

1 Finite element analysis

Here's the big idea: we want a procedure for numerically solving a potential problem with a source $g(\vec{x})$ in a region R and with specified boundary conditions.

Let's look at a Dirichlet problem in 2 dimensions.

We start with two general relations. First, if $\nabla^2\psi = -g$ in R , then

$$\int_R \phi (\nabla^2\psi + g) dx dy = 0$$

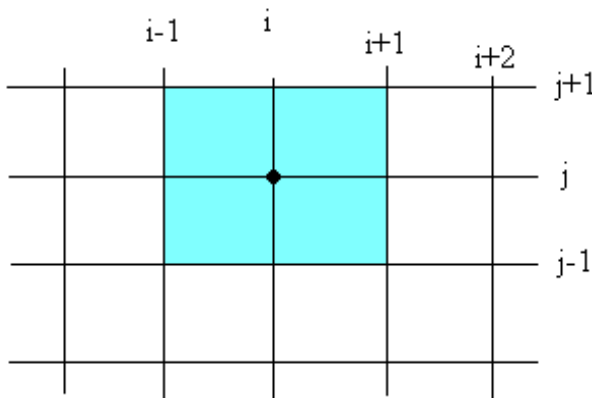
for *any* function ϕ . (If ψ is the potential, then $g = \rho/\epsilon_0$.) The next relation we need is Green's first identity from Chapter 1 ("Formal" notes eqn 1),

$$\int \left(\phi \nabla^2\psi + \vec{\nabla}\phi \cdot \vec{\nabla}\psi \right) dV = \int_S \phi \frac{\partial\psi}{\partial n} dA = 0$$

where we have chosen the function ϕ to be zero on the boundary, so that the surface integral is zero. (Note: in a two dimensional problem the "volume" is an area and the bounding "surface" S is actually a curve.) Now inserting the differential equation for ψ , we get:

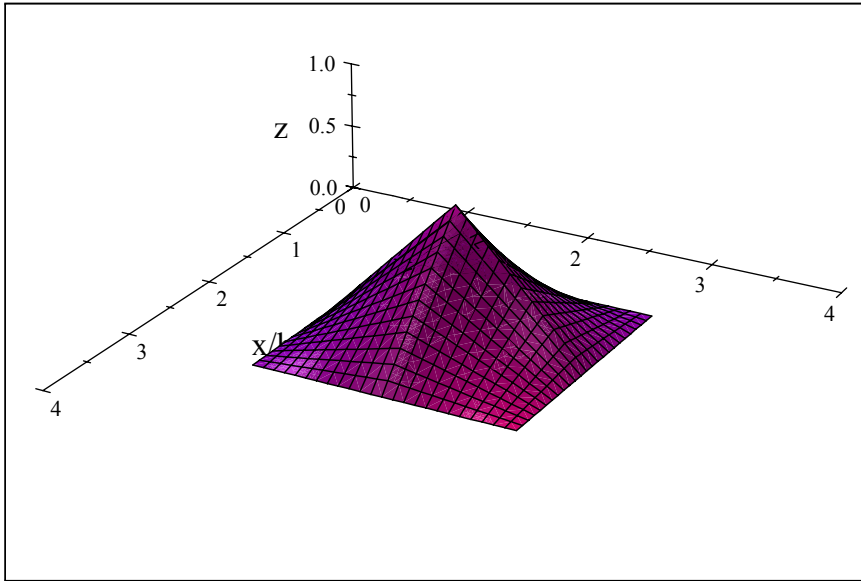
$$\int_R \left(\vec{\nabla}\phi \cdot \vec{\nabla}\psi - g\phi \right) dx dy = 0 \tag{1}$$

The next step is to set up a grid over the region R . We expand the desired solution ψ in a set of functions ϕ_{ij} each of which is zero except on a small region around the grid point (i, j) . For example, let:



$$\phi_{ij} = \begin{cases} \left(1 - \frac{|x-x_i|}{h}\right) \left(1 - \frac{|y-y_j|}{h}\right) & \text{for } |x-x_i| < h, |y-y_j| < h \\ 0 & \text{otherwise} \end{cases}$$

which looks like this:



Then let

$$\psi(x, y) = \sum_{k,l} \psi_{kl} \phi_{kl}(x, y)$$

Effectively, the function ϕ_{kl} smears the potential value ψ_{kl} over one grid square, converting a set of numerical values ψ_{kl} into a continuous function $\psi(x, y)$. Now insert this assumed form into the integral (1), and take the function ϕ to be one of the ϕ_{ij} .

$$\int_R \sum_{k,l} \psi_{kl} \vec{\nabla} \phi_{kl} \cdot \vec{\nabla} \phi_{ij} dx dy = \int_R \phi_{ij} g(x, y) dx dy \quad (2)$$

The integral on the right is non zero only on a square region surrounding the point (x_i, y_j) , and, because the function ϕ_{ij} peaks sharply at the center of the square, we can approximate the integral as:

$$\int_R \phi_{ij} g(x, y) dx dy \simeq g(x_i, y_j) \int_{\text{square}} \phi_{ij} dx dy$$

which may be evaluated as follows. Let $u = x - x_i$ and $v = y - y_j$. Then:

$$\begin{aligned} \int_{\text{square}} \phi_{ij} dx dy &= \left(\int_0^h \left(1 - \frac{u}{h}\right) du + \int_{-h}^0 \left(1 + \frac{u}{h}\right) du \right) \left(\int_0^h \left(1 - \frac{v}{h}\right) dv + \int_{-h}^0 \left(1 + \frac{v}{h}\right) dv \right) \\ &= \left(\frac{h}{2} + \frac{h}{2} \right) \left(\frac{h}{2} + \frac{h}{2} \right) = h^2 \end{aligned}$$

Thus eqn: (2) becomes:

$$\sum_{k,l} \psi_{kl} \int_R \vec{\nabla} \phi_{kl} \cdot \vec{\nabla} \phi_{ij} dx dy = h^2 g(x_i, y_j) \quad (3)$$

where the sum on the left is over all the grid points. However, the integrand is non-zero only on a small square region $2h$ by $2h$ surrounding the grid point (i, j) . First note that inside this square region,

$$\vec{\nabla} \phi_{ij} = \mp \frac{1}{h} \left(1 \mp \frac{v}{h}\right) \hat{\mathbf{x}} + \left(1 \mp \frac{u}{h}\right) \left(\mp \frac{1}{h}\right) \hat{\mathbf{y}}$$

where in the first factor of each term we take the upper sign on the right half of our square ($0 < u < h$) and the bottom sign on the left ($-h < u < 0$); in the second factor we take the top sign on the top of the box ($0 < v < h$) and the bottom sign on the bottom ($-h < v < 0$).

Thus if $k = i$ and $l = j$:

$$\begin{aligned} \int_{\text{box}} \vec{\nabla} \phi_{ij} \cdot \vec{\nabla} \phi_{i,j} dx dy &= \frac{1}{h^2} \int_0^h \int_0^h \left[\left(1 - \frac{v}{h}\right)^2 + \left(1 - \frac{u}{h}\right)^2 \right] dudv \\ &+ \frac{1}{h^2} \int_0^h \int_{-h}^0 \left[\left(1 + \frac{v}{h}\right)^2 + \left(1 - \frac{u}{h}\right)^2 \right] dudv \\ &+ \frac{1}{h^2} \int_{-h}^0 \int_0^h \left[\left(1 - \frac{v}{h}\right)^2 + \left(1 + \frac{u}{h}\right)^2 \right] dudv \\ &+ \frac{1}{h^2} \int_{-h}^0 \int_{-h}^0 \left[\left(1 + \frac{v}{h}\right)^2 + \left(1 + \frac{u}{h}\right)^2 \right] dudv \end{aligned}$$

In the four terms, let $\alpha = 1 - \frac{v}{h}$ (terms 1 and 3), $\alpha = 1 + \frac{v}{h}$ (terms 2 and 4), $\beta = 1 - \frac{u}{h}$ (terms 1 and 2) and $\beta = 1 + \frac{u}{h}$ (terms 3 and 4). Then the integral becomes

$$\begin{aligned} \int_{\text{box}} \vec{\nabla} \phi_{ij} \cdot \vec{\nabla} \phi_{i,j} dx dy &= 4 \int_0^1 \int_0^1 (\alpha^2 + \beta^2) d\alpha d\beta \\ &= \frac{4}{3} (\alpha^3 \beta + \alpha \beta^3) \Big|_0^1 \Big|_0^1 = \frac{8}{3} \end{aligned}$$

However if $k = i + 1$ and $l = j$:

$$\begin{aligned} \phi_{i+1,j} &= \left(1 - \frac{|x - x_i - h|}{h}\right) \left(1 - \frac{|y - y_j|}{h}\right) \\ &= \left(1 - \frac{|u - h|}{h}\right) \left(1 - \frac{|v|}{h}\right) \text{ for } |u - h| < h \text{ and } |v| < h \end{aligned}$$

and zero otherwise. The overlap region is on the right side of our box ($0 < u < h$) where

$$\vec{\nabla} \phi_{i+1,j} = \frac{1}{h} \left(1 \mp \frac{v}{h}\right) \hat{\mathbf{x}} + \frac{u}{h} \left(\mp \frac{1}{h}\right) \hat{\mathbf{y}}$$

giving

$$\begin{aligned}
\int_{\text{box}} \vec{\nabla} \phi_{ij} \cdot \vec{\nabla} \phi_{i+1,j} dx dy &= \frac{1}{h^2} \int_0^h \int_0^h \left(-\left(1 - \frac{v}{h}\right)^2 + \left(1 - \frac{u}{h}\right) \frac{u}{h} \right) dudv \\
&\quad + \frac{1}{h^2} \int_{-h}^0 \int_0^h \left(-\left(1 + \frac{v}{h}\right)^2 + \left(1 - \frac{u}{h}\right) \frac{u}{h} \right) dudv \\
&= 2 \int_0^1 \int_0^1 (-\alpha^2 + \beta(1 - \beta)) d\alpha d\beta \\
&= 2 \left(-\frac{\alpha^3 \beta}{3} + \frac{\alpha \beta^2}{2} - \frac{\alpha \beta^3}{3} \right) \Big|_0^1 \Big|_0^1 \\
&= 2 \left(\frac{1}{2} - \frac{2}{3} \right) = -\frac{1}{3}
\end{aligned}$$

We get the same result in all the overlap regions. Thus equation (3) may be written as a matrix equation:

$$\mathbb{K}\Psi = \mathbb{G}$$

where the matrix \mathbb{K} is described as *sparse* – it has only a few non-vanishing elements and they are all near the diagonal, like this:

$$\mathbb{K} = \frac{1}{3} \begin{pmatrix} 8 & -1 & -1 & 0 & 0 & 0 \\ -1 & 8 & -1 & -1 & 0 & 0 \\ -1 & -1 & 8 & -1 & -1 & 0 \\ 0 & -1 & -1 & 8 & -1 & -1 \\ 0 & 0 & -1 & -1 & 8 & -1 \\ 0 & 0 & 0 & -1 & -1 & 8 \end{pmatrix}$$

Matrices of this type are relatively easy to invert numerically. The column vectors Ψ and \mathbb{G} contain the values of the potential and the source at the grid points. and we have reduced the potential problem to a matrix inversion.

$$\Psi = \mathbb{K}^{-1} \mathbb{G}$$

For regions with odd shapes, the square grid we used above does not fit very well. Triangles of arbitrary size and shape can be fit onto a region of almost any shape. So now we'll modify the method above to use triangles instead of squares. By varying the sizes and shapes of the triangles we can also get better resolution where things change more rapidly.

The basic triangular element has vertices at points that we label 1, 2 and 3 with coordinates (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) . We approximate the potential solution $\psi(x, y)$ in this triangle by a Taylor-series-type expansion of the form:

$$\psi(x, y) = A + Bx + Cy$$

The 3 values of the potential at the 3 vertices give enough information to evaluate the 3 coefficients A, B and C . To make the numerical computation more efficient, it is convenient to define three ‘‘shape functions’’ $N_i(x, y) = a_i + b_i x + c_i y$

that have the properties:

$$N_i(x_i, y_i) = 1$$

and

$$N_i(x_j, y_j) = 0, \quad i \neq j$$

These functions are the analogue of the ϕ_{ij} that we used with the square grid. They have the effect of smearing the potential V_i at node i over the whole triangle. Then, for example, for N_1 we have:

$$\begin{aligned} a_1 + b_1 x_1 + c_1 y_1 &= 1 \\ a_1 + b_1 x_2 + c_1 y_2 &= 0 \\ a_1 + b_1 x_3 + c_1 y_3 &= 0 \end{aligned}$$

This set of equations has a nontrivial solution for the coefficients a_1, b_1 and c_1 if the determinant:

$$D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \neq 0$$

The determinant is:

$$\begin{aligned} D &= x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1 \\ &= (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \end{aligned}$$

This is related to the area of the triangle. To see how, remember that we can write the area using the cross product:

$$A = \frac{1}{2} \left| \vec{\ell}_1 \times \vec{\ell}_2 \right|$$

where $\vec{\ell}_1$ and $\vec{\ell}_2$ are vectors along the sides of the triangle. In terms of the coordinates:

$$\vec{\ell}_1 = (x_2 - x_1) \hat{x} + (y_2 - y_1) \hat{y}$$

and

$$\vec{\ell}_2 = (x_3 - x_1) \hat{x} + (y_3 - y_1) \hat{y}$$

Then

$$2A = |(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|$$

and so

$$|D| = 2A$$

and so is never zero. If we label the vertices so as to make D positive, then the solution for the coefficients is:

$$\begin{aligned} a_1 &= \frac{1}{2A} (x_2 y_3 - x_3 y_2) \\ b_1 &= \frac{1}{2A} (y_2 - y_3) \\ c_1 &= -\frac{1}{2A} (x_2 - x_3) \end{aligned} \tag{4}$$

and similarly for the others.

Now the procedure runs pretty much as before. In equation (1) we take:

$$\psi = \sum_j V_j N_j$$

where V_j are constants (the values of the potentials at the nodes) and the sum is over all the vertices of all the triangles. Next take the function $\phi = N_i$ for one vertex of one triangle. Then relation (1) reduces to an integral over the one triangle where $\vec{\nabla} N_i$ is non-zero:

$$\begin{aligned} \sum_{j=1}^3 V_j \int_{\text{triangle}} \vec{\nabla} N_i \cdot \vec{\nabla} N_j \, dxdy &= \int g N_i \, dxdy \simeq g(\bar{x}, \bar{y}) \int N_i \, dxdy \\ &= g(\bar{x}, \bar{y}) A (a_i + b_i \bar{x} + c_i \bar{y}) \end{aligned} \quad (5)$$

where \bar{x}, \bar{y} are the coordinates of the center of the triangle. Now using our solutions (4) for the coefficients, we have:

$$\begin{aligned} a_1 + b_1 \bar{x} + c_1 \bar{y} &= \frac{1}{2A} \left[(x_2 y_3 - x_3 y_2) + (y_2 - y_3) \frac{(x_1 + x_2 + x_3)}{3} - (x_2 - x_3) \frac{(y_1 + y_2 + y_3)}{3} \right] \\ &= \frac{1}{6A} [x_2 y_3 - x_3 y_2 + x_1 y_2 - x_1 y_3 - x_2 y_1 + x_3 y_1] = \frac{D}{6A} = \frac{1}{3} \end{aligned}$$

On the left hand side of (5), we use the result that

$$\vec{\nabla} N_i = b_i \hat{x} + c_i \hat{y}$$

and is constant over the area of the triangle, so (5) becomes

$$\sum_{j=1}^3 V_j (b_i b_j + c_i c_j) A = g(\bar{x}, \bar{y}) \frac{A}{3}$$

To combine the triangles, we define

$$k_{ij} \equiv (b_i b_j + c_i c_j) A \quad (6)$$

(Note here that if we chose to label our vertices so that D comes out negative, the a, b, c change sign, but we would get the same values for k_{ij} .) If $i = j$, k_{ii} refers to a single node. For each internal node i we sum over all the triangles connected to that node.

$$K_{ii} = \sum_{\text{triangles}} k_{ii} \quad (7)$$

The elements k_{ij} , $i \neq j$, are associated with two nodes, *i.e.* with the side of the triangle connecting the nodes. So we sum over all the triangles with a side along ij (usually two).

$$K_{ij} = \sum_{\text{triangles}} k_{ij} \quad i < j \leq N \quad (8)$$

If a node is on the boundary, the value of the potential there will be known. These terms are moved to the right hand side and serve as source terms. So we have:

$$G_i = \frac{1}{3} \sum_{\text{triangles}} A_t g_t - \sum_{j=N+1}^{N_0} K_{ij} V_j \quad (9)$$

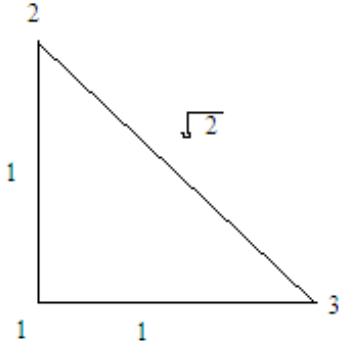
where the numbers $N + 1$ to N_0 label the nodes on the boundary.

Then combining the results for all the triangles, we have the matrix equation:

$$\mathbb{K}\Phi = \mathbb{G} \quad (10)$$

where again \mathbb{K} is a sparse matrix, and the vector Φ contains the values V_j of the potentials at the nodes.

Example of how to get the k_{ij} . Consider an isoceles right triangle of sides 1,1 and $\sqrt{2}$. Put the origin at the right angle, and label the vertices 1 (the origin), 2 (on the y -axis) and 3 (on the x -axis). Then the area of the triangle is $1/2$, and $2A = 1$



$$a_1 = x_2 y_3 - x_3 y_2 = 0 - 1 = -1$$

$$b_1 = y_2 - y_3 = 1$$

$$c_1 = -(x_2 - x_3) = -(-1) = 1$$

$$b_2 = y_3 - y_1 = 0$$

$$c_2 = -(x_3 - x_1) = -1$$

Then from (6) we get

$$k_{11} = \frac{1}{2} (b_1^2 + c_1^2) = \frac{1}{2} (1 + 1) = 1$$

$$k_{12} = \frac{1}{2} (b_1 b_2 + c_1 c_2) = \frac{1}{2} (0 + 1(-1)) = -\frac{1}{2}$$

and so on. (See Jackson Fig 2.16). These coefficients depend on the shape of the triangle but not on its orientation or size.