Green's function notes 2020

# 1 Introduction

Back in the "formal" notes, we derived the potential in terms of the Green's function.

**Dirichlet problem**: Equation (7) in "formal" notes is

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V G_D(\vec{x},\vec{x}')\,\rho(\vec{x}')\ d^3\vec{x}' - \frac{1}{4\pi}\int_S \Phi(\vec{x}')\,\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\ dA' \tag{1}$$

which is also Jackson eqn.1.44. The Dirichlet Green's function is symmetric in $\vec{x}$ and $\vec{x}'$ (see Lea §C.7.1 for the proof).

$$G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$$

**Neumann problem:** Writing $<\Phi>_S = \frac{1}{A}\int_S \Phi\ dA$, the average value of the potential over the surface $S$, equation (9) in "formal" notes is

$$\Phi(\vec{x}) - \ <\Phi>_S = \frac{1}{4\pi\varepsilon_0}\int_V G_N(\vec{x},\vec{x}')\,\rho(\vec{x}')\,d^3\vec{x}' + \frac{1}{4\pi}\int_S G_N\frac{\partial\Phi}{\partial n'}dA' \tag{2}$$

In order to make use of these expressions, we need a form for the Green's function that is relatively easy to integrate. That suggests that we expand the Green's function in orthogonal functions. The Dirichlet Green's function is symmetric in the two sets of coordinates $\vec{x}$ and $\vec{x}'$. (See Lea pg 514 and J problem 1.14.) In the Neumann case it is possible to impose the symmetry as an additional condition. (In problems 1.14 and 3.27 you can see why this does not affect the computed potential). So it does not matter whether we compute $G(\vec{x},\vec{x}')$ or $G(\vec{x}',\vec{x})$ . I find it easier to use $\vec{x}$ as the variable and $\vec{x}'$ as fixed while finding $G$ to reduce the number of primes I have to write. Then the differential equation satisfied by $G$ is ("formal" notes eqn 3):

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \tag{3}$$

This means that $G$ represents the potential at $\vec{x}$ due to a unit point charge[1] at $\vec{x}'$, multiplied by $4\pi\varepsilon_0$. This shows that the physical dimensions of $G$ are 1/length.

The boundary conditions are

**Dirichlet case:**

$$G_D(\vec{x},\vec{x}') = 0 \ \text{ for } \vec{x} \ \text{ on } S \tag{4}$$

and

**Neumann case**:

$$\frac{\partial G_N}{\partial n} = \hat{n}\cdot\vec{\nabla}G_N = -\frac{4\pi}{A} \ \text{ for } \vec{x} \text{ on } S \tag{5}$$

where $A$ is the *total* area of the surface $S$ bounding the volume $V$, and, as usual, $\hat{n}$ is the outward normal.

[1] In 2 dimensions it is a line charge.

## 2 Division of region method

The 10-step method for finding $G$ is outlined in Lea (pg 515). We begin by (Step **1**) drawing the region under consideration and (Step **2**) choosing a coordinate system so that the boundaries are represented by constant values of one or more of the coordinates. In this method (Step **3**) we place the unit point charge at an arbitrary point $\vec{x}'$ within the volume, and (Step **5**) use this point to divide the region into two separate regions, I and II, with $\vec{x}'$ on the boundary between them. Then $G$ satisifies the simpler differential equation

$$\nabla^2 G(\vec{x}, \vec{x}') = 0$$

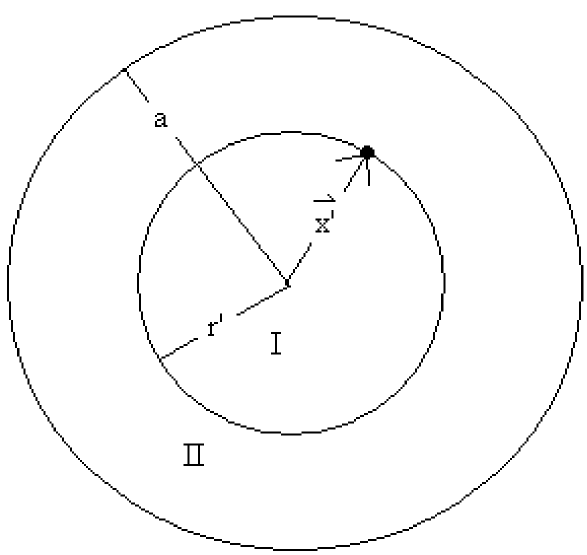
within each of regions I and II, (but not on the boundary between them). This means that we can (Step **4**) make use of the eigenfunctions of Laplace's equation.

### 2.1 Dirichlet Green's Function in Spherical Coordinates

Suppose our volume is the interior of a sphere of radius $a$. We place a unit point charge at an arbitrary (but, for the moment, fixed) point in the region with coordinates $r', \theta', \phi'$. This point then divides the volume into two regions:

Region I: $0 \leq r < r'$

Region II: $r' < r \leq a$



(Step **6**) Then in region I the appropriate solution to Laplace's equation is

$$G_I(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} A_{lm} r^l Y_{lm}(\theta, \phi) \tag{6}$$

where we have used only the functions $r^l$ and excluded $r^{-(l+1)}$ because the potential should be finite at $r = 0$. The dependence on the coordinates $r', \theta'$ and $\phi'$ is contained in the coefficient $A_{lm}$.

Region II contains neither $r = 0$ nor $r \to \infty$, so we need both functions of $r$ :

$$G_{II}(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \left( B_{lm} r^l + \frac{C_{lm}}{r^{l+1}} \right) Y_{lm}(\theta, \phi)$$

However, here we have to deal with the boundary at $r = a$ where $G$ has to be zero:

$$G_{II}(\vec{x}_{\text{on } S}, \vec{x}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \left( B_{lm} a^l + \frac{C_{lm}}{a^{l+1}} \right) Y_{lm}(\theta, \phi) = 0$$

We make use of the orthogonality of the $Y_{lm}$ to argue that each term must separately equal zero:

$$B_{lm} a^l + \frac{C_{lm}}{a^{l+1}} = 0 \Rightarrow C_{lm} = -B_{lm} a^{2l+1}$$

Then

$$G_{II}(\vec{x}, \vec{x}') = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} B_{lm} \left( r^l - \frac{a^{2l+1}}{r^{l+1}} \right) Y_{lm}(\theta, \phi) \tag{7}$$

We still have two sets of unknown constants: the $A_{lm}$ and $B_{lm}$. But we have one more boundary to consider at $r = r'$. (Step **7**) The first boundary condition we need is continuity of the potential ($G$) at $r = r'$.

$$\begin{aligned} G_I(\vec{x}, \vec{x}')_{r=r'} &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} A_{lm} (r')^l Y_{lm}(\theta, \phi) \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} B_{lm} \left[ (r')^l - \frac{a^{2l+1}}{(r')^{l+1}} \right] Y_{lm}(\theta, \phi) = G_{II}(\vec{x}, \vec{x}')_{r=r'} \end{aligned}$$

Again we make use of the orthogonality of the $Y_{lm}$ to argue that

$$A_{lm} (r')^l = B_{lm} \left[ (r')^l - \frac{a^{2l+1}}{(r')^{l+1}} \right]$$

for each $l, m$. Thus

$$\begin{aligned} G_I(\vec{x}, \vec{x}') &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} B_{lm} r^l \left( 1 - \frac{a^{2l+1}}{(r')^{2l+1}} \right) Y_{lm}(\theta, \phi) \\ G_{II}(\vec{x}, \vec{x}') &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} B_{lm} r^l \left( 1 - \frac{a^{2l+1}}{r^{2l+1}} \right) Y_{lm}(\theta, \phi) \end{aligned}$$

We'd like to display the symmetry in $r$ and $r'$ more obviously, and we can do that by relabeling

$$B_{lm} = \beta_{lm} \left(r'\right)^l$$

(Remember that for the moment $r'$ is a constant.) Then

$$\begin{aligned} G_I\left(\vec{x}, \vec{x}'\right) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{\left(r'\right)^{2l+1}}\right) Y_{lm}\left(\theta, \phi\right) \\ G_{II}\left(\vec{x}, \vec{x}'\right) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{r^{2l+1}}\right) Y_{lm}\left(\theta, \phi\right) \end{aligned}$$

or

$$G\left(\vec{x}, \vec{x}'\right) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{r_>^{2l+1}}\right) Y_{lm}\left(\theta, \phi\right) \tag{8}$$

where $r_> = \max\left(r, r'\right)$ and the same expression (8) holds in both regions.

We still need to find the $\beta_{lm}$ and to do this (Step **8**) we make use of the differential equation (3). First we insert expression (8) for $G$ and evaluate the derivatives in $\theta$ and $\phi$. We express the delta function in terms of the spherical coordinates using the result of Jackson Problem 1.2.

$$\begin{aligned} \nabla^2 G &= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} G\right) + \nabla^2_{\text{ang}} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{r_>^{2l+1}}\right) Y_{lm}\left(\theta, \phi\right) \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{r_>^{2l+1}}\right)\right] Y_{lm}\left(\theta, \phi\right) \\ &\quad - \frac{1}{r^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \beta_{lm} \left(r'\right)^l r^l \left(1 - \frac{a^{2l+1}}{r_>^{2l+1}}\right) l\left(l+1\right) Y_{lm}\left(\theta, \phi\right) \\ &= -\frac{4\pi}{r^2} \delta\left(r - r'\right) \delta\left(\mu - \mu'\right) \delta\left(\phi - \phi'\right) \end{aligned}$$

Now we multiply both sides of the equation by $Y^*_{l'm'}\left(\theta, \phi\right)$ and integrate over the whole solid angle of the sphere. On the left hand side we use the orthogonality of the $Y_{lm}$. On the right hand side we use the sifting property.

$$\begin{aligned} &\frac{\beta_{l'm'}}{r^2} \frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r} \left(r'r\right)^{l'} \left(1 - \frac{a^{2l'+1}}{r_>^{2l'+1}}\right)\right] - \beta_{l'm'} \frac{\left(r'r\right)^{l'}}{r^2} \left(1 - \frac{a^{2l'+1}}{r_>^{2l'+1}}\right) l'\left(l'+1\right) \\ = &-\frac{4\pi}{r^2} \delta\left(r - r'\right) Y^*_{l'm'}\left(\theta', \phi'\right) \end{aligned} \tag{9}$$

Now we can drop the primes on $l'$ and $m'$. Thus each $\beta_{lm}$ contains a factor $Y^*_{lm}\left(\theta', \phi'\right)$.

$$\beta_{lm} = \gamma_{lm} Y^*_{lm}\left(\theta', \phi'\right)$$

and eqn (8) becomes

$$G\left(\vec{x},\vec{x}'\right)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\gamma_{lm}\left(r'r\right)^{l}\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)Y_{lm}\left(\theta,\phi\right)Y_{lm}^{*}\left(\theta',\phi'\right)$$

Now we have symmetry in all the coordinates, and the remaining set of constants $\gamma_{lm}$ should be independent of all coordinates. The remaining equation (9) takes the form

$$\frac{\gamma_{lm}}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(r'r\right)^{l}\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)\right]-\gamma_{lm}\frac{\left(r'r\right)^{l}}{r^2}\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)l\left(l+1\right)=-\frac{4\pi}{r^2}\delta\left(r-r'\right)$$

(Step **9**) To make use of this equation we multiply both sides by $r^2$, then integrate across the internal boundary from $r=r'-\varepsilon$ to $r=r'+\varepsilon$.

$$\begin{aligned}&\gamma_{lm}\int_{r'-\varepsilon}^{r'+\varepsilon}\left\{\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(r'r\right)^{l}\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)\right]-\left(r'r\right)^{l}\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)l\left(l+1\right)\right\}dr\\ =\quad&-4\pi\int_{r'-\varepsilon}^{r'+\varepsilon}\delta\left(r-r'\right)dr\end{aligned}$$

The integral on the RHS equals 1. The second term on the LHS $\to 0$ as $\varepsilon\to 0$ because the integrand is continuous by construction and has no singularities in the range of integration. Thus we are left with

$$\lim_{\varepsilon\to 0}\gamma_{lm}r^2\left(r'\right)^{l}\frac{\partial}{\partial r}\left[r^{l}\left(1-\frac{a^{2l+1}}{r_>^{2l+1}}\right)\right]\Bigg|_{r'-\varepsilon}^{r'+\varepsilon}=-4\pi$$

At the upper limit, $r=r'+\varepsilon$ is $>r'$, so $r_>=r$. At the lower limit, $r=r'-\varepsilon$ is $<r'$, so $r_>=r'$. Thus

$$\gamma_{lm}\left(r'\right)^{l}\left\{r^2\frac{\partial}{\partial r}\left[r^{l}-\frac{a^{2l+1}}{r^{l+1}}\right]\Bigg|_{r'}-r^2\frac{\partial}{\partial r}\left[r^{l}\left(1-\frac{a^{2l+1}}{\left(r'\right)^{2l+1}}\right)\right]\Bigg|_{r'}\right\}=-4\pi$$

$$\begin{aligned}\gamma_{lm}\left(r'\right)^{l+2}\left[l\left(r'\right)^{l-1}+\left(l+1\right)\frac{a^{2l+1}}{\left(r'\right)^{l+2}}-l\left(r'\right)^{l-1}\left(1-\frac{a^{2l+1}}{\left(r'\right)^{2l+1}}\right)\right]&=-4\pi\\ \gamma_{lm}\left(r'\right)^{l+2}\left[\left(l+1\right)\frac{a^{2l+1}}{\left(r'\right)^{l+2}}+l\frac{a^{2l+1}}{\left(r'\right)^{l+2}}\right]&=-4\pi\end{aligned}$$

$$\gamma_{lm}=-\frac{4\pi}{2l+1}\frac{1}{a^{2l+1}}$$

$\gamma_{lm}$ is independent of all coordinates, as expected, and thus

$$G\left(\vec{x},\vec{x}'\right)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}\frac{\left(r'\right)^{l}r^{l}}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{2l+1}}-1\right)Y_{lm}\left(\theta,\phi\right)Y_{lm}^{*}\left(\theta',\phi'\right)\qquad(10)$$

(Step **10**) $G$ has dimensions of 1/length, as required, and displays the necessary symmetry. It is also positive, since $a \geq r_>$ throughout the region. In the limit $a \to \infty$ we get

$$\begin{aligned}
G &\to \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} \frac{(r')^l r^l}{r_>^{2l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') \\
&= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') = \frac{1}{|\vec{x} - \vec{x}'|}
\end{aligned}$$

(spherprobnotes eqn 25) which is the correct result for a point charge in infinite space.

## 2.2 Use of the result

Suppose we have a sphere of radius $a$ with potential $V$ on one half and $-V$ on the other half. Inside the sphere, a line charge of length $2b$ ($b < a$) with uniform line charge density $\lambda$ runs along the diameter of the dividing plane and is centered at the center of the sphere. Find the potential inside.

First we have to choose coordinates. No matter what we do, we do not have azimuthal symmetry. I'm going to put the polar axis along the line charge, giving a charge density

$$\rho(\vec{x}) = \frac{\lambda}{2\pi r^2} [\delta(\mu - 1) + \delta(\mu + 1)] S(b - r)$$

We can check this by finding the total charge on a differential piece of the line that runs from $r$ to $r + dr$ with $r < b$. Then

$$\begin{aligned}
dq &= 2\lambda dr = \int_0^{2\pi} \int_{-1}^{+1} \frac{\lambda}{2\pi r^2} [\delta(\mu - 1) + \delta(\mu + 1)] r^2 dr d\mu d\phi \\
&= \frac{\lambda}{2\pi} (2) 2\pi dr = 2\lambda dr \qquad \checkmark
\end{aligned}$$

(Do you understand the factor of 2? Draw a picture and see!) The potential on the surface is then

$$\begin{aligned}
\Phi_S &= +V \quad \text{if} \quad 0 \leq \phi < \pi \\
&= -V \quad \text{if} \quad \pi \leq \phi < 2\pi
\end{aligned}$$

Thus the potential is (eqn 1)

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} \int_{-1}^{+1} \int_0^a G_D(\vec{x}, \vec{x}') \frac{\lambda}{2\pi (r')^2} [\delta(\mu' - 1) + \delta(\mu' + 1)] S(b - r') \ (r')^2 dr' d\mu' d\phi' \\
&\quad - \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{+1} \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'} dA' \qquad (11)
\end{aligned}$$

Now the field point $\vec{x}$ is fixed and $\vec{x}'$ is variable. The first integral gives potential $\Phi_1$ where

$$4\pi\varepsilon_0\Phi_1(\vec{x}) = \int_0^{2\pi}\int_{-1}^{+1}\int_0^a \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} \frac{4\pi}{2l+1}\frac{(r')^l r^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1\right) Y_{lm}(\theta,\phi)\, Y_{lm}^*(\theta',\phi')$$
$$\times \frac{\lambda}{2\pi (r')^2}\left[\delta(\mu'-1) + \delta(\mu'+1)\right] S(b-r') \quad (r')^2\, dr' d\mu' d\phi'$$

By orthogonality of the $e^{im\phi'}$,the integral over $\phi'$ gives zero unless $m = 0$, in which case we get $2\pi$, so we have

$$\int_{-1}^{+1}\int_0^a \sum_{l=0}^{\infty} \frac{4\pi\lambda}{2l+1}\frac{(r')^l r^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1\right) N_{l0}^2 P_l(\mu)\, P_l(\mu')\left[\delta(\mu'-1) + \delta(\mu'+1)\right] S(b-r')\ dr' d\mu'$$

where $N_{l0} = \sqrt{(2l+1)/4\pi}$. Next we use the sifting property of the delta-functions to do the integral over $\mu'$.

$$\int_0^a \sum_{l=0}^{\infty} \lambda \frac{(r')^l r^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1\right) P_l(\mu)\left[P_l(1) + P_l(-1)\right] S(b-r')\ dr'$$

Since $P_l(-1) = (-1)^l$ , the result is zero unless $l$ is even, when the square bracket equals 2. This is expected due to reflection symmetry about the plane $\mu = 0$. Finally we do the integral over $r'$. The integrand is zero for $r' > b$, so we have

$$4\pi\varepsilon_0\Phi_1(\vec{x}) = \sum_{l=0,\text{ even}}^{\infty} 2\lambda \frac{r^l}{a^{2l+1}}\int_0^b (r')^l\left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1\right) dr' P_l(\mu) \qquad (12)$$

Looking at the integral over $r'$ by itself, we note that if $r < b$ we have to split the integral into two parts: $r' < r$ and $r' > r$ .

$$\begin{aligned}
\int_0^b (r')^l\left(\frac{a^{2l+1}}{r_>^{2l+1}} - 1\right) dr' &= \int_0^r (r')^l\left(\frac{a^{2l+1}}{r^{2l+1}} - 1\right) dr' + \int_r^b (r')^l\left(\frac{a^{2l+1}}{(r')^{2l+1}} - 1\right) dr' \\
&= \frac{r^{l+1}}{l+1}\left(\frac{a^{2l+1}}{r^{2l+1}} - 1\right) - \frac{a^{2l+1}}{l}\left(\frac{1}{b^l} - \frac{1}{r^l}\right) - \left(\frac{b^{l+1} - r^{l+1}}{l+1}\right) \\
&= \frac{a^{2l+1}}{r^l}\frac{(2l+1)}{l(l+1)} - \frac{a^{2l+1}}{lb^l} - \frac{b^{l+1}}{l+1} \qquad \text{for} \quad r < b, \quad l > 0
\end{aligned}$$

Note that this fails for $l = 0$ because of the $l$ in the denominator. For $l = 0$, we get

$$\begin{aligned}
\int_0^b \left(\frac{a}{r_>} - 1\right) dr' &= \int_0^r \left(\frac{a}{r} - 1\right) dr' + \int_r^b \left(\frac{a}{r'} - 1\right) dr' \\
&= a - r + a\ln\frac{b}{r} - (b - r) \\
&= a - b + a\ln\frac{b}{r}
\end{aligned}$$

Including the factor of $1/4\pi\varepsilon_0$, the first term in the potential for $r<b$ is

$$\begin{aligned}\Phi_1(\vec{x}) &= \frac{2\lambda}{4\pi\varepsilon_0}\left\{\frac{1}{a}\left(a-b+a\ln\frac{b}{r}\right)P_0(\mu)+\right.\\ &\qquad\left.\sum_{l=2,\text{ even}}^{\infty}\frac{r^l}{a^{2l+1}}\left[\frac{a^{2l+1}}{r^l}\frac{(2l+1)}{l(l+1)}-\frac{a^{2l+1}}{lb^l}-\frac{b^{l+1}}{l+1}\right]P_l(\mu)\right\}\\ &= \frac{\lambda}{2\pi\varepsilon_0}\left\{\left(1-\frac{b}{a}+\ln\frac{b}{r}\right)\right. \qquad (13)\\ &\qquad\left.+\sum_{l=2,\text{ even}}^{\infty}\left[\frac{(2l+1)}{l(l+1)}-\frac{r^l}{lb^l}\left(1+\frac{l}{l+1}\frac{b^{2l+1}}{a^{2l+1}}\right)\right]P_l(\mu)\right\}\end{aligned}$$

The log term in (13) is expected near a line charge, and even $l$ indicates the reflection symmetry about the equatorial $(z=0)$ plane.

If $r>b$ then $r=r_>$ throughout the range of integration, and the integral over $r'$ is

$$\int_0^b (r')^l\left(\frac{a^{2l+1}}{r^{2l+1}}-1\right)dr'=\frac{b^{l+1}}{l+1}\left(\frac{a^{2l+1}}{r^{2l+1}}-1\right)\qquad\text{for}\quad r>b$$

and the potential (first term) is

$$\Phi_1(\vec{x})=\sum_{l=0,\text{even}}^{\infty}\frac{\lambda}{2\pi\varepsilon_0}\frac{r^l}{a^{2l+1}}\frac{b^{l+1}}{l+1}\left(\frac{a^{2l+1}}{r^{2l+1}}-1\right)P_l(\mu) \qquad (14)$$

Separating out the $l=0$ term, we have

$$\Phi_1(\vec{x})=\frac{2\lambda b}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{a}\right)+\frac{\lambda b}{2\pi\varepsilon_0}\sum_{l=2,\text{ even}}^{\infty}\frac{r^l}{a^{2l+1}}\frac{b^l}{l+1}\left(\frac{a^{2l+1}}{r^{2l+1}}-1\right)P_l(\mu)$$

The first term is the potential due to a point charge $q=2\lambda b$ at the center of a grounded sphere of radius $a$. Again this is what we would expect. Finally we note that the potential $\Phi_1$ is continuous at $r=b$.

We still need to evaluate the second term in (11). The outward normal from the volume at the surface $r=a$ is $\hat{n}=+\hat{r}$, so

$$\frac{\partial G}{\partial n'}=\hat{n}'\cdot\vec{\nabla}G=\frac{\partial G}{\partial r'}$$

Thus we have

$$\begin{aligned}-4\pi\Phi_2(\vec{x}) &= \int_0^{2\pi}\int_{-1}^{+1}\Phi(\vec{x}')\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}dA'\\ &= V\left(\int_0^{\pi}-\int_{\pi}^{2\pi}\right)\int_{-1}^{+1}\frac{\partial}{\partial r'}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}\frac{(r')^l r^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{2l+1}}-1\right)\Bigg|_{r'=a}\\ &\qquad\times Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi')\,a^2d\mu'd\phi'\end{aligned}$$

On the outer surface, $r' = a > r$, so $r_> = r'$ and we have

$$\begin{aligned}
\frac{\partial}{\partial r'} \frac{(r')^l r^l}{a^{2l+1}} \left( \frac{a^{2l+1}}{r_>^{2l+1}} - 1 \right) \bigg|_{r'=a} &= \frac{\partial}{\partial r'} \frac{(r')^l r^l}{a^{2l+1}} \left( \frac{a^{2l+1}}{(r')^{2l+1}} - 1 \right) \bigg|_{r'=a} \\
&= \frac{r^l}{a^{2l+1}} \frac{\partial}{\partial r'} \left( \frac{a^{2l+1}}{(r')^{l+1}} - (r')^l \right) \bigg|_{r'=a} \\
&= \frac{r^l}{a^{2l+1}} \left( -(l+1) \frac{a^{2l+1}}{(r')^{l+2}} - l (r')^{l-1} \right) \bigg|_{r'=a} \\
&= -\frac{r^l}{a^{2l+1}} a^{l-1} (2l+1) = -(2l+1) \frac{r^l}{a^{l+2}}
\end{aligned}$$

So

$$\Phi_2(\vec{x}) = V \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{lm}(\theta, \phi) \frac{r^l}{a^l} \left( \int_0^{\pi} - \int_{\pi}^{2\pi} \right) \int_{-1}^{+1} Y_{lm}^*(\theta', \phi') \, d\mu' d\phi'$$

Doing the integral over $\phi'$ first, we get zero if $m = 0$ and for $m \neq 0$

$$\begin{aligned}
\left( \int_0^{\pi} - \int_{\pi}^{2\pi} \right) e^{-im\phi'} d\phi' &= \frac{e^{-im\phi'}}{-im} \bigg|_0^{\pi} - \frac{e^{-im\phi'}}{-im} \bigg|_{\pi}^{2\pi} \\
&= \frac{e^{-im\pi} - 1}{-im} - \frac{e^{-im2\pi} - e^{-im\pi}}{-im} \\
&= \frac{2}{im} \left( 1 - (-1)^m \right)
\end{aligned}$$

So we get zero for $m$ even and $4/im$ for $m$ odd. Since $l \geq m$, the sum over $l$ now starts at 1. Thus the integral has reduced to

$$\Phi_2(\vec{x}) = 4V \sum_{l=1}^{\infty} \sum_{m=-l, \text{ odd}}^{+l} N_{lm} \frac{Y_{lm}(\theta, \phi)}{im} \int_{-1}^{+1} P_l^m(\mu') \, d\mu'$$

$P_l^m(\mu)$ is an even function of $\mu$ if $l + m$ is even and an odd function if $l + m$ is odd. Thus for the integral over $\mu'$ to be non-zero, we need $l + m$ to be even, and thus $l$ odd. Let's call the integral $I_{lm}$, and then we have

$$\Phi_2(\vec{x}) = 4V \sum_{l=1,\text{odd}}^{\infty} \sum_{m=-l, \text{ odd}}^{+l} \left( \frac{r}{a} \right)^l N_{lm} \frac{Y_{lm}(\theta, \phi)}{im} I_{lm}$$

Finally we should tidy this up by combining the positive and negative $m$ terms. Remember that $Y_{l,-m} = (-1)^m Y_{lm}^*$ (spherprobnotes eqn 24) and so $N_{l,-m} P_l^{-m} = (-1)^m N_{lm} P_l^m$. That makes $N_{l,-m} I_{l,-m} = (-1)^m N_{lm} I_{lm}$. Thus,

$$\begin{aligned}
N_{lm} Y_{lm}(\theta, \phi) \frac{I_{lm}}{im} + N_{l,-m} Y_{l,-m} \frac{I_{l,-m}}{-im} &= N_{lm}^2 P_l^m(\mu) e^{im\phi} \frac{I_{lm}}{im} - (-1)^m N_{l,m}^2 P_l^m(\mu) \frac{(-1)^m I_{lm}}{im} e^{-im\phi} \\
&= 2 N_{lm}^2 P_l^m(\mu) \frac{I_{lm}}{m} \sin m\phi
\end{aligned}$$

and so the second term in the potential is

$$
\begin{aligned}
\Phi_2(\vec{x}) &= 8V \sum_{l=1,\text{odd}}^{\infty} \sum_{m=1,\text{ odd}}^{+l} \left(\frac{r}{a}\right)^l N_{lm}^2 P_l^m(\mu) \frac{I_{lm}}{m} \sin m\phi \\
&= 8V \sum_{l=1,\text{odd}}^{\infty} \sum_{m=1,\text{ odd}}^{+l} \left(\frac{r}{a}\right)^l \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} P_l^m(\mu) \frac{I_{lm}}{m} \sin m\phi \\
&= \frac{2V}{\pi} \sum_{\substack{l=1,\\ \text{odd}}}^{\infty} \sum_{m=1,\text{ odd}}^{+l} \left(\frac{r}{a}\right)^l \frac{2l+1}{m} \frac{(l-m)!}{(l+m)!} I_{lm} P_l^m(\mu) \sin m\phi \qquad (15)
\end{aligned}
$$

Compare $\Phi_2(\vec{x})$, in method and result, with pages 393-395 in Lea.

The total potential is the sum of the two terms (13 or 14 and 15):

$$
\begin{aligned}
\Phi(\vec{x}) &= \frac{\lambda}{2\pi\varepsilon_0} \left\{ \left(1 - \frac{b}{a} + \ln\frac{b}{r}\right) + \sum_{l=2,\text{ even}}^{\infty} \left[ \frac{(2l+1)}{l(l+1)} - \frac{r^l}{lb^l} \left(1 + \frac{l}{l+1}\left(\frac{b}{a}\right)^{2l+1}\right)\right] P_l(\mu) \right\} \\
&\quad + \frac{2V}{\pi} \sum_{l=1,\text{odd}}^{\infty} \sum_{m=1,\text{ odd}}^{+l} \frac{(2l+1)}{m} \frac{(l-m)!}{(l+m)!} I_{lm} \left(\frac{r}{a}\right)^l P_l^m(\mu) \sin m\phi \; \dots r < b
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi(\vec{x}) &= \frac{\lambda b}{2\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{a}\right) + \frac{\lambda b}{2\pi\varepsilon_0} \sum_{l=2,\text{ even}}^{\infty} \frac{r^l}{a^{2l+1}} \frac{b^l}{l+1} \left(\frac{a^{2l+1}}{r^{2l+1}} - 1\right) P_l(\mu) \\
&\quad + \frac{2V}{\pi} \sum_{l=1,\text{odd}}^{\infty} \sum_{m=1,\text{ odd}}^{+l} \frac{(2l+1)}{m} \frac{(l-m)!}{(l+m)!} I_{lm} \left(\frac{r}{a}\right)^l P_l^m(\mu) \sin m\phi \; \dots r > b
\end{aligned}
$$

Let's evaluate the first few terms in our result.

$$
I_{11} = \int_0^\pi (-\sin\theta)\sin\theta d\theta = -\frac{\pi}{2}
$$

Thus for $r < b$

$$
\begin{aligned}
\Phi(\vec{x}) &= \frac{\lambda}{2\pi\varepsilon_0} \left\{ \left(1 - \frac{b}{a} + \ln\frac{b}{r}\right) + \left[\frac{5}{6} - \frac{r^2}{2b^2} - \frac{r^2}{a^5}\frac{b^3}{3}\right] \frac{1}{2}\left(3\mu^2 - 1\right) \right\} \\
&\quad - \frac{2V}{\pi} 3 \frac{1}{2} \frac{\pi}{2} \left(\frac{r}{a}\right)(-\sin\theta)\sin\phi + \cdots \\
&= \frac{\lambda}{4\pi\varepsilon_0} \left\{ \left(1 - \frac{b}{a} + \ln\frac{b}{r}\right) + \left[5 - \frac{r^2}{b^2}\left(3 - 2\frac{b^3}{a^3}\right)\right] \frac{\left(3\cos^2\theta - 1\right)}{12} \right\} \\
&\quad + \frac{3V}{2}\frac{r}{a}\sin\theta\sin\phi + \cdots \; \dots r < b
\end{aligned}
$$

and for $r > b$

$$\begin{aligned}\Phi(\vec{x}) &= \frac{\lambda b}{2\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{a}\right)+\frac{\lambda}{2\pi\varepsilon_0}\frac{r^2}{a^5}\frac{b^3}{3}\left(\frac{a^5}{r^5}-1\right)\frac{1}{2}\left(3\mu^2-1\right)+\frac{3V}{2}\frac{r}{a}\sin\theta\sin\phi+\cdots \\ &= \frac{\lambda b}{2\pi\varepsilon_0}\left\{\frac{1}{r}-\frac{1}{a}+\frac{r^2b^2}{6a^5}\left(\frac{a^5}{r^5}-1\right)\left(3\cos^2\theta-1\right)\right\}+\frac{3V}{2}\frac{r}{a}\sin\theta\sin\phi+\cdots\end{aligned}$$

Lea 8.67 gives the value of $I_{lm}$ for larger odd values of $l$ and $m$.

## 2.3 Dirichlet Green's function in Cylindrical coordinates

Here we will find the Green's function for the interior of an infinitely long tube of radius $a$. Because we are going to use the result to find a potential that does depend on $z$, we need the three-dimensional Green's function. (Step **1:** see Lea Figure C.6) (Step **2**) We choose cylindrical coordinates with $z$-axis along the axis of the tube, and (Step **3**) place our unit point charge at $\vec{x}'$ with (fixed) coordinates $(\rho', \phi', z')$. (Step **4**) The solutions of Laplace's equation in this coordinate system are (Bessels notes §1.7)

$$e^{im\phi}\left\{\begin{array}{c} J_m(k\rho) \\ N_m(k\rho) \end{array}\right\}e^{\pm kz}$$

or

$$e^{im\phi}\left\{\begin{array}{c} I_m(k\rho) \\ K_m(k\rho) \end{array}\right\}\left\{\begin{array}{c} \sin kz \\ \cos kz \end{array} \quad \text{or} \quad e^{\pm ikz}\right\}$$

The sets of functions $\{e^{im\phi}$ times $J_m(k\rho)$ or $N_m(k\rho)\}$ and $\{e^{im\phi}$ times $\sin kz$ or $\cos kz\}$ form complete orthogonal sets of functions, so we may divide our space in either $z$ or $\rho$. (Step **5**) Let's divide in $\rho$. (The other choice is in Lea pg 521-525.) (Step **6**) Then in region I, $\rho < \rho'$, we need the function that is finite at $\rho = 0$, which is $I$. Here we have no boundaries at finite $z$, so we have no way to determine specific values for $k$. We have an integral instead of a sum:

$$G_I(\vec{x}, \vec{x}') = \sum_{m=-\infty}^{+\infty}\int_{-\infty}^{\infty} A_m(k)\, I_m(k\rho)\, e^{ikz}e^{im\phi}dk \tag{16}$$

In region II, $a > \rho > \rho'$, we have both modified Bessel functions.

$$G_{II}(\vec{x}, \vec{x}') = \sum_{m=-\infty}^{+\infty}\int_{-\infty}^{\infty} B_m(k)\left[I_m(k\rho)+C_mK_m(k\rho)\right]e^{ikz}e^{im\phi}dk$$

At $\rho = a$, $G_{II} = 0$, so

$$0 = \sum_{m=-\infty}^{+\infty}\int_{-\infty}^{\infty} B_m(k)\left[I_m(ka)+C_mK_m(ka)\right]e^{ikz}e^{im\phi}dk$$

By orthogonality of the $e^{ikz}e^{im\phi}$ in $z$ and $\phi$, we can equate each term separately to zero, so we get

$$C_m = -\frac{I_m(ka)}{K_m(ka)}$$

and thus

$$G_{II}\left(\vec{x},\vec{x}'\right)=\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{\infty}B_m\left(k\right)\left[I_m\left(k\rho\right)-\frac{I_m\left(ka\right)}{K_m\left(ka\right)}K_m\left(k\rho\right)\right]e^{ikz}e^{im\phi}dk \tag{17}$$

(Step **7**) Now we use continuity of the potential at $\rho=\rho'$ to get

$$\begin{aligned}&\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{\infty}A_m\left(k\right)I_m\left(k\rho'\right)e^{ikz}e^{im\phi}dk\\=\;&\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{\infty}B_m\left(k\right)\left[I_m\left(k\rho'\right)-\frac{I_m\left(ka\right)}{K_m\left(ka\right)}K_m\left(k\rho'\right)\right]e^{ikz}e^{im\phi}dk\end{aligned}$$

and again by orthogonality we have

$$A_m(k)=B_m\left(k\right)\left[1-\frac{I_m\left(ka\right)}{I_m\left(k\rho'\right)}\frac{K_m\left(k\rho'\right)}{K_m\left(ka\right)}\right]$$

Thus (16) becomes

$$G_I\left(\vec{x},\vec{x}'\right)=\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{\infty}B_m\left(k\right)\left[1-\frac{I_m\left(ka\right)}{I_m\left(k\rho'\right)}\frac{K_m\left(k\rho'\right)}{K_m\left(ka\right)}\right]I_m\left(k\rho\right)e^{ikz}e^{im\phi}dk \tag{18}$$

We may combine the two results (18) and (17) to get

$$G\left(\vec{x},\vec{x}'\right)=\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{\infty}B_m\left(k\right)g_m\left(\rho,\rho'\right)e^{ikz}e^{im\phi}dk \tag{19}$$

where

$$g_{mI}\equiv\left[1-\frac{I_m\left(ka\right)}{I_m\left(k\rho'\right)}\frac{K_m\left(k\rho'\right)}{K_m\left(ka\right)}\right]I_m\left(k\rho\right) \tag{20}$$

and

$$g_{mII}\equiv I_m\left(k\rho\right)-\frac{I_m\left(ka\right)}{K_m\left(ka\right)}K_m\left(k\rho\right) \tag{21}$$

(Step **8**) Next we make use of the differential equation (3), expressing the delta function on the right in cylindrical coordinates (*cf* J Problem 1.2):

$$\nabla^2G=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho G\right)+\frac{1}{\rho^2}\frac{\partial^2G}{\partial\phi^2}+\frac{\partial^2G}{\partial z^2}=-\frac{4\pi}{\rho}\delta\left(\rho-\rho'\right)\delta\left(\phi-\phi'\right)\delta\left(z-z'\right)$$

$$\begin{aligned}&\sum_{m=-\infty}^{+\infty}\int_{0}^{\infty}B_m\left(k\right)e^{ikz}e^{im\phi}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g_m\left(\rho\right)}{\partial\rho}-\frac{m^2}{\rho^2}g_m-k^2g_m\right\}dk\\=\;&-\frac{4\pi}{\rho}\delta\left(\rho-\rho'\right)\delta\left(\phi-\phi'\right)\delta\left(z-z'\right)\end{aligned}$$

Now we multiply both sides by $e^{-im'\phi}$ and integrate from 0 to $2\pi$ in $\phi$. On the RHS we use the sifting property, and on the left hand side, we use orthogonality of the $e^{im\phi}$.

$$2\pi\int_{-\infty}^{\infty}B_{m'}\left(k\right)e^{ikz}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g_{m'}\left(\rho\right)}{\partial\rho}-\frac{\left(m'\right)^2}{\rho^2}g_{m'}-k^2g_{m'}\right\}dk=-\frac{4\pi}{\rho}\delta\left(\rho-\rho'\right)\delta\left(z-z'\right)e^{-im'\phi'}$$

Now we can drop the prime on $m$. Similarly, we multiply by $e^{-ik'z}$ and integrate over all $z$. We use Lea eqn 6.16 to get

$$\begin{aligned}\int_{-\infty}^{\infty}B_m\left(k\right)2\pi\delta\left(k-k'\right)\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g_m\left(\rho\right)}{\partial\rho}-\frac{m^2}{\rho^2}g_m-k^2g_m\right\}dk &= -\frac{2}{\rho}\delta\left(\rho-\rho'\right)e^{-ik'z'}e^{-im\phi'}\\ B_m\left(k'\right)\pi\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial g_m\left(\rho\right)}{\partial\rho}-\frac{m^2}{\rho^2}g_m-\left(k'\right)^2g_m\right\} &= -\frac{1}{\rho}\delta\left(\rho-\rho'\right)e^{-ik'z'}e^{-im\phi'}\end{aligned}$$

Drop the primes on the $k'$, and relabel, remembering that $z'$ and $\phi'$ are fixed for the moment.

$$B_m\left(k\right)\equiv\beta_m\left(k\right)e^{-ikz'}e^{-im\phi'}$$

(Step **9**) Finally we multiply both sides by $\rho$ and integrate across the boundary at $\rho=\rho'$

$$\pi\beta_m\left(k\right)\int_{\rho'-\varepsilon}^{\rho'+\varepsilon}\left\{\frac{\partial}{\partial\rho}\rho\frac{\partial g_m\left(\rho\right)}{\partial\rho}-\frac{m^2}{\rho}g_m-k^2\rho g_m\right\}d\rho=-\int_{\rho'-\varepsilon}^{\rho'+\varepsilon}\delta\left(\rho-\rho'\right)d\rho$$

Making use of the fact that $g_m/\rho$ and $\rho g_m$ are well behaved in the range of integration, in the limit $\varepsilon\to 0$ we get

$$\lim_{\varepsilon\to 0}\beta_m\left(k\right)\rho\frac{\partial g_m\left(\rho\right)}{\partial\rho}\bigg|_{\rho'-\varepsilon}^{\rho'+\varepsilon}=-\frac{1}{\pi}$$

At the upper limit we are in region II where $g=g_{II}$ (21), and at the lower limit we are in region I where $g=g_I$ (20), so

$$\begin{aligned}\beta_m\left(k\right)k\rho'\left\{\left[I_m'\left(k\rho'\right)-\frac{I_m\left(ka\right)}{K_m\left(ka\right)}K_m'\left(k\rho'\right)\right]-\left[1-\frac{I_m\left(ka\right)}{I_m\left(k\rho'\right)}\frac{K_m\left(k\rho'\right)}{K_m\left(ka\right)}\right]I_m'\left(k\rho'\right)\right\} &= -\frac{1}{\pi}\\ \beta_m\left(k\right)k\rho'\left\{-\frac{I_m\left(ka\right)}{K_m\left(ka\right)}K_m'\left(k\rho'\right)+I_m\left(ka\right)\frac{K_m\left(k\rho'\right)}{K_m\left(ka\right)}\frac{I_m'\left(k\rho'\right)}{I_m\left(k\rho'\right)}\right\} &= -\frac{1}{\pi}\end{aligned}$$

Thus

$$\beta_m\left(k\right)=\frac{1}{\pi k\rho'}\frac{K_m\left(ka\right)I_m\left(k\rho'\right)}{I_m\left(ka\right)\left[K_m'\left(k\rho'\right)I_m\left(k\rho'\right)-I_m'\left(k\rho'\right)K_m\left(k\rho'\right)\right]}$$

The denominator contains the Wronskian $W\left(k\rho'\right)$ of the modified Bessel differential equation. To evaluate it, let's use the large argument form of the functions. We can do this because $W(x)$ is the same function for all $x$. (Also see http://www.physics.sfsu.edu/~lea/courses/grad/Besselwronskian.pdf. )

$$K_m(x) \simeq \sqrt{\frac{\pi}{2x}} e^{-x}; \quad I_m(x) \simeq \frac{1}{\sqrt{2\pi x}} e^{x}$$

$$\begin{aligned} W(x) &= K_m'(x) I_m(x) - I_m'(x) K_m(x) \\ &= \sqrt{\frac{\pi}{2}} e^{-x} \left(-\frac{1}{\sqrt{x}} - \frac{1}{2x^{3/2}}\right) \sqrt{\frac{1}{2\pi x}} e^{x} - \sqrt{\frac{\pi}{2}} e^{x} \left(\frac{1}{\sqrt{x}} - \frac{1}{2x^{3/2}}\right) \frac{e^{-x}}{\sqrt{2\pi x}} \\ &= -\frac{1}{x} \end{aligned} \tag{22}$$

Thus

$$\beta_m(k) = -\frac{1}{\pi} \frac{K_m(ka) I_m(k\rho')}{I_m(ka)}$$

and then (18) becomes

$$\begin{aligned} G_I(\vec{x}, \vec{x}') &= \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{-1}{\pi} \frac{K_m(ka) I_m(k\rho')}{I_m(ka)} e^{ik(z-z')} \\ &\qquad \times \left[1 - \frac{I_m(ka)}{I_m(k\rho')} \frac{K_m(k\rho')}{K_m(ka)}\right] I_m(k\rho) e^{im(\phi-\phi')} dk \\ &= \frac{1}{\pi} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} e^{ik(z-z')} \left[I_m(ka) K_m(k\rho') - K_m(ka) I_m(k\rho')\right] \frac{I_m(k\rho)}{I_m(ka)} e^{im(\phi-\phi')} dk \end{aligned}$$

for $\rho < \rho'$ and (17)

$$\begin{aligned} G_{II}(\vec{x}, \vec{x}') &= \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{-1}{\pi} \frac{K_m(ka) I_m(k\rho')}{I_m(ka)} e^{ik(z-z')} \left[I_m(k\rho) - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho)\right] e^{im(\phi-\phi')} dk \\ &= \frac{1}{\pi} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{I_m(k\rho')}{I_m(ka)} e^{ik(z-z')} \left[I_m(ka) K_m(k\rho) - K_m(ka) I_m(k\rho)\right] e^{im(\phi-\phi')} dk \end{aligned}$$

for $\rho > \rho'$, or, in both cases,

$$G(\vec{x}, \vec{x}') = \frac{1}{\pi} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{I_m(k\rho_<)}{I_m(ka)} \left[I_m(ka) K_m(k\rho_>) - K_m(ka) I_m(k\rho_>)\right] e^{ik(z-z')} e^{im(\phi-\phi')} dk \tag{23}$$

(Step **10**) The Green's function has dimensions 1/length (from the integral over $k$) as expected. It also displays the necessary symmetry. As $a \to \infty$, $K_m(ka) \to 0$ and so

$$G \to \frac{1}{\pi} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} I_m(k\rho_<) K_m(k\rho_>) e^{ik(z-z')} e^{im(\phi-\phi')} dk = \frac{1}{|\vec{x} - \vec{x}'|}$$

(Jackson 3.148), as required.

Now let's solve the same problem that is in the math phys book (pg 524) with $V_0$ on the wall between $z=-a$ and $z=+a$, and the rest of the cylinder grounded. The outward normal is $\hat{n}=+\hat{\rho}$, and at $\rho'=a$, $\rho_>=\rho'$, so

$$\begin{aligned}
\Phi(\vec{x}) &= -\frac{1}{4\pi}\int_S \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}dA \\
&= -\frac{V_0}{4\pi}\int_{-a}^{a}\int_0^{2\pi}\int_{-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\frac{1}{\pi}\frac{I_m(k\rho)}{I_m(ka)}e^{ik(z-z')}e^{im(\phi-\phi')} \\
&\qquad \times \frac{\partial}{\partial\rho'}\left[I_m(ka)K_m(k\rho')-K_m(ka)I_m(k\rho')\right]\Big|_{\rho'=a} dkad\phi'dz'
\end{aligned}$$

The integral over $\phi'$ gives zero unless $m=0$, so $\Phi$ is independent of $\phi$, as expected.

$$\Phi(\vec{x}) = -\frac{V_0}{4\pi^2}2\pi\int_{-\infty}^{+\infty}\int_{-a}^{a}\frac{I_0(k\rho)}{I_0(ka)}e^{ik(z-z')}k\left[I_0(ka)K_0'(ka)-K_0(ka)I_0'(ka)\right]dkadz'$$

Here again we have the Wronskian of the modified Bessel functions $W(ka)$ (22), so

$$\begin{aligned}
\Phi(\vec{x}) &= +\frac{V_0}{2\pi}\int_{-\infty}^{+\infty}\int_{-a}^{a}\frac{I_0(k\rho)}{I_0(ka)}e^{ik(z-z')}\frac{k}{ka}dkadz' \\
&= \frac{V_0}{2\pi}\int_{-\infty}^{+\infty}\frac{I_0(k\rho)}{I_0(ka)}\left.\frac{e^{ik(z-z')}}{-ik}\right|_{-a}^{+a}dk \\
&= \frac{V_0}{2\pi}\int_{-\infty}^{+\infty}\frac{I_0(k\rho)}{I_0(ka)}\frac{e^{ikz}\left(e^{-ika}-e^{ika}\right)}{-ik}dk \\
&= \frac{V_0}{\pi}\int_{-\infty}^{+\infty}\frac{I_0(k\rho)}{I_0(ka)}\frac{e^{ikz}\sin ka}{k}dk \\
&= \frac{2V_0}{\pi}\int_0^{+\infty}\frac{I_0(k\rho)}{I_0(ka)}\frac{\sin ka\cos kz}{k}dk
\end{aligned}$$

The result is dimensionally correct. Since $\rho\le a$, $I_0(k\rho)\le I_0(ka)$, and so the integral converges. The potential is even in $z$, also as expected. Let's see how this looks for $\rho\to a$.

$$\begin{aligned}
\Phi(a,z) &= \frac{V_0}{\pi}\int_{-\infty}^{+\infty}\frac{e^{ikz}\sin ka}{k}dk = \frac{V_0}{\pi}\int_0^a\int_{-\infty}^{+\infty}e^{ikz}\cos ku\;dkdu \\
&= \frac{V_0}{\pi}2\pi\int_0^a\frac{\delta(z+u)+\delta(z-u)}{2}du \\
&= V_0 \text{ if } 0\le z\le a \;\; \text{(from the second delta function)} \\
&= V_0 \;\text{ if } -a\le z\le 0 \;\; \text{(from the first delta function)} \\
&= 0 \;\text{ otherwise}
\end{aligned}$$

So we get back the correct boundary values.

Comparing with the result in the book (pg 525),

$$\Phi(\vec{x}) = 2V_0 \sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)}{x_{0n} J_1(x_{0n})} \left[1 - e^{-x_{0n}} \cosh\left(x_{0n}\frac{z}{a}\right)\right],$$

we see that the main difference is that we have an integral instead of an infinite sum. We can evaluate the integral numerically. For $z = a/2$, we have

$\rho = 0.1a : \frac{2}{\pi}\int_0^{+\infty} \frac{I_0(.1k)}{I_0(k)} \frac{\sin k \cos(k/2)}{k} dk = 0.768\,66$

$\rho = 0.3a : \frac{2}{\pi}\int_0^{+\infty} \frac{I_0(.3k)}{I_0(k)} \frac{\sin k \cos k/2}{k} dk = 0.785\,18$

$\rho = 0.5a : \frac{2}{\pi}\int_0^{+\infty} \frac{I_0(.5k)}{I_0(k)} \frac{\sin k \cos k/2}{k} dk = 0.820\,58$

$\rho = 0.7a : \frac{2}{\pi}\int_0^{+\infty} \frac{I_0(.7k)}{I_0(k)} \frac{\sin k \cos k/2}{k} dk = 0.878\,66$

$\rho = 0.9a : \frac{2}{\pi}\int_0^{+\infty} \frac{I_0(.9k)}{I_0(k)} \frac{\sin k \cos k/2}{k} dk = 0.957\,96$

For $z/a = 2$, $\rho = 0.1a$ we have $\frac{2}{\pi}\int_0^{+\infty} \frac{I_0(.1k)}{I_0(k)} \frac{\sin k \cos(2k)}{k} dk = 6.877\,8 \times 10^{-2}$

Compare with Lea Figure C.7. The results are the same.

# 3 Expansion in eigenfunctions without division.

This method is outlined in Jackson section 3.12 and Lea sections C.2 and C.7.5. The result is a single function, valid everywhere in the region, at the expense of an extra sum. We'll begin with a one-dimensional problem, and then extend the result to three dimensions.

We start with the Sturm-Liouville equation

$$\frac{d}{dx}\left(f(x)\frac{dy}{dx}\right) - g(x)\,y + \lambda w(x)\,y = 0 \tag{24}$$

valid for $a \leq x \leq b$, where we either have boundary conditions (Dirichlet or Neumann) that guarantee orthogonality of the solutions, or else $f(x) = 0$ on the boundary. The solutions are the eigenfunctions $y_n(x)$ with eigenvalues $\lambda_n$. Let's normalize the eigenfunctions (as we did with the $Y_{lm}$) so that they satisfy the orthonormality relation

$$\int_a^b w(x)\,y_n(x)\,y_m(x)\,dx = \delta_{nm} \tag{25}$$

Now we look for a Green's functions that satisfies the differential equation

$$\frac{d}{dx}\left(f\frac{dG(x,x')}{dx}\right) - g(x)\,G(x,x') + \lambda w(x)\,G(x,x') = -4\pi\delta(x-x') \tag{26}$$

where the constant $\lambda$ does not equal $\lambda_n$ for any $n$. Since $G$ is expected to be well-behaved, we may expand $G(x,x')$ in the eigenfunctions $y_n(x)$ :

$$G(x,x') = \sum_n \gamma_n(x')\,y_n(x)$$

and substitute into the differential equation (26).

$$\begin{aligned}&\sum_n \gamma_n(x') \frac{d}{dx}\left(f\frac{dy_n(x)}{dx}\right) - g(x)\sum_n \gamma_n(x')\, y_n(x) + \lambda w(x) \sum_n \gamma_n(x')\, y_n(x) \\ = \quad & -4\pi\delta(x - x')\end{aligned}$$

We use the eigenfunction equation (24) to express the first two terms in terms of $\lambda_n y_n$:

$$\sum_n \gamma_n(x')\left[-\lambda_n w(x)\, y_n(x)\right] + \lambda w(x) \sum_n \gamma_n(x')\, y_n(x) = -4\pi\delta(x - x')$$

Now we multiply both sides by $y_m(x)$ and integrate over the range $x = a$ to $b$. We use orthonormality (eqn 25) on the LHS and the sifting property on the RHS.

$$(\lambda - \lambda_m)\, \gamma_m(x') = -4\pi y_m(x')$$

Thus

$$\gamma_m(x') = -4\pi \frac{y_m(x')}{(\lambda - \lambda_m)}$$

Now it is clear why $\lambda$ cannot equal any $\lambda_n$. Then

$$G(x.x') = 4\pi \sum_n \frac{y_n(x')\, y_n(x)}{\lambda_n - \lambda} \qquad (27)$$

To extend this result to potential problems in 3-d, we start with equation (3). The eigenvalue $\lambda$ in this equation is zero, so we need eigenfunctions that satisfy an equation with non-zero $\lambda$, and that is the Helmholtz equation.

$$\nabla^2 f(\vec{x}) + k^2 f(\vec{x}) = 0$$

(We have solved this equation before: See *e.g* waveguide notes pages 8, 9 and 11.)

Let's find the Dirichlet Green's function for the interior of a rectangular box measuring $a$ by $b$ by $c$. We put the origin at one corner, with Cartesian axes along the three sides, and solve by separation of variables: $f(x, y, z) = X(x)\, Y(y)\, Z(z)$. Then the eigenfunctions we want are the solutions of

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + k^2 = 0$$

with $X(0) = X(a) = 0$, $Y(0) = Y(b) = 0$ and $Z(0) = Z(c) = 0$. Then $X = \sin n\pi x/a$, $Y = \sin m\pi y/b$ and $Z = \sin p\pi z/c$, and

$$k^2_{nmp} = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{c^2}\right)$$

We still need to normalize the eigenfunctions. The orthogonality integral is

$$\int_0^a \sin^2 \frac{n\pi x}{a} dx = \frac{a}{2}$$

so the normalized function is

$$X_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

and so the normalized eigenfunctions are

$$f_{nmp}(x,y,z) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \sqrt{\frac{2}{b}} \sin \frac{m\pi y}{b} \sqrt{\frac{2}{c}} \sin \frac{p\pi z}{c}$$

Now we just put this result into (27) (extended to 3-d) with $\lambda = 0$.

$$\begin{aligned} G(\vec{x},\vec{x}') &= 4\pi \left(\sqrt{\frac{8}{abc}}\right)^2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \frac{\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{p\pi z}{c}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y'}{b}\sin\frac{p\pi z'}{c}}{\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{c^2}\right)} \\ &= \frac{32}{\pi abc}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \frac{\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{p\pi z}{c}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y'}{b}\sin\frac{p\pi z'}{c}}{\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{c^2}} \end{aligned}$$

Again check that $G$ has dimensions of [1/length]. The benefit of this approach is that we have a single expression that we can use in the whole region. The disadvantage is that we have an extra sum.

**Example:** Suppose the box, with its walls remaining grounded, contains a sheet of charge with uniform surface charge density $\sigma$ that extends from $x = a/4$ to $x = 3a/4$ and $y = b/4$ to $y = 3b/4$ at $z = c/2$. Find the potential inside the box.

In this case we have the volume integral in (1), but the surface integral is zero. The charge density is

$$\begin{aligned} \rho(\vec{x}) &= \sigma\delta\left(z-\frac{c}{2}\right) \quad a/4 < x < 3a/4,\ \ b/4 < y < 3b/4 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

So

$$\begin{aligned} \Phi(\vec{x}) &= \frac{1}{4\pi\varepsilon_0}\int_0^a\int_0^b\int_0^c G(\vec{x},\vec{x}')\,\rho(\vec{x}')\,dx'dy'dz' \\ &= \frac{\sigma}{4\pi\varepsilon_0}\int_{a/4}^{3a/4}\int_{b/4}^{3b/4}\int_0^c \frac{32}{\pi abc}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \frac{\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{p\pi z}{c}\sin\frac{n\pi x'}{a}\sin\frac{m\pi y'}{b}\sin\frac{p\pi z'}{c}}{\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{c^2}} \\ &\qquad \times\delta\left(z'-\frac{c}{2}\right)dx'dy'dz' \\ &= \frac{\sigma}{4\pi\varepsilon_0}\frac{32}{\pi abc}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\frac{\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{p\pi z}{c}\sin\frac{p\pi}{2}}{\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{c^2}} \\ &\qquad \times\int_{a/4}^{3a/4}\sin\frac{n\pi x'}{a}dx'\int_{b/4}^{3b/4}\sin\frac{m\pi y'}{b}dy' \end{aligned}$$

The result is zero if $p$ is even, so

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{8\sigma}{\varepsilon_0 \pi^2 abc} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1,\text{odd}}^{\infty} (-1)^{\left(\frac{p-1}{2}\right)} \frac{\sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{p\pi z}{c}}{\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{c^2}} \\
&\qquad \times \frac{a}{n\pi} \cos\frac{n\pi x'}{a}\Bigg|_{a/4}^{3a/4} \frac{b}{m\pi} \cos\frac{m\pi x'}{b}\Bigg|_{b/4}^{3b/4} \\
&= \frac{8\sigma}{\varepsilon_0 \pi^4 c} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1,\text{odd}}^{\infty} (-1)^{\left(\frac{p-1}{2}\right)} \frac{\sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{p\pi z}{c}}{nm\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{c^2}\right)} \\
&\qquad \times \left(\cos\frac{3n\pi}{4} - \cos\frac{n\pi}{4}\right)\left(\cos\frac{3m\pi}{4} - \cos\frac{m\pi}{4}\right)
\end{aligned}$$

Now we combine the cosines to get

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{32\sigma}{\varepsilon_0 \pi^4 c} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{p=1,\text{odd}}^{\infty} (-1)^{(p-1)/2} \frac{\sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{p\pi z}{c}}{nm\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{c^2}\right)} \sin\frac{n\pi}{2} \sin\frac{n\pi}{4} \sin\frac{m\pi}{2} \sin\frac{m\pi}{4} \\
&= \frac{32\sigma}{\varepsilon_0 \pi^4 c} \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \sum_{\substack{m=1\\ \text{odd}}}^{\infty} \sum_{\substack{p=1\\ \text{odd}}}^{\infty} (-1)^{(p-1)/2+(n-1)/2+(m-1)/2} \frac{\sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{p\pi z}{c}}{nm\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{c^2}\right)} \sin\frac{n\pi}{4} \sin\frac{m\pi}{4}
\end{aligned}$$

If $n = 2l + 1$, then

$$\begin{aligned}
\sin\frac{n\pi}{4} &= \sin\frac{(2l+1)\pi}{4} = \sin\left(\frac{l\pi}{2}+\frac{\pi}{4}\right) = \sin\frac{l\pi}{2}\cos\frac{\pi}{4} + \cos\frac{l\pi}{2}\sin\frac{\pi}{4} \\
&= \frac{\sqrt{2}}{2}\left(\sin\frac{l\pi}{2} + \cos\frac{l\pi}{2}\right) \\
&= \frac{\sqrt{2}}{2}(-1)^{(l-1)/2} \ \text{ if } l \text{ is odd} \\
&= \frac{\sqrt{2}}{2}(-1)^{l/2} \ \text{ if } l \text{ is even}
\end{aligned}$$

Thus replacing $n$ (odd) with $2n + 1$ and similarly for $m$ and $p$ we get

$$\begin{aligned}
\Phi(\vec{x}) &= \frac{\sigma}{\varepsilon_0} \frac{16}{\pi^4 c} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{p+n+m+q}}{\left(\frac{(2n+1)^2}{a^2}+\frac{(2m+1)^2}{b^2}+\frac{(2p+1)^2}{c^2}\right)} \\
&\qquad \times \sin\frac{(2n+1)\pi x}{a} \sin\frac{(2m+1)\pi y}{b} \sin\frac{(2p+1)\pi z}{c}
\end{aligned}$$

where

$$q = \text{integer part}\left(\frac{n}{2}\right) + \text{integer part}\left(\frac{m}{2}\right)$$

Check the dimensions!

At $z = c/4$ we have

$$\frac{\Phi\left(x, y, c/4\right)}{16\sigma a^2/\left(\varepsilon_0 \pi^4 c\right)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(-1)^{p+n+m+q}}{(2n+1)\,(2m+1)\left[(2n+1)^2 + \frac{(2m+1)^2 a^2}{b^2} + \frac{(2p+1)^2 a^2}{c^2}\right]} \times \sin\frac{(2n+1)\,\pi x}{a} \sin\frac{(2m+1)\,\pi y}{b} \sin\frac{(2p+1)\,\pi}{4} \qquad (28)$$

The diagram shows the dimensionless potential (28), computed using values of $n, m$ and $p$ from zero to five, and with $a = b = c$.