec{A}'$ and Φ' will also if we choose ψ and C to satisfy the equation:

$$\nabla^2\psi + \frac{1}{c^2}\frac{\partial}{\partial t}\chi = \nabla^2\psi + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial \psi}{\partial t} + C\right) = 0$$

Then ψ satisfies the source-free wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi = 0$$

with χ given by (45).

5 Magnetic scalar potential and permanent magnets

The relation between $\vec{B}$ and $\vec{H}$ is

$$\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right)$$

Then, since $\vec{\nabla} \cdot \vec{B} = 0$,

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} \tag{46}$$

Thus $-\vec{\nabla} \cdot \vec{M}$ acts like a "magnetic charge density" in producing $\vec{H}$. In the time-independent case,

$$\vec{\nabla} \times \vec{H} = \vec{j}_{\text{free}}$$

So if $\vec{j}_{\text{free}} = 0$, we may write $\vec{H}$ as the gradient of a potential, $\vec{H} = -\vec{\nabla}\Psi_m$. Comparing equations (32) and (46), we may immediately write the solution:

$$\Psi_m(\vec{x}) = -\frac{1}{4\pi} \int_V \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \tag{47}$$

Then we may "integrate by parts" using relation (22):

$$\begin{aligned}
\Psi_m(\vec{x}) &= -\frac{1}{4\pi} \int_V \vec{\nabla}' \cdot \left[\frac{\vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right] d^3\vec{x}' + \frac{1}{4\pi} \int_V \vec{M}(\vec{x}') \cdot \vec{\nabla}' \frac{1}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \\
&= -\frac{1}{4\pi} \int_{S_\infty} \frac{\vec{M}(\vec{x}') \cdot \hat{n}'}{|\vec{x} - \vec{x}'|} d^2\vec{x}' - \frac{1}{4\pi} \int_V \vec{M}(\vec{x}') \cdot \vec{\nabla} \frac{1}{|\vec{x} - \vec{x}'|} d^3\vec{x}' \qquad (48) \\
&= 0 - \frac{1}{4\pi} \vec{\nabla} \cdot \int_V \frac{\vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}'
\end{aligned}$$

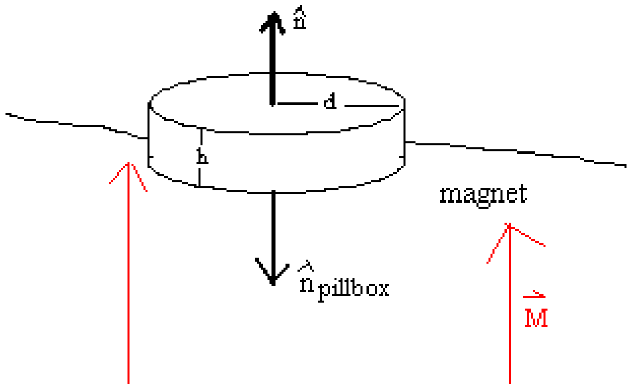
The surface integral is zero because $\vec{M}$ is confined to a finite region of space. Then at a large distance from the magnet, $|\vec{x} - \vec{x}'| \simeq \frac{1}{r}$, and

$$\Psi_m(\vec{x}) \simeq -\vec{\nabla} \frac{1}{4\pi r} \cdot \int_V \vec{M}(\vec{x}')\, d^3\vec{x}' = \frac{\vec{m} \cdot \vec{r}}{4\pi r^3} \tag{49}$$

This is a dipole potential (*cf* equation 33) , with magnetic dipole moment

$$\vec{m} = \int_V \vec{M}(\vec{x}')\, d^3\vec{x}'.$$

When the magnetization is uniform within a volume V, $\vec{\nabla} \cdot \vec{M}$ (and hence the magnetic charge density) is zero except at the edge of the volume where $\vec{M}$ comes abruptly to zero, and $\vec{\nabla} \cdot \vec{M} \to \infty$. Then if we put a pillbox over the surface, we find



$$\int_{V\ \text{pillbox}} \rho_M\ dV = \int_{V\ \text{pillbox}} -\vec{\nabla}\cdot\vec{M}\ dV = \int_{S\ \text{pillbox}} -\vec{M}\cdot\hat{n}_{\text{pillbox}}\ dA = \int_{S\ \text{"cookie"}} \sigma_M\ dA$$

where the pillbox has cut a "cookie" from the surface of the volume. Notice that $\vec{M}$ is non-zero only inside the magnet, so the contribution to the integral over the surface of the pillbox comes from inside the magnet. The normal $\hat{n}_{\text{pillbox}}$ is outward from the pillbox, but on the face inside the magnet $\hat{n}_{\text{pillbox}} = -\hat{n}$, where $\hat{n}$ is the outward normal from the magnetized volume. Thus the magnetic surface charge density is

$$\sigma_M = \vec{M}\cdot\hat{n} \tag{50}$$

In such cases the potential is

$$\Psi_m(\vec{x}) = \frac{1}{4\pi}\int_S \frac{\vec{M}(\vec{x}')\cdot\hat{n}}{|\vec{x}-\vec{x}'|} d^2\vec{x}' \tag{51}$$

where the integral is over all parts of the surface where $\vec{M}\cdot\hat{n}$ is not zero.