# Lagrangians etc

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## 1 Non-relativistic Lagrangian for electromagnetism

We begin with the definition of the Lagrangian

$$\mathcal{L} = T - V$$

where $T$ is the system's kinetic energy and $V$ is the potential energy. The system action is defined as

$$A = \int_{t_1}^{t_2} \mathcal{L} dt$$

and the system's motion between time $t_1$ and $t_2$ will be such as to make the action an extremum. Thus

$$\delta A = \delta \int_{t_1}^{t_2} \mathcal{L}\left(\vec{x}, \vec{v}\right) dt = 0$$

Now in general the kinetic energy is a function of the velocities, while the potential energy is a function of the coordinates. So we obtain:

$$\int_{t_1}^{t_2} \sum_i \left( \frac{\partial \mathcal{L}\left(\vec{x}, \vec{v}\right)}{\partial x_i} \delta x_i + \frac{\partial \mathcal{L}\left(\vec{x}, \vec{v}\right)}{\partial v_i} \delta v_i \right) dt = 0$$

Integrating the second term by parts, we get:

$$\sum_i \left. \frac{\partial \mathcal{L}\left(\vec{x}, \vec{v}\right)}{\partial v_i} \delta x_i \right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_i \left( \frac{\partial \mathcal{L}\left(\vec{x}, \vec{v}\right)}{\partial x_i} - \frac{d}{dt} \frac{\partial \mathcal{L}\left(\vec{x}, \vec{v}\right)}{\partial v_i} \right) \delta x_i dt = 0$$

Now if the path begins and ends at fixed points, then the integrated term is zero, and since the variation $\delta x_i$ is otherwise arbitrary, we obtain Lagrange's equations:

$$\frac{d}{dt} \frac{\partial \mathcal{L}\left(\vec{x}, \vec{v}\right)}{\partial v_i} - \frac{\partial \mathcal{L}\left(\vec{x}, \vec{v}\right)}{\partial x_i} = 0$$

Now we want to extend the usual formalism to allow for velocity-dependent potential energies. Thus for a particle with $K = \frac{1}{2} m v^2$, Lagrange's equations become:

$$\frac{d}{dt} \left( m v_i - \frac{\partial V}{\partial v_i} \right) + \frac{\partial V}{\partial x_i} = 0$$

If the system is a particle in an electrostatic field, then $V = q\Phi$, and we obtain:

$$\begin{aligned} m\frac{dv_i}{dt} + q\frac{\partial\Phi}{\partial x_i} &= 0 \\ m\frac{dv_i}{dt} &= qE_i \end{aligned}$$

as expected.

But in an electromagnetic field, we want to include the magnetic force. The invariant combination

$$\frac{q}{c}A^{\alpha}v_{\alpha} = \frac{q}{c}\left(\Phi, \vec{A}\right)\cdot(\gamma c, \gamma\tilde{\mathbf{v}}) = q\gamma\left(\Phi - \vec{A}\cdot\vec{v}\right)$$

suggests that in the non-relativistic case we choose $q\left(\Phi - \vec{A}\cdot\vec{v}\right)$ as our velocity dependent potential. The new term $-q\vec{A}\cdot\vec{v}$ is called the interaction term. Then we have:

$$\frac{\partial V}{\partial v_i} = -qA_i$$

and the equation of motion is

$$\frac{d}{dt}\left(mv_i - \frac{\partial V}{\partial v_i}\right) + \frac{\partial V}{\partial x_i} = 0$$

$$m\frac{dv_i}{dt} - \frac{d}{dt}(-qA_i) + q\frac{\partial}{\partial x_i}\left(\Phi - \vec{v}\cdot\vec{A}\right) = 0$$

Now

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)A_i$$

and thus

$$\begin{aligned} m\frac{dv_i}{dt} &= -q\frac{dA_i}{dt} - q\frac{\partial}{\partial x_i}\left(\Phi - \vec{v}\cdot\vec{A}\right) \\ &= -q\left(\frac{\partial}{\partial x_i}\left(\Phi - \vec{v}\cdot\vec{A}\right) + \frac{\partial A_i}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)A_i\right) \end{aligned} \tag{1}$$

Next we use the relations between the fields and the potentials to show that the RHS of (1) is the Lorentz force.

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

and

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

So the Lorentz force has components

$$
\begin{aligned}
F_i &= q\left(E_i + \varepsilon_{ijk}\frac{v_j}{c}B_k\right) \\
&= q\left(E_i + \varepsilon_{ijk}\frac{v_j}{c}\varepsilon_{klm}\partial_l A_m\right) \\
&= q\left(-\partial_i\Phi - \frac{1}{c}\frac{\partial A_i}{\partial t} + \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{v_j}{c}\partial_l A_m\right) \\
&= q\left(-\partial_i\Phi - \frac{1}{c}\frac{\partial A_i}{\partial t} + \frac{v_m}{c}\partial_i A_m - \frac{v_l}{c}\partial_l A_i\right) \\
&= q\left(-\partial_i\Phi - \frac{1}{c}\frac{\partial A_i}{\partial t} + \frac{v_m}{c}\partial_i A_m - \left(\frac{\vec{v}\cdot\vec{\nabla}}{c}\right)A_i\right)
\end{aligned}
$$

Since the coordinates $x_i$ and the velocities $v_i$ are independent variables, we can move $v_m$ across the derivative to obtain:

$$
\begin{aligned}
F_i &= \frac{q}{c}\left(-\partial_i c\Phi - \frac{\partial A_i}{\partial t} + \partial_i\left(v_m A_m\right) - \left(\vec{v}\cdot\vec{\nabla}\right)A_i\right) \\
&= \frac{q}{c}\left(-\partial_i\left(c\Phi - \vec{v}\cdot\vec{A}\right) - \frac{\partial A_i}{\partial t} - \left(\vec{v}\cdot\vec{\nabla}\right)A_i\right) \qquad (2)
\end{aligned}
$$

which shows that the right hand side of equation (1) is indeed the Lorentz force (2).

We can also derive the generalized force directly from the Lagrangian:

$$Q_j = -\frac{\partial V}{\partial x_i}$$

in the usual case. With velocity dependent potentials, this relation becomes:

$$
\begin{aligned}
Q_j &= -\frac{\partial V}{\partial x_i} + \frac{d}{dt}\left(\frac{\partial V}{\partial v_j}\right) \\
&= -q\partial_i\left(\Phi - \frac{\vec{v}}{c}\cdot\vec{A}\right) + \frac{1}{c}\frac{d}{dt}\left(-qA_i\right) \qquad (3)
\end{aligned}
$$

which gives the same result (2).

Once we have the Lagrangian we can construct the canonical momenta and the Hamiltonian. The canonical momentum is:

$$P_i = \frac{\partial \mathcal{L}}{\partial v_i} = mv_i + q\frac{A_i}{c} = p_i + q\frac{A_i}{c} \qquad (4)$$

where $p_i$ is the usual mechanical momentum. The Hamiltonian is constructed from the relation

$$H = \sum_k P_k v_k - \mathcal{L}$$

where we must express $\mathcal{L}$ as a function of the canonical momenta:

$$
\begin{aligned}
\mathcal{L} &= \sum_i \frac{1}{2}mv_i^2 - q\left(\Phi - \sum_i \frac{v_i A_i}{c}\right) \\
&= \sum_i \frac{1}{2m}\left(P_i - \frac{q}{c}A_i\right)^2 - q\Phi + \sum_i \frac{qA_i}{cm}\left(P_i - \frac{q}{c}A_i\right)
\end{aligned}
$$

and so

$$\begin{aligned}
H &= \sum_k P_k \frac{1}{m}\left(P_k - \frac{q}{c}A_k\right) - \sum_i \frac{1}{2m}\left(P_i - \frac{q}{c}A_i\right)^2 + q\Phi - \sum_i \frac{qA_i}{mc}\left(P_i - \frac{q}{c}A_i\right) \\
&= \frac{1}{m}\sum_k \left(P_k - \frac{qA_k}{c}\right)\left(P_k - \frac{q}{c}A_k\right) - \sum_i \frac{1}{2m}\left(P_i - \frac{q}{c}A_i\right)^2 + q\Phi \\
&= \sum_i \frac{1}{2m}\left(P_i - \frac{q}{c}A_i\right)^2 + q\Phi \qquad (5)
\end{aligned}$$

Happily, the Hamiltonian still equals the total energy of the particle.

Then Hamilton's equations are:

$$\frac{\partial H}{\partial P_k} = v_k$$

and

$$\frac{\partial H}{\partial x_k} = -\frac{d}{dt}P_k$$

With the Hamiltonian (5), we get:

$$\frac{\partial H}{\partial P_i} = \frac{1}{m}\left(P_i - \frac{q}{c}A_i\right) = v_i$$

which is no surprise, and

$$\frac{\partial H}{\partial x_k} = -\sum_i \left(P_i - \frac{q}{c}A_i\right)\frac{q}{mc}\frac{\partial A_i}{\partial x_k} + q\frac{\partial \Phi}{\partial x_k} = -\frac{d}{dt}P_k$$

Expanding out the $P_i$ again, we can show that this relation is consistent with the Lorentz force law:

$$\begin{aligned}
-\sum_i v_i \frac{q}{c}\frac{\partial A_i}{\partial x_k} + q\frac{\partial \Phi}{\partial x_k} &= -\frac{d}{dt}\left(mv_k + \frac{q}{c}A_i\right) \\
q\frac{\partial}{\partial x_k}\left(\Phi - \frac{\vec{v}}{c}\cdot\vec{A}\right) &= -m\frac{dv_k}{dt} - \frac{q}{c}\left(\frac{\partial A_i}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)A_i\right) \\
q\frac{\partial}{\partial x_k}\left(\Phi - \frac{\vec{v}}{c}\cdot\vec{A}\right) + \frac{q}{c}\left(\frac{\partial A_i}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)A_i\right) &= -m\frac{dv_k}{dt}
\end{aligned}$$

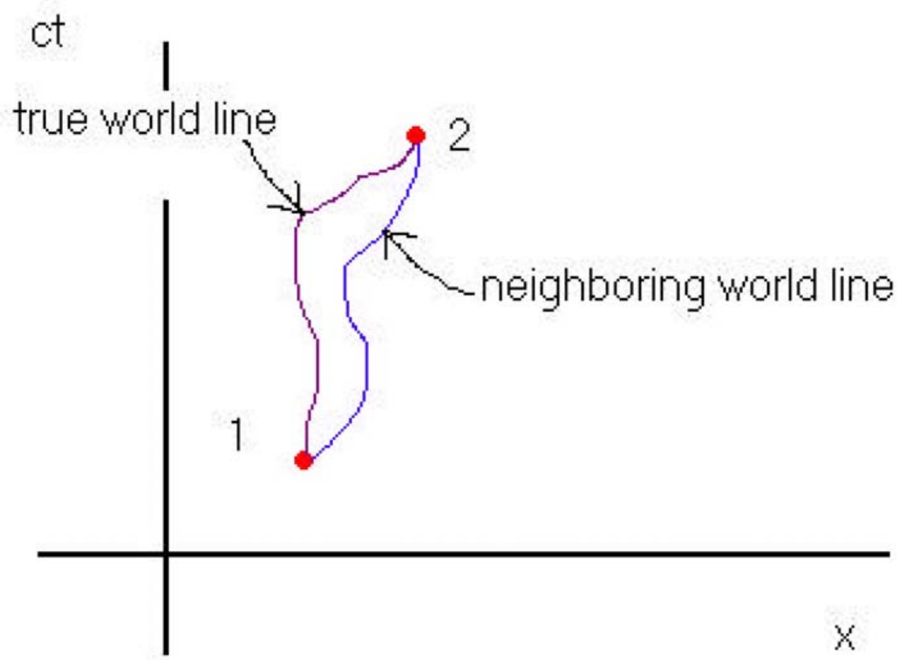
and once again we retrieve the Lorentz force (2).

## 2 Fully covariant theory

Now the problem with this formalism is that it contains the coordinate time $t$. To have a fully covariant theory we will need to reformulate it in terms of the proper time $\tau$. We start with the action for a free particle (a particle moving in a field-free region). The integral is now taken over paths in space-time. By analogy with Fermat's principle from optics, we assert that the action integral is:

$$S = \int_1^2 -mc^2 d\tau$$

where the integral is taken over a path in space-time between the two events labelled 1 and 2. One of the paths is the true world line of the particle, and on that path the action is an extremum.




To see that this idea makes sense, notice that in any given reference frame in which the particle has speed $v$, $d\tau = dt/\gamma$, and expanding $1/\gamma$ we get the integrand

$$-mc^2\left(\sqrt{1-\beta^2}\right) = -mc^2 + \frac{1}{2}mc^2\beta^2 = -mc^2 + \frac{1}{2}mv^2$$

which is the particle's kinetic energy minus its rest energy. This looks like a Lagrangian in which rest energy plays the role of the potential energy in the previous theory.

The coordinates on the true world line are given by $x_0^\alpha(\theta)$ where $\theta$ is a parameter that varies continuously as we move along the world line from event 1 to event 2. On a neighboring path, the coordinates are

$$x_a^\alpha(\theta) = x_0^\alpha(\theta) + ag^\alpha(\theta) \tag{6}$$

where $g^\alpha(\theta)$ is a vector function such that $g^\alpha(\theta) \equiv 0$ at the two events labelled 1 and 2, and $a$ is a small parameter. The proper time interval between two neighboring events on the path is given by the invariant

$$c^2 d\tau^2 = dx^\alpha dx_\alpha$$

and thus we may express the action in terms of the parameter $\theta$ as

$$S_a = \int_1^2 -mc\sqrt{\frac{dx_a^\alpha}{d\theta}\frac{dx_{a\alpha}}{d\theta}}d\theta$$

(Strictly, the invariant line interval equals the proper time only on the true world line.)

Now we extend this result to include the interaction between the particle and fields by adding the interaction energy:

$$S_a = \int_1^2 \left(-mc\sqrt{\frac{dx_a^\alpha}{d\theta}\frac{dx_{a\alpha}}{d\theta}} - \frac{q}{c}A_\alpha\frac{dx_a^\alpha}{d\theta}\right)d\theta$$

(On the true world line $\theta$ goes over to the proper time, and $dx_a^\alpha/d\theta$ becomes the 4-velocity.)

Now the variation of the integral gives:

$$\delta S = \frac{\partial S}{\partial a}\delta a = \int_1^2 \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_a^\alpha}\frac{\partial \dot{x}_a^\alpha}{\partial a} + \frac{\partial \mathcal{L}}{\partial x_a^\alpha}\frac{\partial x_a^\alpha}{\partial a}\right)\delta a d\theta$$

where here the dot means $d/d\theta$. Now we may use our expression for the coordinates $x_a^\alpha$ (6) to obtain:

$$\delta S = \int_1^2 \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_a^\alpha}\frac{dg^\alpha}{d\theta} + \frac{\partial \mathcal{L}}{\partial x_a^\alpha}g^\alpha\right)\delta a d\theta$$

Now integrate the first term by parts:

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{x}_a^\alpha}g^\alpha\bigg|_1^2 - \int_1^2 \left(\frac{d}{d\theta}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_a^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial x_a^\alpha}\right)g^\alpha \delta a d\theta$$

and since the function $g^\alpha$ is arbitrary, except that it must be zero at events 1 and 2, we have Lagrange's equations:

$$\frac{d}{d\theta}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_a^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial x_a^\alpha} = 0$$

where on the true world line, $\theta \to \tau$. Thus

$$\mathcal{L} = -\, mc\sqrt{u^\alpha u_\alpha} - \frac{q}{c}A_\alpha u^\alpha \tag{7}$$

and Lagrange's equations are.

$$\frac{d}{d\tau}\left(\frac{\partial}{\partial \dot{x}^\alpha}\left(-mc\sqrt{\frac{dx^\beta}{d\tau}\frac{dx_\beta}{d\tau}} - \frac{q}{c}A_\beta\frac{dx^\beta}{d\tau}\right)\right) - \frac{\partial}{\partial x^\alpha}\left(-mc\sqrt{\frac{dx^\beta}{d\tau}\frac{dx_\beta}{d\tau}} - \frac{q}{c}A_\beta\frac{dx^\beta}{d\tau}\right) = 0$$

where now the dot means $d/d\tau$.

Expanding out the derivatives,

$$\frac{d}{d\tau}\left(\frac{-mcu_\alpha}{\sqrt{u^\beta u_\beta}} - \frac{q}{c}A_\alpha\right) + \frac{q}{c}u^\beta\frac{\partial A_\beta}{\partial x^\alpha} = 0$$

Now we can write the invariant $u^\beta u_\beta = c^2$ to get:

$$\begin{aligned} -m\frac{du_\alpha}{d\tau} - \frac{q}{c}u^\beta\frac{\partial A_\alpha}{\partial x^\beta} + \frac{q}{c}u^\beta\frac{\partial A_\beta}{\partial x^\alpha} &= 0 \\ -m\frac{du_\alpha}{d\tau} &= \frac{q}{c}\left(\partial_\beta A_\alpha - \partial_\alpha A_\beta\right)u^\beta \\ &= \frac{q}{c}F_{\beta a}u^\beta \end{aligned}$$

Raising indices, we have:

$$m\frac{du^\alpha}{dt} = -\frac{q}{c}F^{\beta\alpha}u_\beta = \frac{q}{c}F^{\alpha\beta}u_\beta$$

and again the right hand side is the Lorentz force.

Now we proceed to calculate the canonical momenta:

$$\begin{aligned} P_\alpha &= -\frac{\partial\mathcal{L}}{\partial u^\alpha} = mc\frac{u_\alpha}{\sqrt{u^\beta u_\beta}} + \frac{q}{c}A_\alpha \\ &= mu_\alpha + \frac{q}{c}A_\alpha = p_\alpha + \frac{q}{c}A_\alpha \end{aligned} \tag{8}$$

which is what we might have expected from the non-relativistic results.

Why the minus sign? Well, it has to do with the metric. The upper index contravariant momentum $P^\alpha$ has to be the derivative with respect to the lower index, covariant velocity $u_\alpha$, and vice versa. The space part changes sign under the mapping contravariant→covariant, and hence the minus sign.

Now we try to form the Hamiltonian:

$$H = \frac{1}{2}\left(P^\alpha u_\alpha + L\right)$$

Why is it a plus not a minus? Well, remember that the space part of the 4-vector dot-product is $-\vec{P}\cdot\vec{u}$, and that's where the minus sign is. The 1/2 is an irrelevant normalization that gives us the correct form for Hamilton's equations. (See below.)

Then

$$\begin{aligned} H &= \frac{1}{2}\left(P^\alpha u_\alpha - mc\sqrt{u^\alpha u_\alpha} - \frac{q}{c}A_\alpha u^\alpha\right) \\ &= \frac{1}{2}\left(-mc^2 + u_\alpha\left(P^\alpha - \frac{q}{c}A^\alpha\right)\right) \end{aligned}$$

Now we can use equation (8) to replace $u_\alpha$ with $P_\alpha$ :

$$H = \frac{1}{2}\left(-mc^2 + \frac{1}{m}\left(P_\alpha - \frac{q}{c}A_\alpha\right)\left(P^\alpha - \frac{q}{c}A^\alpha\right)\right)$$

(Aside- if we replace $P_\alpha$ with $u_\alpha$ instead we can see that $H \equiv 0$! This is very odd! See Oliver Johns' paper in AJP for more on this.)

Pressing on, we write Hamilton's equations.

$$\frac{\partial H}{\partial P^\alpha} = u_\alpha$$

and this works out as it did before.

$$\begin{aligned}
\frac{\partial H}{\partial x^\alpha} &= -\frac{dP_\alpha}{d\tau} \\
\frac{1}{m}\left(-\frac{q}{c}\frac{\partial A_\beta}{\partial x^\alpha}\right)\left(P^\beta - \frac{q}{c}A^\beta\right) &= -\frac{d}{d\tau}\left(p_\alpha + \frac{q}{c}A_\alpha\right) \\
\frac{q}{c}\left(\frac{\partial A_\beta}{\partial x^\alpha}u^\beta - u^\beta \partial_\beta A_\alpha\right) &= \frac{dp_\alpha}{d\tau} \\
\frac{q}{c}F_{\alpha\beta}u^\beta &= \frac{dp_\alpha}{d\tau}
\end{aligned}$$

and again we retrieve the expected result.