Waves in plasmas

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1 Plasma as an example of a dispersive medium

We shall now discuss the propagation of electromagnetic waves through a hydrogen plasma – an electrically neutral fluid of protons and electrons. This will allow us to develop a specific expression for the dielectric constant as a function of frequency. Maxwell's equations include the charge density $\rho = e\left(n_i - n_e\right)$ and the current density $\vec{j} = -e\left(n_e\vec{v}_e - n_i\vec{v}_i\right)$. We will be looking for normal modes of the complete system of particles plus fields. If we assume that each field has the form $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$, as usual, (or equivalently, take the Fourier transform of the equations) then we find:

$$i\vec{k}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\vec{k}\cdot\vec{B} = 0 \tag{2}$$

$$\vec{k}\times\vec{E} = \omega\vec{B} \tag{3}$$

and

$$i\vec{k}\times\vec{B} = \mu_0\vec{j} - i\frac{\omega}{c^2}\vec{E} \tag{4}$$

We also need the equation of motion for the electrons:

$$m\frac{d\vec{v}_e}{dt} = -e\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

which becomes:

$$-i\omega m\vec{v}_e = -e\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{5}$$

We consider first high frequency waves. (We will see later what "high frequency" means in this context.) Because of their greater mass, the ions accelerate much more slowly than the electrons, and do not have time to respond to the wave fields before the fields reverse again. Thus for high-frequency waves, we may assume that the ions remain at rest:

$$\vec{v}_i \simeq 0; \ \ n_i = n_0 = \text{ constant}$$

where n_0 is the original, uniform, particle number density. Then the current density is due to the electrons alone:

$$\vec{j} = -n_e e\vec{v}_e \tag{6}$$

and equation (4) becomes:

$$i\vec{k}\times\vec{B} = \mu_0\left(-n_e e\vec{v}_e\right) - i\frac{\omega}{c^2}\vec{E}$$

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The electron density does not remain constant:

$$n_e\left(\vec{x}, t\right) = n_0 + n_1\left(\vec{x}, t\right)$$

where n_1 is the perturbation to the electron density. We now assume that the waves have small amplitude, which means that $n_1 \ll n_0$, $v \ll c$, and so on. Thus we will ignore all products of wave amplitudes in what follows. (We *linearize* the equations.)

The equation for charge conservation is (Notes 1 eqn 7):

$$\frac{\partial n_e}{\partial t} + \vec{\nabla} \cdot \left(n_e \vec{v}_e\right) = 0$$

which becomes (ignoring the term in $n_1 \vec{v}$, which is a product of two small quantities):

$$-i\omega n_1 + i\vec{k} \cdot \vec{v}_e n_0 = 0 \tag{7}$$

Now let's look for transverse waves $\left(\vec{k} \cdot \vec{v}_e = 0\right)$. For such waves equation (7) shows that the electron density perturbation n_1 is zero. Thus the right hand side of equation (1) is zero and $\vec{E}$ is perpendicular to $\vec{k}$ just as in a vacuum, and the waves are transverse in that sense too. The equation of motion (5) relates the electron velocity to the electric field. For small amplitude waves, the term in $\vec{v} \times \vec{B}$ is second order, and we neglect it. Then:

$$-i\omega m \vec{v}_e = -e\vec{E}$$

and then the current density (6) is

$$\vec{j} = -n_0 e \frac{e\vec{E}}{i\omega m} = i\frac{n_0 e^2}{\omega m}\vec{E} = \sigma\vec{E} \tag{8}$$

The conductivity is imaginary, indicating a 90° phase shift between $\vec{E}$ and $\vec{j}$. Putting this result into Ampere's law (4), we get:

$$i\vec{k} \times \vec{B} = \frac{1}{c^2 \varepsilon_0} i\frac{n_0 e^2}{\omega m}\vec{E} - i\frac{\omega}{c^2}\vec{E}$$

$$\vec{k} \times \vec{B} = -\frac{\omega}{c^2}\left(1 - \frac{n_0 e^2}{\omega^2 m \varepsilon_0}\right)\vec{E} \tag{9}$$

The quantity

$$\frac{n_0 e^2}{\varepsilon_0 m} \equiv \omega_p^2,$$

where ω_p is the *plasma frequency*, the frequency of natural electrostatic oscillations in the ionized plasma[1]. Now we may interpret equation (9) in terms of the dielectric constant ε for the plasma ("brewster" notes eqn 5)

$$\vec{k} \times \vec{B} = -\frac{\omega}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\vec{E} = -\frac{\omega}{c^2}\frac{\varepsilon}{\varepsilon_0}\vec{E}$$

with

$$\varepsilon = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\varepsilon_0 \tag{10}$$

Recall that we may express the wave speed in terms of the dielectric constant ("brewster"

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[1] We'll veryify this later in these notes. (Pg 7)

notes eqn 6):

$$v_\phi = \frac{\omega}{k} = c\sqrt{\frac{\varepsilon_0}{\varepsilon}} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \tag{11}$$

Since $\varepsilon/\varepsilon_0 < 1$, the wave phase speed is greater than c! This is physically possible, because the speed at which information travels, the group speed $d\omega/dk$, is less than c. Squaring equation (11), we get

$$\frac{\omega^2}{k^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) = c^2$$
$$\omega^2 - \omega_p^2 = c^2 k^2 \tag{12}$$

Now we differentiate, to get

$$2\omega\frac{d\omega}{dk} = 2c^2 k$$
$$v_g = \frac{d\omega}{dk} = \frac{c^2}{v_\phi} \tag{13}$$

So v_g is less than c when v_ϕ is greater than c.

Eqn (12) shows that the wave number k becomes imaginary when $\omega < \omega_p$ and the wave ceases to propagate. Wave energy is dissipated by the fields as the electrons gain kinetic energy. The rate of energy dissipation per unit volume is

$$P = \vec{j} \cdot \vec{E}$$

where we have to take the real part before multiplying. Since σ is imaginary (eqn 8), there is a phase shift of $\pi/2$ and so for real k

$$\begin{aligned}
P(x,t) &= \mathrm{Re}\left(\sigma\vec{E}\right) \cdot \mathrm{Re}\left(\vec{E}\right) = \mathrm{Re}\left(i\left|\sigma\right|\vec{E}\right) \cdot \mathrm{Re}\left(\vec{E}\right) \\
&= -\left|\sigma\right| E_0 \sin\left(\vec{k}\cdot\vec{x} - \omega t\right) E_0 \cos\left(\vec{k}\cdot\vec{x} - \omega t\right) \\
&= -\frac{\omega_p^2}{\omega}\frac{\varepsilon_0 E_0^2}{2} \sin 2\left(\vec{k}\cdot\vec{x} - \omega t\right)
\end{aligned}$$

Looking at $\vec{x} = 0$ for simplicity:

$$P(0,t) = \omega u_{E,0}\frac{\omega_p^2}{\omega^2}\sin 2\omega t$$

where $u_{E,0} = \frac{1}{2}\varepsilon_0 E_0^2$ is the initial energy density. If k is imaginary, we get

$$P(x,t) = -\left|\sigma\right| E_0 e^{-2\hat{k}\cdot\vec{x}|k|}\sin\left(-\omega t\right) E_0 \cos\left(-\omega t\right) = \omega u_{E,0}\frac{\omega_p^2}{\omega^2}\sin 2\omega t \text{ at } x = 0$$

In both cases, the energy converted in time t is

$$\mathcal{E}(t) = \int_0^t P(0,t)\,dt = \frac{u_{E,0}}{2}\frac{\omega_p^2}{\omega^2}\left(1 - \cos 2\omega t\right)$$

which shows that electric field energy is converted to electron kinetic energy fast enough to dissipate all the wave energy in less than a quarter of a wave period if $\omega < \omega_p$. For $\omega > \omega_p$ the time-averaged energy transfer does not increase with time, and is less than $u_{E,0}/2$.

The plot shows $\left(\mathcal{E}/u_{E,0}\right)\left(\omega^2/\omega_p^2\right) = y$ versus t/T.
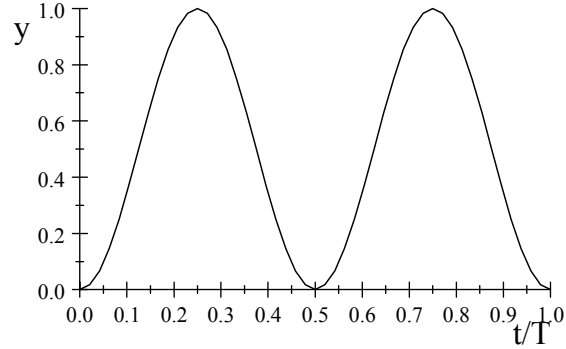
2   Waves in magnetized plasmas

If the plasma is magnetized with a uniform magnetic field $\vec{B}_0$, there is an additional term in the equation of motion:

$$-i\omega m\vec{v}_e = -e\left(\vec{E} + \vec{v}_e \times \vec{B}_0\right)$$

To simplify the solution of this equation for $\vec{v}_e$, choose the $z-$axis along $\vec{B}_0$. Then we have:

$$-i\omega m v_x = -eE_x - ev_y B_0$$
$$-i\omega m v_y = -eE_y + ev_x B_0$$
$$-i\omega m v_z = -eE_z$$

and thus solving for the components of $\vec{v}$, we have:

$$v_z = -i\frac{e}{\omega m}E_z \tag{14}$$

$$-i\omega m v_x = -eE_x - eB_0\left(\frac{-eE_y + ev_x B_0}{-i\omega m}\right)$$

Write $\Omega = eB_0/m$, the cyclotron frequency, to get

$$
\begin{aligned}
-i\omega m v_x &= -eE_x + i\frac{\Omega}{\omega}eE_y - i\frac{\Omega^2}{\omega}mv_x \\
-i\omega m v_x\left(1 - \frac{\Omega^2}{\omega^2}\right) &= -e\left(E_x - i\frac{\Omega}{\omega}E_y\right) \\
v_x &= -i\frac{e}{\omega m}\frac{\left(E_x - i\frac{\Omega}{\omega}E_y\right)}{1 - \Omega^2/\omega^2} \tag{15}
\end{aligned}
$$

and then

$$v_y = \frac{-eE_y + \Omega mv_x}{-i\omega m}$$

4

$$v_y = -i\frac{e}{\omega m}\left[E_y + \frac{\Omega m}{-e}\left(-i\frac{e}{\omega m}\frac{(E_x - i\frac{\Omega}{\omega}E_y)}{1 - \Omega^2/\omega^2}\right)\right]$$

$$= -i\frac{e}{\omega m}\frac{(E_y + i\frac{\Omega}{\omega}E_x)}{1 - \Omega^2/\omega^2} \tag{16}$$

With $\vec{j} = -n_0 e\vec{v}$, (eqn 6), we see that j_x is related to both E_x and E_y components. Then $\vec{j}$ is related to $\vec{E}$ by the tensor relation:

$$j_i = -n_0 e v_i = \sigma_{ij} E_j$$

with the conductivity tensor

$$\widetilde{\sigma} = \frac{-n_0 e\left(-i\frac{e}{\omega m}\right)}{1 - \Omega^2/\omega^2}\begin{pmatrix} 1 & -i\Omega/\omega & 0 \\ i\Omega/\omega & 1 & 0 \\ 0 & 0 & 1 - \Omega^2/\omega^2 \end{pmatrix}$$

$$= \frac{i\varepsilon_0\omega_p^2/\omega}{1 - \Omega^2/\omega^2}\begin{pmatrix} 1 & -i\Omega/\omega & 0 \\ i\Omega/\omega & 1 & 0 \\ 0 & 0 & 1 - \Omega^2/\omega^2 \end{pmatrix} \tag{17}$$

Putting this back into Ampere's law (4), we get the dielectric "constant", which is also now a rank 2 tensor:

$$i\vec{k} \times \vec{B} = \mu_0 \overleftrightarrow{\sigma} \cdot \vec{E} - i\frac{\omega}{c^2}\vec{E} = -i\frac{\omega}{c^2}\frac{\overleftrightarrow{\varepsilon}}{\varepsilon_0} \cdot \vec{E} \tag{18}$$

The dielectric tensor has components

$$\varepsilon_{ij} = \varepsilon_0\left(\delta_{ij} + i\frac{\mu_0}{\omega}c^2\sigma_{ij}\right)$$

$$\frac{\varepsilon_{ij}}{\varepsilon_0} = \delta_{ij} + \frac{i}{\varepsilon_0\omega}\frac{i\varepsilon_0\omega_p^2/\omega}{1 - \Omega^2/\omega^2}\begin{pmatrix} 1 & -i\Omega/\omega & 0 \\ i\Omega/\omega & 1 & 0 \\ 0 & 0 & 1 - \Omega^2/\omega^2 \end{pmatrix}$$

$$= \delta_{ij} - \frac{\omega_p^2/\omega^2}{1 - \Omega^2/\omega^2}\begin{pmatrix} 1 & -i\Omega/\omega & 0 \\ i\Omega/\omega & 1 & 0 \\ 0 & 0 & 1 - \Omega^2/\omega^2 \end{pmatrix} \tag{19}$$

We can tidy this up by rewriting the matrix as

$$\begin{pmatrix} 1 & -i\frac{\Omega}{\omega} & 0 \\ i\frac{\Omega}{\omega} & 1 & 0 \\ 0 & 0 & 1 - \frac{\Omega^2}{\omega^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{\omega^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Omega^2 \end{pmatrix} - i\frac{\Omega}{\omega}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now define a vector cyclotron frequency:

$$\vec{\Omega} = \frac{e}{mc}\vec{B}_0$$

With our coordinate choice, $\vec{\Omega}$ has only one component: Ω_3. Then we can write the dielectric tensor in coordinate-free form as:

$$\frac{\varepsilon_{ij}}{\varepsilon_0} = \delta_{ij} - \frac{\omega_p^2}{(\omega^2 - \Omega^2)}\left(\delta_{ij} - \frac{\Omega_i\Omega_j}{\omega^2} - i\epsilon_{ijk}\frac{\Omega_k}{\omega}\right) \tag{20}$$

See also Jackson P 7.17.

2.1 Normal modes

We have now reduced Ampere's law to the form (18):

$$\epsilon_{ijn}k_j B_n = -\frac{\omega}{c^2}\frac{\varepsilon_{ij}}{\varepsilon_0}E_j$$

Now take $\vec{k} \times$ (equation 3) and use this result to get:

$$k_i k_j E_j - k^2 E_i = \omega\epsilon_{ijn}k_j B_n = -\frac{\omega^2}{c^2}\frac{\varepsilon_{ij}}{\varepsilon_0}E_j$$

or

$$\left[\frac{\varepsilon_{ij}}{\varepsilon_0} - \frac{c^2 k^2}{\omega^2}\left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)\right]E_j = 0 \tag{21}$$

This equation has a non-trivial solution for $\vec{E}$ only if

$$\det\left[\frac{\varepsilon_{ij}}{\varepsilon_0} - \frac{c^2 k^2}{\omega^2}\left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)\right] = 0 \tag{22}$$

Again, with the $z-$axis along $\vec{B}_0$, and defining the dimensionless quantities:

$$\kappa_0 \equiv 1 - \frac{\omega_p^2}{\omega^2} \tag{23}$$

$$\kappa_1 \equiv 1 - \frac{\omega_p^2/\omega^2}{1 - \frac{\Omega^2}{\omega^2}} \tag{24}$$

and

$$\kappa_2 \equiv \frac{\omega_p^2\Omega/\omega^3}{1 - \frac{\Omega^2}{\omega^2}} \tag{25}$$

we can write (19) as:

$$\frac{\varepsilon_{ij}}{\varepsilon_0} = \begin{pmatrix} \kappa_1 & i\kappa_2 & 0 \\ -i\kappa_2 & \kappa_1 & 0 \\ 0 & 0 & \kappa_0 \end{pmatrix}$$

Choose coordinate axes so that $\hat{k}$ lies in the $y - z$ plane:

$$\hat{k} = \begin{pmatrix} 0, & \sin\theta, & \cos\theta \end{pmatrix}$$

Then equation (22) becomes:

$$\begin{vmatrix} \kappa_1 - \frac{c^2 k^2}{\omega^2} & i\kappa_2 & 0 \\ -i\kappa_2 & \kappa_1 - \frac{c^2 k^2}{\omega^2}\left(1 - \sin^2\theta\right) & \frac{c^2 k^2}{\omega^2}\sin\theta\cos\theta \\ 0 & \frac{c^2 k^2}{\omega^2}\sin\theta\cos\theta & \kappa_0 - \frac{c^2 k^2}{\omega^2}\left(1 - \cos^2\theta\right) \end{vmatrix} = 0 \tag{26}$$

This is the dispersion relation for electromagnetic waves in a magnetized plasma. Rather than solve this equation in the most general case, we'll look here at the specific cases of propagation along and perpendicular to $\vec{B}_0$.

2.2 Propagation along $\vec{B}_0$ ($\theta = 0$)

With $\theta = 0$, equation (26) becomes:

$$\begin{vmatrix} \kappa_1 - \frac{c^2 k^2}{\omega^2} & i\kappa_2 & 0 \\ -i\kappa_2 & \kappa_1 - \frac{c^2 k^2}{\omega^2} & 0 \\ 0 & 0 & \kappa_0 \end{vmatrix} = 0$$

$$\left[\left(\kappa_1 - \frac{c^2 k^2}{\omega^2} \right)^2 - \kappa_2^2 \right] \kappa_0 = 0$$

Thus either $\kappa_0 = 0$, which corresponds to electrostatic (longitudinal) waves at frequency $\omega = \omega_p$ (eqn 23) (see below), or

$$\kappa_1 - \frac{c^2 k^2}{\omega^2} = \pm \kappa_2 \tag{27}$$

Substituting in for the κ from (24) and (25), we get the dispersion relation for transverse waves:

$$\begin{aligned} \frac{c^2 k^2}{\omega^2} &= 1 - \frac{\omega_p^2}{\omega^2 - \Omega^2} \mp \frac{\omega_p^2 \Omega / \omega}{\omega^2 - \Omega^2} \\ &= 1 - \frac{\omega_p^2}{\omega} \frac{(\omega \pm \Omega)}{\omega^2 - \Omega^2} \\ \frac{c^2 k^2}{\omega^2} &= 1 - \frac{\omega_p^2}{\omega (\omega \mp \Omega)} \end{aligned} \tag{28}$$

There is a resonance ($n^2 = c^2 k^2 / \omega^2 \to \infty$) when $\omega \to \Omega$ with the upper sign. (More on this below.)

To show that these are the transverse (electromagnetic waves), let's solve for the corresponding electric field vectors. Equation (21) is:

$$\begin{pmatrix} \kappa_1 - \frac{c^2 k^2}{\omega^2} & i\kappa_2 & 0 \\ -i\kappa_2 & \kappa_1 - \frac{c^2 k^2}{\omega^2} & 0 \\ 0 & 0 & \kappa_0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = 0$$

Now we use equation (27) to simplify:

$$\begin{pmatrix} \pm\kappa_2 & i\kappa_2 & 0 \\ -i\kappa_2 & \pm\kappa_2 & 0 \\ 0 & 0 & \kappa_0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} \pm\kappa_2 E_1 + i\kappa_2 E_2 \\ -i\kappa_2 E_1 \pm \kappa_2 E_2 \\ \kappa_0 E_3 \end{pmatrix} = 0$$

The $z-$component shows that either $\kappa_0 = 0$ and $E_3 \neq 0$, ($\vec{E}$ is parallel to $\vec{k}$, so these are longitudinal waves[2]) or if $\kappa_0 \neq 0$, then $E_3 = 0$. This is the transverse wave. In this case, the $x-$component gives either $\kappa_2 = 0$ (which is not possible unless $\omega \to \infty$) or

$$E_2 = \pm i E_1$$

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[2] When $\kappa_0 = 0$, the $x-$ and $y-$ components give $E_1 = E_2 = 0$. This result verifies our previous claim about the meaning of ω_p.

We get the same result from the $y-$component. These solutions corrrespond to right hand (upper sign) and left hand (lower sign) circular polarization. (Polarization notes eqn 4.) Note that the RHC mode (top sign of the pair) corresponds to the resonance in eqn (28). An electron gyrates counter-clockwise around $\vec{B}$. In this mode, the electric field vector rotates in the same sense, and at $\omega = \Omega$, it also rotates at the same rate. Energy may be transferred continuously from the fields to the electrons.

The wave ceases to propagate when $k^2 < 0$. $k^2 = 0$ when

$$\begin{aligned}
\frac{c^2 k^2}{\omega^2} &= 1 - \frac{\omega_p^2}{\omega\left(\omega \mp \Omega\right)} = 0 \\
\omega^2 \mp \omega\Omega - \omega_p^2 &= 0 \\
\omega &= \frac{\pm\Omega \pm \sqrt{\Omega^2 + 4\omega_p^2}}{2}
\end{aligned}$$

(29)

Only the + sign on the square root leads to positive frequencies, so

$$\omega_R, \omega_L = \frac{\sqrt{\Omega^2 + 4\omega_p^2} \pm \Omega}{2}$$

are the roots for the right and left hand circular polarizations. These frequencies are called the cut-off frequencies. Thus the dispersion relation is

$$n_{R,L}^2 = \frac{\left(\omega - \omega_{R,L}\right)\left(\omega + \omega_{L,R}\right)}{\omega\left(\omega \mp \Omega\right)}$$

For the LHC mode (lower sign), the denominator is always positive and $n^2 > 0$ for $\omega > \omega_L$. For the RHC mode, there is a resonance ($k \to \infty$) at $\omega = \Omega$. Since $\omega_R > \Omega$, the square of the refractive index n^2 is positive for $\omega > \omega_R$ or $\omega < \Omega$, but is negative for $\Omega < \omega < \omega_R$.

We may plot n^2 versus ω/Ω for the two modes in the case $\Omega = 2\omega_p$.

$$\omega_{R,L} = \frac{\Omega}{2}\left(\sqrt{1+1} \pm 1\right) = \frac{\Omega}{2}\left(\sqrt{2} \pm 1\right) = 1.207\Omega,\ 0.207\Omega$$

(Black curve LHC, red curve RHC). The frequencies where $n^2 < 0$ are called stop bands, and the wave does not propagate at these frequencies. Note the resonance at $\omega = \Omega$ for the RHC mode. The resonance at $\omega \to 0$ is spurious due to our neglect of ion motion.
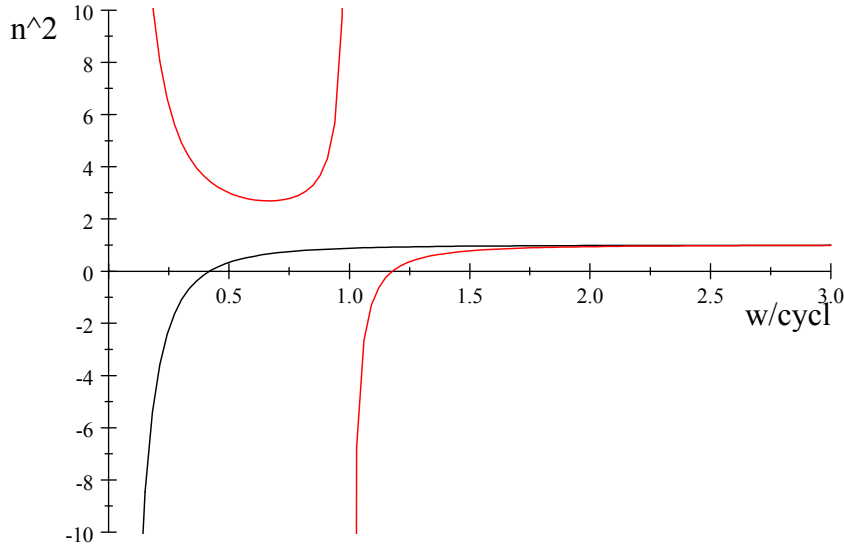
Black: LHC Red: RHC

2.3 Propagation of a plane polarized wave along $\vec{B}_0$

Since the normal modes are circularly polarized, to understand the propagation of a linearly polarized wave we must split it up into two circular polarizations. Let's put the $x-$axis along the direction of the linear polarization. Then (polarization notes eqn 4):

$$
\begin{aligned}
\vec{E}_0 &= E_0\hat{x} \\
&= \frac{1}{2}E_0\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right) + \frac{1}{2}E_0\left(\hat{\mathbf{x}}-i\hat{\mathbf{y}}\right)
\end{aligned}
$$

The two circular polarizations have different phase velocities. Thus, after travelling a distance z, the wave is described by:

$$
\vec{E} = \frac{1}{2}E_0\left(\hat{\mathbf{x}}+i\hat{\mathbf{y}}\right)e^{ik_R z - i\omega t} + \frac{1}{2}E_0\left(\hat{\mathbf{x}}-i\hat{\mathbf{y}}\right)e^{ik_L z - i\omega t}
$$

where (equation 28):

$$
\frac{c^2 k_R^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega\left(\omega - \Omega\right)} \tag{30}
$$

and

$$
\frac{c^2 k_L^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega\left(\omega + \Omega\right)} \tag{31}
$$

9

At time t, the electric field vector makes an angle ϕ with the $x-$axis, where

$$
\begin{aligned}
\tan \phi &= \frac{E_y}{E_x} = \frac{\operatorname{Re}\left(i\left[e^{ik_R z - i\omega t} - e^{ik_L z - i\omega t}\right]\right)}{\operatorname{Re}\left(e^{ik_R z - i\omega t} + e^{ik_L z - i\omega t}\right)} \\
&= \frac{-\sin\left(k_R z - \omega t\right) + \sin\left(k_L z - \omega t\right)}{\cos\left(k_R z - \omega t\right) + \cos\left(k_L z - \omega t\right)} \\
&= \frac{2\cos\left(\frac{k_L + k_R}{2} z - \omega t\right) \sin\left(\frac{k_L - k_R}{2} z\right)}{2\cos\left(\frac{k_L + k_R}{2} z - \omega t\right) \cos\left(\frac{k_L - k_R}{2} z\right)} \\
&= \tan\left(\frac{k_L - k_R}{2} z\right)
\end{aligned}
$$

Thus:
$$
\phi = \frac{k_L - k_R}{2} z \pm 2n\pi \tag{32}
$$

For high frequency waves, $\omega \gg \omega_p,\ \Omega$, we may write equations (30) and (31) in terms of the small quantities Ω/ω and ω_p/ω to get:

$$
\begin{aligned}
k_{R,L} &= \frac{\omega}{c} \sqrt{1 - \frac{\omega_p^2}{\omega^2}\left(1 \mp \frac{\Omega}{\omega}\right)^{-1}} \simeq \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}\left(1 \pm \frac{\Omega}{\omega}\right)} \\
&\simeq \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\left(1 \pm \frac{\Omega}{\omega}\right)\right]
\end{aligned}
$$

and thus
$$
k_L - k_R = \frac{\omega}{c}\frac{\omega_p^2}{\omega^2}\frac{\Omega}{\omega}
$$

giving

$$
\begin{aligned}
\phi &= \frac{\omega_p^2 \Omega}{2\omega^2}\frac{z}{c} \\
&= \frac{z}{2c\omega^2}\frac{n_0 e^2}{\varepsilon_0 m}\frac{eB_0}{m} \\
&= \frac{e^3}{2\left(2\pi\right)^2 m^2 c}\frac{n_0 B_0 z}{\nu^2} \tag{33}
\end{aligned}
$$

Thus the direction of polarization rotates as the wave travels. The rotation angle ϕ is proportional to the electron density n_0, the magnetic field strength B_0, the distance travelled z, and inversely proportional to the square of the frequency ν. This effect is known as *Faraday Rotation,* and it offers an important method for measuring magnetic field strength.

2.4 Low frequency waves

When the wave frequency[3] $\omega \ll \Omega, \omega_p$ the dispersion relation (28) simplifies:

$$
\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega\left(\mp\Omega\right)} = 1 \pm \frac{\omega_p^2}{\omega\Omega} \approx \frac{\omega_p^2}{\omega\Omega}
$$

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[3] We also need $\omega \gg \Omega_p = \sqrt{\frac{ne^2}{\varepsilon_0 M_p}}$, the ion plasma frequency, to remain in our original "high-frequency" regime in which ion motion may be ignored.

Note: only the positive sign (RHC waves) makes sense in this frequency range. With the minus sign (LHC waves) , $k^2 < 0$ and this mode does not propagate. For RHC waves, we get:

$$\frac{c^2 k^2}{\omega^2} \approx \frac{\omega_p^2}{\omega\Omega}$$

$$k \approx \sqrt{\frac{\omega}{\Omega}} \frac{\omega_p}{c} \tag{34}$$

For these waves, the phase speed

$$v_\phi = \frac{\omega}{k} = \frac{\sqrt{\omega\Omega}}{\omega_p} c \tag{35}$$

increases with the frequency ω. Because a signal's high frequencies will arrive before the low frequencies, resulting in a declining pitch "whistle", these waves are called *whistlers*.

The graph shows the square of the refractive index n versus ω/ω_p for the RHC wave in the case $\Omega = \omega_p/2$. The Whistler branch is the left (low frequency) side of the upper curve.
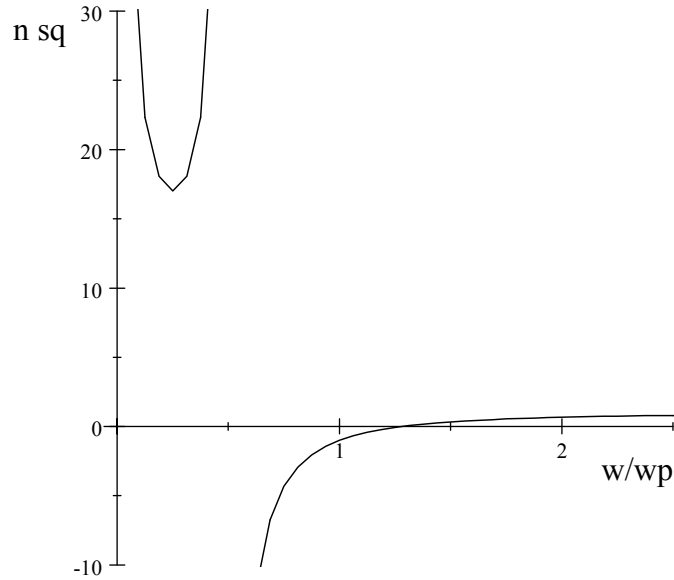
2.5 Propagation perpendicular to $\vec{B}$

We start with the general dispersion relation (26) and this time set $\theta = \pi/2$, so that the wave propagates in the $y-$direction.

$$\begin{vmatrix} \kappa_1 - \frac{c^2 k^2}{\omega^2} & i\kappa_2 & 0 \\ -i\kappa_2 & \kappa_1 - \frac{c^2 k^2}{\omega^2}\left(1 - \sin^2\theta\right) & \frac{c^2 k^2}{\omega^2}\sin\theta\cos\theta \\ 0 & \frac{c^2 k^2}{\omega^2}\sin\theta\cos\theta & \kappa_0 - \frac{c^2 k^2}{\omega^2}\left(1 - \cos^2\theta\right) \end{vmatrix} = 0$$

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$$
\begin{vmatrix}
\kappa_1 - \frac{c^2 k^2}{\omega^2} & i\kappa_2 & 0 \\
-i\kappa_2 & \kappa_1 & 0 \\
0 & 0 & \kappa_0 - \frac{c^2 k^2}{\omega^2}
\end{vmatrix} = 0
$$

$$
\left(\kappa_0 - \frac{c^2 k^2}{\omega^2} \right) \left[\left(\kappa_1 - \frac{c^2 k^2}{\omega^2} \right) \kappa_1 - \kappa_2^2 \right] = 0
$$

One solution is

$$
\kappa_0 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}
$$

This is the same relation (12) that we obtained in the unmagnetized plasma. This wave is called the ordinary (O-)wave.

The alternative solutions are

$$
\kappa_1^2 - \frac{c^2 k^2}{\omega^2} \kappa_1 - \kappa_2^2 = 0
$$

$$
n^2 = \frac{c^2 k^2}{\omega^2} = \kappa_1 - \frac{\kappa_2^2}{\kappa_1}
$$

$$
= 1 - \frac{\omega_p^2/\omega^2}{1 - \frac{\Omega^2}{\omega^2}} - \left(\frac{\omega_p^2 \Omega/\omega^3}{1 - \frac{\Omega^2}{\omega^2}} \right)^2 \frac{1}{1 - \frac{\omega_p^2/\omega^2}{1 - \frac{\Omega^2}{\omega^2}}}
$$

Define a new frequency, the upper hybrid frequency, by

$$
\omega_H^2 \equiv \omega_p^2 + \Omega^2
$$

Then

$$
n^2 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1 - \frac{\Omega^2}{\omega^2}} \left[1 + \frac{\omega_p^2 \Omega^2/\omega^4}{1 - \frac{\omega_H^2}{\omega^2}} \right]
$$

$$
= 1 - \frac{\omega_p^2}{(\omega^2 - \omega_H^2)} \left[\frac{\omega^2 - \omega_H^2 + \omega_p^2 \Omega^2/\omega^2}{\omega^2 - \Omega^2} \right]
$$

$$
= 1 - \frac{\omega_p^2/\omega^2}{(\omega^2 - \omega_H^2)} \left[\frac{\left(\omega^2 - \omega_p^2 - \Omega^2 \right) \omega^2 + \omega_p^2 \Omega^2}{\omega^2 - \Omega^2} \right]
$$

$$
= 1 - \frac{\omega_p^2}{\omega^2} \left(\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_H^2} \right)
$$

This is the extra-ordinary (X-)wave. The cut-off frequencies where $n = 0$ are given by

$$
\omega^2 \left(\omega^2 - \omega_H^2 \right) = \omega_p^2 \left(\omega^2 - \omega_p^2 \right)
$$

$$
\omega^4 - \omega^2 \omega_H^2 - \omega_p^2 \omega^2 + \omega_p^4 = 0
$$

$$
\omega^4 - \omega^2 \Omega^2 - 2\omega_p^2 \omega^2 + \omega_p^4 = 0
$$

$$
\omega^2 - \omega_p^2 = \pm \omega \Omega
$$

This is eqn (29) with solutions

$$
\omega = \frac{\pm \Omega \pm \sqrt{\Omega^2 + 4\omega_p^2}}{2} = \omega_R, \omega_L
$$

Now let's solve for the fields:

$$\begin{pmatrix} \kappa_1 - \frac{c^2 k^2}{\omega^2} & i\kappa_2 & 0 \\ -i\kappa_2 & \kappa_1 & 0 \\ 0 & 0 & \kappa_0 - \frac{c^2 k^2}{\omega^2} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = 0$$

From the last component, we see that the O-wave has $E_3 \neq 0$. The other two components show that $E_1 = E_2 = 0$. In this mode, the electrons ocscillate back and forth along $\vec{B}$, and the magnetic force plays no role. For the other mode $E_3 = 0$, and the first component gives

$$\left(\kappa_1 - \frac{c^2 k^2}{\omega^2} \right) E_1 + i\kappa_2 E_2 = 0$$

or

$$\begin{aligned} E_2 &= -\frac{1}{i\kappa_2} \left(\kappa_1 - \frac{c^2 k^2}{\omega^2} \right) E_1 \\ &= \frac{i\kappa_2}{\kappa_1} E_1 \end{aligned}$$

and we get the same result from the second component. Since there is an electric field component (and hence a velocity component) along the direction of propagation (in this case the $y-$direction), there is a non-zero charge density (eqn. 7) and this wave is not purely electromagnetic.

The plot shows n^2 for $\Omega = 2\omega_p$, $\omega_H = \sqrt{5}\omega_p$. Note the stop bands for $\omega < \omega_L$ and $\omega_H < \omega < \omega_R$
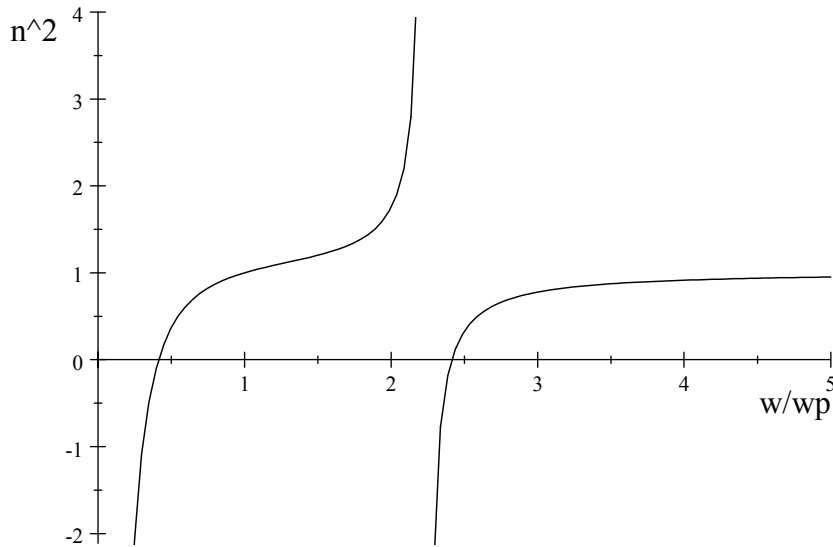
For more on wave propagation in plasmas see P 7.17, 18, 25,
http://www.physics.sfsu.edu/~lea/courses/grad/plaswav.PDF, and books on Plasma Physics.