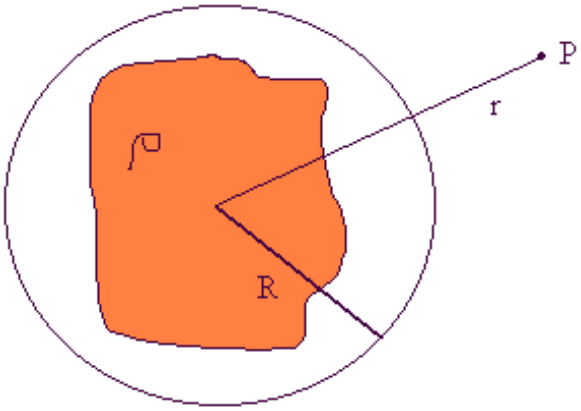
Jackson notes 2020

1 Spherical multipole moments



Suppose we have a charge distribution $\rho(\vec{x})$ where all of the charge is contained within a spherical region of radius R, as shown in the diagram. Then there is no charge in the region $r > R$ and so we may write the potential in that region as a solution of Laplace's equation in spherical coordinates. With the constraint that $\Phi \to 0$ as $r \to \infty$, we have

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}\frac{q_{lm}}{r^{l+1}}Y_{lm}(\theta,\phi) \qquad r > R \tag{1}$$

where the constants $4\pi/(2l+1)$ have been included for future convenience. We may also write the potential in terms of the charge density as (Notes 1 eqn.29):

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' \tag{2}$$

and then expand the factor $1/|\vec{x}-\vec{x}'|$ in spherical harmonics using J eqn 3.70 (spherprobnotes eqn 25):

$$\begin{aligned}\Phi(r,\theta,\phi) &= \frac{1}{4\pi\varepsilon_0}\int\rho(\vec{x}')\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\phi)\,Y_{lm}^*(\theta',\phi')\;d^3x' \\ &= \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)\int\rho(\vec{x}')\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')\;d^3x'\end{aligned}$$

The integral is over the entire volume where ρ is not zero, which is inside the sphere of radius R. Then since $r' \leq R$ and $r > R$, we have $r_< = r'$ and $r_> = r$, so

$$\Phi\left(r > R, \theta, \phi\right) = \frac{1}{4\pi\varepsilon_0} \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}\left(\theta, \phi\right)}{r^{l+1}} \int \rho\left(\vec{x}'\right) \left(r'\right)^l Y_{lm}^*\left(\theta', \phi'\right) \; d^3x' \quad (3)$$

and then comparing equations (1) and (3) we find

$$q_{lm} = \int \rho\left(\vec{x}'\right) \left(r'\right)^l Y_{lm}^*\left(\theta', \phi'\right) \; d^3x' \quad (4)$$

These are the *spherical multipole moments* of the source. With $l = 1$ we have the dipole, $l = 2$ the quadrupole etc.

2 Cartesian multipole moments

This time we expand $\frac{1}{|\vec{x}-\vec{x}'|}$ in expression (2) in a Taylor series about $\vec{x}' = 0$:

$$\Phi\left(\vec{x}\right) = k \int \rho\left(\vec{x}'\right) \left(\frac{1}{r} + \vec{x}' \cdot \vec{\nabla}' \frac{1}{R}\bigg|_{\vec{x}'=0} + \frac{1}{2} \sum_{i,j=1}^{3} x_i' x_j' \; \frac{\partial}{\partial x_i'} \frac{\partial}{\partial x_j'} \frac{1}{R}\bigg|_{\vec{x}'=0} + \cdots \right) \; d^3x'$$

Now let's evaluate the derivatives:

$$\vec{\nabla}' \frac{1}{R}\bigg|_{\vec{x}'=0} = \frac{\left(\vec{x} - \vec{x}'\right)}{R^3}\bigg|_{\vec{x}'=0} = \frac{\vec{x}}{r^3}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_i'} \frac{\partial}{\partial x_j'} \frac{1}{R}\bigg|_{\vec{x}'=0} &= \frac{\partial}{\partial x_i'} \frac{\left(x_j - x_j'\right)}{R^3}\bigg|_{\vec{x}'=0} = \frac{-\delta_{ij}}{R^3} + 3\frac{\left(x_i - x_i'\right)\left(x_j - x_j'\right)}{R^5}\bigg|_{\vec{x}'=0} \\ &= \frac{-\delta_{ij}}{r^3} + 3\frac{x_i x_j}{r^5} \end{aligned} \quad (5)$$

Putting these expressions back into the Taylor series, we get:

$$\Phi\left(\vec{x}\right) = k \int \rho\left(\vec{x}'\right) \left[\frac{1}{r} + \vec{x}' \cdot \frac{\vec{x}}{r^3} + \frac{1}{2} \sum_{i,j} x_i' x_j' \left(3\frac{x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3} \right) + \cdots \right] d^3x'$$

We may integrate term by term because the Taylor series is uniformly convergent.

$$\Phi\left(\vec{x}\right) = k \left\{ \frac{1}{r} \int \rho\left(\vec{x}'\right) d^3x' + \frac{\vec{x}}{r^3} \cdot \int \vec{x}' \rho\left(\vec{x}'\right) d^3x' + \frac{1}{2} \sum_{i,j} \int \left(3\frac{x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3} \right) \rho\left(\vec{x}'\right) x_i' x_j' d^3x' + \cdots \right\}$$

The first two terms may be re-expressed to give:

$$\Phi(\vec{x}) = k\frac{q}{r} + k\frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{k}{r^5}\sum_{i,j}\int \left(3x_i x_j - r^2\delta_{ij}\right)\rho(\vec{x}')\,\frac{x_i' x_j'}{2}d^3x' + \cdots \qquad (6)$$

where q is the total charge in the distribution and

$$\vec{p} \equiv \int \rho(\vec{x}')\,\vec{x}'\; d^3x' \qquad (7)$$

is the Cartesian dipole moment. The last term in (6) needs some work, since we would like to express the result in terms of the quadrupole tensor, with components defined as

$$Q_{ij} \equiv \int \rho(\vec{x}')\left[3x_i' x_j' - (r')^2\,\delta_{ij}\right] d^3x', \qquad (8)$$

but our integral has unprimed variables where we need primes. However, the first term in (6) is symmetric in primed and unprimed variables, while the second is non-zero only if $i = j$. The second term is:

$$r^2(x')^2 + r^2(y')^2 + r^2(z')^2 = (r')^2 r^2 = (r')^2 x^2 + (r')^2 y^2 + (r')^2 z^2$$

and so we may interchange prime and unprime and rewrite the integral as:

$$\begin{aligned}\int \rho(\vec{x}')\sum_{i,j} x_i' x_j'\left(3x_i x_j - r^2\delta_{ij}\right)d^3x' &= \sum_{i,j} x_i x_j \int \rho(\vec{x}')\left(3x_i' x_j' - \left(r'\right)^2\delta_{ij}\right)d^3x' \\ &= \sum_{i,j} x_i x_j Q_{ij}\end{aligned}$$

Thus we have for the potential

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j=1}^{3}\frac{x_i x_j}{r^5}Q_{ij} + \cdots\right\} \qquad (9)$$

The quadrupole tensor Q_{ij} (8) is symmetric, real-valued and traceless. It has three real eigenvalues.

The total charge is independent of our choice of origin. In general, the lowest non-zero multipole is independent of origin, but higher order multipoles do depend on the origin. (See Jackson Problem 4.4.)

3 Relations between the two sets of multipoles

Comparing the expressions (1) and (9) allows us to relate the two sets of multipoles. Both expressions are series in increasing powers of $1/r$. We'll need expressions for the spherical harmonics (Jackson page 109, Lea page 388).

The $l = 0$ $(1/r)$ term is the monopole term. From equation (4) with $l = m = 0$,

$$q_{00} = \int \rho(\vec{x}') Y_{00}^*(\theta', \phi') d^3x' = \frac{1}{\sqrt{4\pi}} \int \rho(\vec{x}') d^3x' = \frac{q}{\sqrt{4\pi}} \tag{10}$$

and the first term in the spherical multipole expansion (1) is

$$\Phi_0 = \frac{1}{4\pi\varepsilon_0} 4\pi \frac{q_{00}}{r} Y_{00} = \frac{1}{\varepsilon_0} \frac{q}{\sqrt{4\pi}} \frac{1}{r} \frac{1}{\sqrt{4\pi}} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$$

which is also the first term in expression (9).

With $l = 1$ ($1/r^2$ term) there are three contributions, with $m = \pm 1$ and zero. First note that

$$\begin{aligned} q_{l,-m} &= \int \rho(\vec{x}') (r')^l Y_{l-m}^*(\theta', \phi') d^3x' \\ &= (-1)^m \int \rho(\vec{x}') (r')^l Y_{lm}(\theta', \phi') d^3x' \\ &= (-1)^m q_{lm}^* \end{aligned} \tag{11}$$

So there are actually only two values to calculate.

With $l = 1$, $m = 0$

$$\begin{aligned} q_{10} &= \int \rho(\vec{x}') r' Y_{10}^*(\theta', \phi') d^3x' \\ &= \int \rho(\vec{x}') r' \sqrt{\frac{3}{4\pi}} \cos\theta' \ d^3x' \\ q_{10} &= \sqrt{\frac{3}{4\pi}} \int \rho(\vec{x}') z' \ d^3x' = \sqrt{\frac{3}{4\pi}} p_z \end{aligned} \tag{12}$$

where we used the $z-$component of (7).

With $l = 1$, $m = 1$

$$\begin{aligned} q_{11} &= \int \rho(\vec{x}') r' Y_{11}^*(\theta', \phi') d^3x' \\ &= -\int \rho(\vec{x}') r' \sqrt{\frac{3}{8\pi}} \sin\theta' e^{-i\phi'} \ d^3x' \\ &= -\int \rho(\vec{x}') r' \sqrt{\frac{3}{8\pi}} \sin\theta' (\cos\phi' - i\sin\phi') \ d^3x' \\ &= -\sqrt{\frac{3}{8\pi}} \int \rho(\vec{x}') (x' - iy') \ d^3x' \\ q_{11} &= \sqrt{\frac{3}{8\pi}} (ip_y - p_x) \end{aligned} \tag{13}$$

Equivalently,

$$p_x = -\sqrt{\frac{2\pi}{3}}\left(q_{11} + q_{11}^*\right), \quad p_y = \frac{1}{i}\sqrt{\frac{2\pi}{3}}\left(q_{11} - q_{11}^*\right), \qquad p_z = \sqrt{\frac{4\pi}{3}} q_{10} \tag{14}$$

The corresponding $l = 1$ term in the potential (1) is

$$\begin{aligned}
\Phi_1 &= \frac{1}{4\pi\varepsilon_0}\sum_{m=-1}^{+1}\frac{4\pi}{3}\frac{q_{1m}}{r^2}Y_{1m}(\theta,\phi) \\
&= \frac{1}{4\pi\varepsilon_0 r^2}\frac{4\pi}{3}\left(q_{11}Y_{11}(\theta,\phi) + q_{10}Y_{10} + q_{1,-1}Y_{1,-1}\right) \\
&= \frac{1}{4\pi\varepsilon_0 r^2}\frac{4\pi}{3}\left(q_{11}Y_{11}(\theta,\phi) + q_{10}Y_{10} + (-1)\,q_{11}^*(-1)\,Y_{11}^*\right) \\
&= \frac{1}{3\varepsilon_0 r^2}\left\{2\,\mathrm{Re}\left[q_{11}Y_{11}(\theta,\phi)\right] + q_{10}Y_{10}\right\}
\end{aligned}$$

Inserting our expressions for q_{1m} in terms of the components of $\vec{p}$,we have

$$\begin{aligned}
\Phi_1 &= \frac{1}{3\varepsilon_0 r^2}\left\{2\,\mathrm{Re}\left[\sqrt{\frac{3}{8\pi}}\left(ip_y - p_x\right)\sqrt{\frac{3}{8\pi}}\left(-\sin\theta\right)e^{i\phi}\right] + \sqrt{\frac{3}{4\pi}}p_z\sqrt{\frac{3}{4\pi}}\cos\theta\right\} \\
&= \frac{1}{4\pi\varepsilon_0 r^2}\left(p_x\sin\theta\cos\phi + p_y\sin\theta\sin\phi + p_z\cos\theta\right) = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}
\end{aligned}$$

which is the second term in (9). (See also Notes 1 eqn 33.)

With $l = 2$ ($1/r^3$ term) we have $m = 2, 1$, and 0. The multipole moment q_{20} is

$$\begin{aligned}
q_{20} &= \int \rho(\vec{x}')\left(r'\right)^2 Y_{20}^*\left(\theta',\phi'\right)\, d^3x' \\
&= \int \rho(\vec{x}')\left(r'\right)^2 \frac{1}{2}\sqrt{\frac{5}{4\pi}}\left(3\cos^2\theta' - 1\right)\, d^3x' \\
&= \frac{1}{2}\sqrt{\frac{5}{4\pi}}\int \rho(\vec{x}')\left[3\left(z'\right)^2 - \left(r'\right)^2\right]\, d^3x' \\
q_{20} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}}Q_{33}
\end{aligned} \tag{15}$$

where we used the (3,3) component of (8). Similarly, we may show that

$$q_{21} = \frac{1}{3}\sqrt{\frac{15}{8\pi}}\left(iQ_{23} - Q_{13}\right) \tag{16}$$

and

$$q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\left(Q_{11} - Q_{22} - 2iQ_{12}\right) \tag{17}$$

Equivalently:

$$Q_{11} - Q_{22} = \mathrm{Re}\left(12\sqrt{\frac{2\pi}{15}}q_{22}\right)$$

and so on. We have 3 independent values of q_{2m}, and q_{21} and q_{22} each have a real and imaginary part, giving us 5 real numbers, but there are 6 independent values Q_{ij} in a general symmetric rank 2 tensor. However, in this case the six values are not all independent as we have the additional constraint that Q_{ij} is traceless, leaving us with 5 independent values. For the extension of this discussion to higher multipoles, see Jackson Problem 4.3.

4 Example

A ring of charge of radius a carries linear charge density that varies with angle ϕ measured around the ring: $\lambda = \lambda_0 \cos\phi$. Let's find the multipole moments and the potential for $r > a$.

The ring is most easily described in spherical coordinates, so let's first find the spherical multipoles. With polar axis along the axis of the ring, the charge density is (Lea Example 6.7)

$$\rho(\vec{x}) = \frac{\lambda_0 \cos\phi\ \delta(\mu)\,\delta(r-a)}{r}$$

and thus

$$\begin{aligned} q_{lm} &= \int \rho(\vec{x})\, r^l Y_{lm}^*(\theta,\phi)\ d^3x \\ &= \int \frac{\lambda_0 \cos\phi\ \delta(\mu)\,\delta(r-a)}{r} r^l Y_{lm}^*(\theta,\phi)\ r^2\ d\mu d\phi dr \\ &= \lambda_0 \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \int_0^\infty \int_0^{2\pi} \int_{-1}^{+1} \cos\phi\ \delta(\mu)\,\delta(r-a)\, r^{l+1} P_l^m(\mu)\, e^{-im\phi}\ d\mu d\phi dr \end{aligned}$$

Using the sifting property, we have immediately

$$\begin{aligned} q_{lm} &= \lambda_0 \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} a^{l+1} P_l^m(0) \int_0^{2\pi} \cos\phi e^{-im\phi}\ d\phi \\ &= \lambda_0 \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} a^{l+1} P_l^m(0) \int_0^{2\pi} \left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right) e^{-im\phi}\ d\phi \end{aligned}$$

The integral over ϕ is zero unless $m = \pm 1$, so

$$q_{lm} = \lambda_0 \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} a^{l+1} P_l^m(0)\,\pi\,(\delta_{m1} + \delta_{m,-1})$$

Then

$$q_{l,1} = \lambda_0 \pi \sqrt{\frac{2l+1}{4\pi}} a^{l+1} \sqrt{\frac{(l-1)!}{(l+1)!}} P_l^1(0)$$

We must have $l > 0$, since with $l = 0$ there is no $m = 1$. We also need $l+1$ to be even so that $P_l^1(0)$ is not zero. Thus we have all multipoles of odd order, and $m = 1$.

To find the potential, we use (11):

$$q_{l,-1} = -q_{l1}^*$$

So, since q_{l1} is real in this case,

$$q_{l,-1}Y_{l,-1} = (-1)^2 q_{l,1}^* Y_{l1}^* = q_{l,1}Y_{l1}^*$$

Thus

$$\begin{aligned} q_{l1}Y_{l1} + q_{l,-1}Y_{l,-1} &= q_{l1}\left(Y_{l1} + Y_{l1}^*\right) = 2q_{l1}\,\mathrm{Re}\left(Y_{l1}\right) \\ &= 2\lambda_0\pi\sqrt{\frac{2l+1}{4\pi}}a^{l+1}\sqrt{\frac{(l-1)!}{(l+1)!}}P_l^1(0)\sqrt{\frac{2l+1}{4\pi}\frac{(l-1)!}{(l+1)!}}P_l^1(\mu)\cos\phi \\ &= \lambda_0 a^{l+1}\frac{2l+1}{2}\frac{(l-1)!}{(l+1)!}P_l^1(0)\,P_l^1(\mu)\cos\phi \end{aligned}$$

Thus the potential for $r > a$ is (eqn 1)

$$\begin{aligned} \Phi &= \frac{1}{4\pi\varepsilon_0}\sum_{l=1,\ \mathrm{odd}}^{\infty}\frac{4\pi}{2l+1}\lambda_0\frac{a^{l+1}}{r^{l+1}}\frac{2l+1}{2}\frac{(l-1)!}{(l+1)!}P_l^1(0)\,P_l^1(\mu)\cos\phi \\ &= \frac{\lambda_0}{2\varepsilon_0}\sum_{l=1,\ \mathrm{odd}}^{\infty}\frac{a^{l+1}}{r^{l+1}}\frac{(l-1)!}{(l+1)!}P_l^1(0)\,P_l^1(\mu)\cos\phi \\ &= \frac{\lambda_0\cos\phi}{2\varepsilon_0}\left[\frac{a^2}{r^2}\frac{1}{2}\sin\theta + \frac{a^4}{r^4}\frac{2}{4!}\frac{3}{2}\frac{3}{2}\left(1-5\cos^2\theta\right)\sin\theta + \cdots\right] \\ &= \frac{\lambda_0\sin\theta\cos\phi}{4\varepsilon_0}\frac{a^2}{r^2}\left[1 + \frac{3}{8}\frac{a^2}{r^2}\left(1-5\cos^2\theta\right) + \cdots\right] \end{aligned}$$

The dominant term is a dipole, because the ring has zero net charge. Comparing with (9), we may read off the dipole moment:

$$\vec{p} = \pi\lambda_0 a^2\hat{x}$$

We may also compute $\vec{p}$ using (7).

$$\begin{aligned} \vec{p} &= \int_0^{\infty}\int_0^{2\pi}\int_{-1}^{+1}\frac{\lambda_0\cos\phi\;\delta(\mu)\,\delta(r-a)}{r}r\left[\mu\hat{z} + \sqrt{1-\mu^2}\left(\cos\phi\hat{x} + \sin\phi\hat{y}\right)\right]r^2 dr d\mu d\phi \\ &= \lambda_0 a^2\pi\hat{x} \end{aligned}$$

The two results agee.

The next term is the octupole ($l = 3$) term. The potential has the same azimuthal dependence ($\cos\phi$) as the charge density.

Multipole moments are very important in computing the radiation from a time-dependent charge distribution. (See Ch 9.)

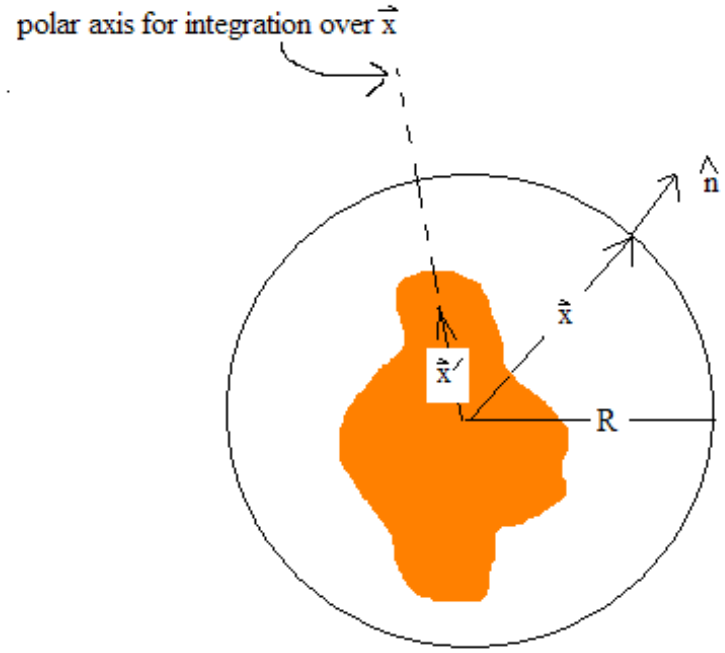
5 A surprising result

Let us evaluate the integral:

$$\int_{\text{sphere}} \vec{E}\,(\vec{x})\ d^3x$$

where the integral is over a sphere of radius R that contains all the sources of $\vec{E}$, as shown in the diagram below. We express the electric field in terms of the potential, and convert to a surface integral:

$$\int_{\text{sphere}} \vec{E}\,(\vec{x})\ d^3x = \int_{\text{sphere}} -\vec{\nabla}\Phi\ dV = -\oint_S \Phi\hat{\mathbf{n}}\ dA = -\oint_S \Phi\hat{\mathbf{n}}R^2\ d\Omega$$



Insert the integral expression (2) for the potential, and expand $1/|\vec{x}-\vec{x}'|$ in Legendre Polynomials:

$$\begin{aligned}\int_{\text{sphere}} \vec{E}\,(\vec{x})\ d^3x &= -k\oint_S \int_{\text{all space}} \frac{\rho\,(\vec{x}')}{|\vec{x}-\vec{x}'|}dV'\hat{\mathbf{n}}R^2\ d\Omega \\ &= -k\oint_S \int_{\text{all space}} \rho\,(\vec{x}')\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l\,(\cos\gamma)\,dV'\hat{\mathbf{n}}R^2\ d\Omega\end{aligned}$$

where $k = 1/4\pi\varepsilon_0$ and γ is the angle between $\vec{x}$ and $\vec{x}'$. Now we interchange the order of integration, and put the polar axis along the vector $\vec{x}'$, so that the polar angle is γ. The unit vector $\hat{n} = \hat{r}$ depends on the angles, so we write $\hat{n}$ in Cartesian components as:

$$\hat{\mathbf{n}} = \hat{\mathbf{x}} \sin\gamma\cos\phi + \hat{\mathbf{y}} \sin\gamma\sin\phi + \hat{\mathbf{z}} \cos\gamma$$

to obtain:

$$\int \vec{E}\; d^3x = -k \int_{\substack{\text{all}\\ \text{space}}} \int_0^{2\pi} \int_{-1}^{+1} \rho\left(\vec{x}'\right) \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l\left(\mu_\gamma\right) \left(\hat{\mathbf{x}} \sin\gamma\cos\phi + \hat{\mathbf{y}} \sin\gamma\sin\phi + \hat{\mathbf{z}} \cos\gamma\right) R^2 \; d\mu_\gamma d\phi dV'$$

The $x-$and $y-$components vanish upon integration over ϕ, leaving:

$$\int \vec{E}\; d^3x = -k\hat{\mathbf{z}} \int_{\text{all space}} 2\pi \int_{-1}^{+1} \rho\left(\vec{x}'\right) \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l\left(\cos\gamma\right) \cos\gamma R^2 \; d\mu_\gamma dV'$$

Since $\cos\gamma = P_1\left(\cos\gamma\right)$, orthogonality of the P_l requires that only the $l = 1$ term survive the integration over γ. Then (Jackson eqn 3.21, Lea 8.33) with $l = 1$ gives

$$\int \vec{E}\; d^3x = -k\hat{\mathbf{z}} \int_{\text{all space}} 2\pi \frac{2}{3} \rho\left(\vec{x}'\right) \frac{r_<}{r_>^2} R^2 \; dV'$$

Now $\vec{x}$ is on the surface of the sphere, where $|\vec{x}| = r = R$, and $|\vec{x}'| = r' < r$ (because all of the charge is inside the sphere). Also, we chose our polar axis along $\vec{x}'$, so $\hat{z} = \hat{r}'$. Thus:

$$\begin{aligned} \int_{\text{sphere}} \vec{E}\; d^3x &= -\frac{4\pi}{3} k \int_{\text{all space}} \rho\left(\vec{x}'\right) \frac{r'}{R^2} \hat{r}' R^2 dV' \\ &= -\frac{1}{3\varepsilon_0} \int_{\text{all space}} \rho\left(\vec{x}'\right) \vec{x}' dV' = -\frac{\vec{p}}{3\varepsilon_0} \end{aligned} \tag{18}$$

where $\vec{p}$ is the dipole moment with respect to the center of the sphere (eqn 7). .

This is a completely general result: we did not need to say anything about the details of the charge distribution.

Now let's look at the usual expression for the potential due to an ideal "point" dipole at the origin (Notes 1 eqn 33) :

$$\Phi\left(\vec{x}\right) = k \frac{\vec{p} \cdot \vec{x}}{r^3}$$

The electric field is:

$$\vec{E} = -\vec{\nabla}\Phi = -k \left\{ \frac{\vec{p}}{r^3} - 3\frac{\vec{x}}{r^5} \vec{p} \cdot \vec{x} \right\} \tag{19}$$

(Here we used the expression for $\vec{\nabla}\left(\vec{a} \cdot \vec{b}\right)$ from J's front cover, together with the fact that $\vec{p}$ is a constant vector, $\vec{x}/r^3 = -\vec{\nabla}\left(1/r\right)$, and the curl of a gradient is zero. See also Jackson 4.13.) With this electric field, we have:

$$\int_{\text{sphere}} \vec{E}\; d^3x = \int_{\text{sphere}} -k \left[\frac{\vec{p}}{r^3} - 3\frac{\vec{x}}{r^5} \left(\vec{p} \cdot \vec{x}\right) \right] d^3x$$

Put the polar axis along $\vec{p}$, so that:

$$\int_{\rm sphere} \vec{E}\; d^3x = -\int_{\rm sphere} k\left[\frac{\vec{p}}{r^3} - 3\frac{\hat{r}}{r^3}p\cos\theta\right] r^2\; dr d\Omega$$

Now

$$\hat{r} = \hat{p}\cos\theta + \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi$$

Again the $x-$ and $y-$components vanish when we perform the $\phi-$integration. We do the θ integration next, to get

$$\begin{aligned}\int_{\rm sphere} \vec{E}\; d^3x &= -2\pi k\int_0^R\int_{-1}^{+1}\frac{\vec{p}}{r}\left(1-3\mu^2\right)d\mu dr \\ &= -2\pi k\vec{p}\int_0^R\frac{1}{r}\left(\mu-\mu^3\right)\Big|_{-1}^{+1}dr = 0\end{aligned}$$

How can this be? We should have obtained $-k\frac{4\pi}{3}\vec{p}$ (equation 18). The problem is that when we took the derivatives to get the field (eqn. 19), those operations are not valid at the origin. There is a missing delta-function! You can also understand this by looking at the field line diagram for a pair of equal and opposite point charges. As the separation goes to zero, the field lines between the charges (whose direction is opposite $\vec{p}$) get packed into zero space– the field becomes infinite!

For consistency, the dipole field must be:

$$\vec{E}_{\rm dipole} = -k\left\{\frac{\vec{p}}{r^3} - 3\frac{\vec{x}}{r^5}\vec{p}\cdot\vec{x}\right\} - \frac{4\pi}{3}k\vec{p}\delta\left(\vec{x}\right) \tag{20}$$

To see why this result makes sense, let's look again at the derivative of the potential:

$$\Phi = k\frac{\vec{p}\cdot\vec{x}}{r^3} = -k\vec{p}\cdot\vec{\nabla}\left(\frac{1}{r}\right)$$

Thus, with $z-$axis along $\vec{p}$,

$$-\vec{\nabla}\Phi = k\vec{\nabla}\left\{\vec{p}\cdot\vec{\nabla}\left(\frac{1}{r}\right)\right\} = kp\frac{\partial}{\partial z}\vec{\nabla}\left(\frac{1}{r}\right)$$

We already have the relation

$$\nabla^2\frac{1}{r} = \vec{\nabla}\cdot\left\{\vec{\nabla}\left(\frac{1}{r}\right)\right\} = -4\pi\delta\left(\vec{x}\right), \tag{21}$$

but

$$\frac{\partial}{\partial z}\vec{\nabla}\left(\frac{1}{r}\right)$$

is only one of the three second derivatives in the standard delta-function result (21), which helps us to understand the factor of 1/3 in equation (20). We can understand the direction by looking at the field line diagram.

To understand the magnitude another way, look at the elecric field half way between two point charges $\pm q$ separated by a displacement $\vec{d}$, where $\vec{p} = q\vec{d}$.

$$\begin{aligned} \vec{E} &= -2\frac{kq}{(d/2)^2}\hat{d} = -\frac{kq}{(d/2)^3}\vec{d} \\ &= -\frac{k\vec{p}}{(d/2)^3} = -\frac{4\pi k}{3}\frac{\vec{p}}{\frac{4}{3}\pi\left(\frac{d}{2}\right)^3} \end{aligned}$$

Now in the limit $d \to 0$ the dipole moment density $\vec{p}/\left(\frac{4\pi(d/2)^3}{3}\right) \to \vec{p}\delta(\vec{x})$ and so

$$\vec{E} \to -\frac{4\pi k}{3}\vec{p}\delta(\vec{x})$$

which is what we obtained in (20).

We can write our result (20) as a purely mathematical statement:

$$\frac{\partial}{\partial z}\vec{\nabla}\left(\frac{1}{r}\right) = \vec{\nabla}\left(\frac{\partial}{\partial z}\frac{1}{r}\right) = -\left(\frac{\hat{z}}{r^3} - 3\frac{\vec{x}}{r^5}\hat{z}\cdot\vec{x}\right) - \frac{4\pi}{3}\delta(\vec{x})\ \ \hat{z} \qquad (22)$$

You'll need this result in Problem 6.20.

6 Energy

If our charge distribution is now placed in an *external* field $\vec{E}_{\rm ext} = -\vec{\nabla}\Phi_{\rm ext}$, the energy of the system is (Notes 2 eqn 3. Do you understand why there is no factor of 1/2 here?)

$$U = \int_V \rho(\vec{x})\,\Phi_{\rm ext}(\vec{x})\ \ d^3\vec{x}$$

where the integral is over a volume V that contains all of the charge. To exhibit the result in terms of the multipoles of the charge distribution, we expand the external potential in a Taylor series about the origin[1];

$$\begin{aligned} U &= \int_V \rho(\vec{x})\left(\Phi_{\rm ext}(0) + \vec{x}\cdot\vec{\nabla}\Phi_{\rm ext}\Big|_0 + \frac{1}{2}x_i x_j\ \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Phi_{\rm ext}\Big|_0 + \cdots\right)\ d^3\vec{x} \\ &= q\Phi_{\rm ext}(0) - \vec{p}\cdot\vec{E}_{\rm ext}(0) - \frac{1}{2}\ \frac{\partial E_{{\rm ext},j}}{\partial x_i}\Big|_0 \int_V \rho(\vec{x})\,x_i x_j\ d^3\vec{x} + \cdots \end{aligned}$$

where $\vec{p}$ is the dipole moment with respect to the same origin O. We want to express the third term using the Q_{ij}, but we are missing a term. We can add it in because it is zero! Since the sources of the external field $\vec{E}_{\rm ext}$ are not in the volume V that contains the charge density ρ, then

$$\vec{\nabla}\cdot\vec{E}_{\rm ext} = 0 \ \ \text{in } V$$

[1] Summation convention is used in this section.

Thus

$$\frac{\partial E_{\text{ext},j}}{\partial x_i} r^2 \delta_{ij} = \frac{\partial E_{\text{ext},j}}{\partial x_j} r^2 = \vec{\nabla} \cdot \vec{E}_{\text{ext}} r^2 = 0$$

and so

$$\begin{aligned} \frac{1}{2} \left. \frac{\partial E_{\text{ext}.j}}{\partial x_i} \right|_0 \int_V \rho(\vec{x}) x_i x_j \, d^3\vec{x} &= \frac{1}{2} \left. \frac{\partial E_{\text{ext}.j}}{\partial x_i} \right|_0 \int_V \rho(\vec{x}) \left(x_i x_j - \frac{r^2}{3} \delta_{ij} \right) d^3\vec{x} \\ &= \frac{1}{6} \left. \frac{\partial E_{\text{ext}.j}}{\partial x_i} \right|_0 Q_{ij} \end{aligned}$$

Thus

$$U = q\Phi_{\text{ext}}(0) - \vec{p} \cdot \vec{E}_{\text{ext}}(0) - \frac{1}{6} Q_{ij} \left. \frac{\partial E_{\text{ext}.j}}{\partial x_i} \right|_0 + \cdots \tag{23}$$

The second term shows that a dipole is in stable equilbrium in an external field when it is aligned parallel to $\vec{E}_{\text{ext}}$. (This is the minimum energy state.)

Similar techniques may be used to express the force and torque on a charge distribution in terms of its multipoles, as in J Problem 4.5.