Waveguide notes 2020

Electromagnetic waves in free space

We start with Maxwell's equations for an LIH medum in the case that the source terms ρ_f and $\vec{j}_f$ are both zero.

$$\vec{\nabla}\cdot\vec{D}=\vec{\nabla}\cdot\varepsilon\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{H}=\frac{\partial\vec{D}}{\partial t}$$

Take the curl of Faraday's law, and then use Ampere's law:

$$\begin{aligned}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) &= -\frac{\partial\vec{\nabla}\times\vec{B}}{\partial t}=-\mu\frac{\partial\vec{\nabla}\times\vec{H}}{\partial t}\\ \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\nabla^2\vec{E} &= -\mu\frac{\partial}{\partial t}\frac{\partial\vec{D}}{\partial t}\end{aligned}$$

Use the first Maxwell equation (the "H" in LIH assures us that spatial derivatives of ε are zero[1]), and we obtain the wave equation with wave speed $v_\phi=1/\sqrt{\mu\varepsilon}$

$$\begin{aligned}\nabla^2\vec{E} &= \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}\\ \frac{\partial^2\vec{E}}{\partial t^2} &= v_\phi^2\nabla^2\vec{E}\end{aligned}$$

A similar derivation gives the same equation for $\vec{B}$. Now let's look at a plane wave solution:

$$\begin{aligned}\vec{E} &= \vec{E}_0\exp\left(i\vec{k}\cdot\vec{x}-i\omega t\right)\\ \vec{B} &= \vec{B}_0\exp\left(i\vec{k}\cdot\vec{x}-i\omega t+\phi\right)\end{aligned}$$

where $\omega/k=v_\phi$, the wave phase speed. By including the phase constant ϕ in the expression for $\vec{B}$ we allow for a possible phase shift bewteen $\vec{E}$ and $\vec{B}$. Inserting these expressions into Maxwell's equations, we have

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= 0\Rightarrow\vec{k}\cdot\vec{E}_0=0 \qquad (1)\\ \vec{\nabla}\cdot\vec{B} &= 0\Rightarrow\vec{k}\cdot\vec{B}_0=0\end{aligned}$$

Thus both $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation. From Faraday's law

$$\vec{k}\times\vec{E}_0\exp\left(i\vec{k}\cdot\vec{x}-i\omega t\right)=\omega\vec{B}_0\exp\left(i\vec{k}\cdot\vec{x}-i\omega t+\phi\right)$$

[1] Here we also assume that ε is independent of t.

Since this relation must be true for all $\vec{x}$ and t, and k is real[2], we have $\phi = 0$ ($\vec{E}$ and $\vec{B}$ oscillate in phase) and

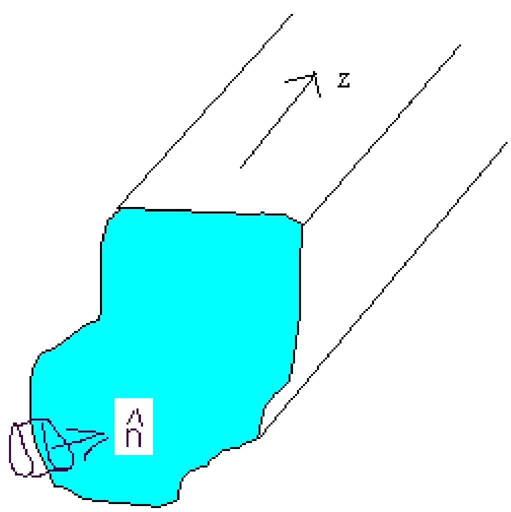
$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 = \frac{1}{v_\phi} \hat{k} \times \vec{E}_0 \tag{2}$$

Thus $\vec{B}$ is also perpendicular to $\vec{E}$, and its magnitude is E/v_ϕ.

If the waves propagate in a vacuum, the derivation goes through in the same way and the only difference is that the wave speed is $c = 1/\sqrt{\mu_0 \varepsilon_0}$. In an LIH medium, $v_\phi = c/n$, where the refractive index $n = \sqrt{\varepsilon\mu/\varepsilon_0\mu_0} \simeq \sqrt{\varepsilon/\varepsilon_0}$.

Electromagnetic fields in a wave guide

A wave guide is a region with a conducting boundary inside which EM waves are caused to propagate. In this confined region, the boundary conditions create constraints on the wave fields. We shall idealize, and assume that the walls are *perfect* conductors. If they are not, currents flowing in the walls lead to energy loss. See Jackson Ch 8 for a discussion of this case, especially sections 1 and 5.



The boundary conditions at the walls of our perfectly conducting guide are (see notes 1 eqns 10,12,13 and 15):

$$\hat{n} \cdot \vec{D} = \Sigma \tag{3}$$

where Σ is the free surface charge density on the wall,

$$\hat{n} \times \vec{E} = 0 \tag{4}$$

[2] k is not always real. We will see later some situations where k is complex. In these cases $\vec{E}$ and $\vec{B}$ may be out of phase.

$$\hat{n} \cdot \vec{B} = 0 \tag{5}$$

and

$$\hat{n} \times \vec{H} = \vec{K} \tag{6}$$

where $\vec{K}$ is the free surface current density. (Note that we have taken $\vec{H} = 0$ inside the conducting material. This is true here because all our fields are *time-dependent*, and then non-zero $\partial\vec{B}/\partial t$ implies non-zero $\vec{E}$, by Faraday's law. Non-zero $\vec{E}$ is not allowed inside a perfect conductor, and so $\vec{H}$ must be zero too.) Since we do not know Σ or $\vec{K}$, equations (4) and (5) will be most useful.

Now we use cylindrical coordinates with $\hat{z}$ along the guide in the direction of wave propagation. The transverse coordinates will be chosen to match the cross-sectional shape of the guide – Cartesian for a rectangular guide and polar for a circular guide. Next we assume that all fields may be written in the form

$$\vec{E} = \vec{E}_0\left(\vec{x}\right) e^{-i\omega t}$$

We are not making any special assumptions about the time variation, because we can always Fourier transform the fields to get combinations of terms of this form. Then Maxwell's equations in the guide take the form:

$$\vec{\nabla} \times \vec{E} = i\omega\vec{B}$$

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

and

$$\vec{\nabla} \times \vec{H} = -i\omega\vec{D}$$

Taking the curl of Faraday's law, and inserting $\vec{\nabla} \times \vec{B}$ from Ampere's law, we get:

$$\begin{aligned} \vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) &= \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2\vec{E} = i\omega\vec{\nabla} \times \vec{B} = i\omega\left(-i\omega\mu\varepsilon\vec{E}\right) \\ \nabla^2\vec{E} &= -\omega^2\mu\varepsilon\vec{E} \end{aligned} \tag{7}$$

This equation is the same as we obtained for free space. Note that ε is usually a function of ω.

Next we look for solutions that take the form of waves propagating in the $z-$direction, that is:

$$\vec{E}_0\left(\vec{x}\right) = \vec{E}_a\left(x, y\right) e^{ikz}$$

The wave equation (7) then becomes:

$$\nabla_{\mathrm{t}}^2\vec{E} + \omega^2\mu\varepsilon\vec{E} - k^2\vec{E} = 0 \tag{8}$$

where ∇_{t}^2 is the Laplacian operator in the two transverse coordinates (x and y, or ρ and ϕ, for example.) Thus equation (8) is an equation for the function $\vec{E}_a$ of the two transverse coordinates.

Since we were able to simplify the equations by separating the function $\vec{E}$ into its dependence on the coordinates along and transverse to the guide, we now try to do the same thing with the components. At first glance, and based on equation (1), you might want to jump to the conclusion that there is no $z-$component of a wave propagating in the $z-$direction, but in general there is. The waves are propagating between conducting boundaries, and we have to allow for the possibility that waves travel at an angle to the guide center-line, and bounce back and forth off the walls as they travel. Since $\vec{E}$ is perpendicular to the wave vector, $\vec{E}$ in such a bouncing wave has a $z-$component. The total electromagnetic disturbance in the guide is a sum of such waves. The sum is a combination of waves that interfere constructively. Thus we take

$$\vec{E}_a = E_z \hat{z} + \vec{E}_{\mathrm{t}}$$

and similarly

$$\vec{B}_a = B_z \hat{z} + \vec{B}_{\mathrm{t}}$$

This decomposition simplifies the boundary conditions, since the normal $\hat{n}$ on the boundary has no $z-$component. Then eqn (5) becomes

$$\hat{n} \cdot \vec{B}_{\mathrm{t}} = 0 \text{ on } S \tag{9}$$

However equation (4) has two components. The transverse component gives

$$E_z = 0 \text{ on } S \tag{10}$$

while the $z-$ component gives

$$\hat{n} \times \vec{E}_{\mathrm{t}} = 0 \quad \text{on} \quad S \tag{11}$$

Next we put these components into Maxwell's equations. The "divergence" equations are scalar equations, so let's start with them:

$$\vec{\nabla} \cdot \vec{D} = 0 = \varepsilon \left(\frac{\partial E_z}{\partial z} + \vec{\nabla}_{\mathrm{t}} \cdot \vec{E}_{\mathrm{t}} \right)$$

and evaluating the $z-$derivative, we get

$$ikE_z + \vec{\nabla}_{\mathrm{t}} \cdot \vec{E}_{\mathrm{t}} = 0 \tag{12}$$

Similarly:

$$ikB_z + \vec{\nabla}_{\mathrm{t}} \cdot \vec{B}_{\mathrm{t}} = 0 \tag{13}$$

We separate the curl equations into transverse and $z-$components. Take the dot product of Faraday's law with $\hat{z}$:

$$\hat{z} \cdot \left(\vec{\nabla} \times \vec{E} \right) = i\omega B_z = \hat{z} \cdot \left(\vec{\nabla}_{\mathrm{t}} \times \vec{E}_{\mathrm{t}} \right) \tag{14}$$

and also the cross product:

$$\hat{z} \times \left(\vec{\nabla} \times \vec{E}\right) = \hat{z} \times i\omega\vec{B}$$

Let's investigate the triple cross product on the left. Since $\hat{z}$ is a constant, we may move it through the ∇ operator in the BAC-CAB rule:

$$\hat{z} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\hat{z} \cdot \vec{E}\right) - \left(\hat{z} \cdot \vec{\nabla}\right)\vec{E} = \hat{z} \times i\omega\vec{B}_{\mathrm{t}}$$

The derivative $\partial E_z/\partial z$ times $\hat{z}$ appears in both terms in the middle, and so cancels, leaving:

$$\vec{\nabla}_{\mathrm{t}} E_z - \frac{\partial}{\partial z}\vec{E}_{\mathrm{t}} = i\omega\hat{z} \times \vec{B}_{\mathrm{t}}$$

and evaluating the $z-$derivative, we get

$$\vec{\nabla}_{\mathrm{t}} E_z - ik\vec{E}_{\mathrm{t}} = i\omega\hat{z} \times \vec{B}_{\mathrm{t}} \tag{15}$$

Similarly, from Ampere's law, we have the transverse component:

$$\vec{\nabla}_{\mathrm{t}} B_z - ik\vec{B}_{\mathrm{t}} = -i\omega\varepsilon\mu\hat{z} \times \vec{E}_{\mathrm{t}} \tag{16}$$

and the $z-$component

$$-i\omega\varepsilon\mu E_z = \hat{z} \cdot \left(\vec{\nabla}_{\mathrm{t}} \times \vec{B}_{\mathrm{t}}\right) \tag{17}$$

Equations (12), (13), (17) and (14) show that the longitudinal components E_z and B_z act as sources of the transverse fields $\vec{B}_{\mathrm{t}}$ and $\vec{E}_{\mathrm{t}}$.

Now we can simplify by looking at the normal modes of the system.

Transverse Electric (TE) (or magnetic) **modes.**

In these modes there is no longitudinal component of $\vec{E}$:

$$E_z \equiv 0 \text{ everywhere}$$

Thus boundary condition (10) is automatically satisfied. The remaining boundary conditions are (9) and (11), and we can find the version that we need by taking the dot product of $\hat{n}$ with equation (16):

$$\hat{n} \cdot \vec{\nabla}_{\mathrm{t}} B_z - ik\hat{n} \cdot \vec{B}_{\mathrm{t}} = -i\omega\varepsilon\mu\hat{n} \cdot \left(\hat{z} \times \vec{E}_{\mathrm{t}}\right)$$

The second term is zero on S (eqn 9), and we rearrange the triple scalar product on the right, leaving:

$$\frac{\partial B_z}{\partial n} = i\omega\varepsilon\mu\hat{z} \cdot \left(\hat{n} \times \vec{E}_{\mathrm{t}}\right) = 0 \text{ on } S \tag{18}$$

where we used the other boundary condition (11) for $\vec{E}$.

Transverse Magnetic (TM) (or electric) **modes.**

In these modes there is no longitudinal component of $\vec{B}$:

$$B_z \equiv 0 \text{ everywhere}$$

Thus the boundary condition (18) is trivially satisfied, and we must impose the remaining condition (10)

$$E_z = 0 \text{ on S.}$$

Since the Maxwell equations are linear, we can form superpositions of these two sets of modes to obtain fields in the guide with non-zero longitudinal components of both $\vec{E}$ and $\vec{B}$. These modes are the result of the constructive interference mentioned above.

Transverse electromagnetic (TEM) modes

In these modes both E_z and B_z are zero everywhere. Then from (12) and (14), $\vec{\nabla}_{\mathrm{t}} \times \vec{E}_{\mathrm{t}}$ and $\vec{\nabla}_{\mathrm{t}} \cdot \vec{E}_{\mathrm{t}}$ are zero everywhere. This means we can express $\vec{E}_{\mathrm{t}}$ as the gradient of a scalar function Φ that satisfies Laplace's equation in two dimensions. The boundary condition (11) becomes

$$\hat{n} \times \vec{\nabla}\Phi = \hat{n} \times \hat{s}\left(\frac{\partial \Phi}{\partial s}\right) = 0 \ \text{ on } S$$

where s is a coordinate parallel to the surface S. Since $\hat{n}$ and $\hat{s}$ are perpendicular, $\hat{n} \times \hat{s}$ is not zero, and so $\Phi =$ constant on S and therefore is constant everywhere inside the volume V, making $\vec{E}_{\mathrm{t}} = 0$. Thus these modes cannot exist inside a hollow guide. They may exist, and in fact become the dominant modes, inside a guide with a separate inner boundary, like a coaxial cable. We will not consider them further here.

Now let's see how the equations simplify for the TE and TM modes..

TM modes

We start by finding an equation for E_z. Since $B_z \equiv 0$, equation (16) simplifies to:

$$\begin{aligned} -ik\vec{B}_{\mathrm{t}} &= -i\omega\varepsilon\mu\hat{z} \times \vec{E}_{\mathrm{t}} \\ \vec{B}_{\mathrm{t}} &= \frac{\omega}{k}\varepsilon\mu\hat{z} \times \vec{E}_{\mathrm{t}} \end{aligned} \tag{19}$$

and we substitute this result back into equation (15).

$$\begin{aligned} -ik\vec{E}_{\mathrm{t}} + \vec{\nabla}_{\mathrm{t}}E_z &= i\omega\hat{z} \times \left(\frac{\omega}{k}\varepsilon\mu\hat{z} \times \vec{E}_{\mathrm{t}}\right) \\ \vec{\nabla}_{\mathrm{t}}E_z &= ik\vec{E}_{\mathrm{t}} - i\frac{\omega^2}{k}\varepsilon\mu\vec{E}_{\mathrm{t}} \\ &= ik\left(1 - \frac{\omega^2}{k^2}\varepsilon\mu\right)\vec{E}_{\mathrm{t}} \end{aligned} \tag{20}$$

and finally we substitute this result for $\vec{E}_t$ back into equation (12):

$$\begin{aligned} ikE_z + \vec{\nabla}_{\mathrm{t}} \cdot \frac{\vec{\nabla}_{\mathrm{t}}E_z}{ik\left(1 - \frac{\omega^2}{k^2}\varepsilon\mu\right)} &= 0 \\ \nabla_{\mathrm{t}}^2 E_z + \left(\omega^2\varepsilon\mu - k^2\right)E_z &= 0 \end{aligned}$$

or

$$\nabla_{\mathrm{t}}^2 E_z + \gamma^2 E_z = 0 \tag{21}$$

with

$$\gamma^2 \equiv \omega^2 \varepsilon\mu - k^2 \tag{22}$$

Equation[3] (21) is the defining differential equation for E_z. Once we have solved for E_z, we can find $\vec{E}_{\mathrm{t}}$ from equation (20) and then $\vec{B}_{\mathrm{t}}$ from equation (19).

TE modes

The argument proceeds similarly. We start with equation (15) with $E_z = 0$, to get:

$$\vec{E}_{\mathrm{t}} = -\frac{\omega}{k}\hat{z} \times \vec{B}_{\mathrm{t}} \tag{23}$$

and substitute into equation (16)

$$\begin{aligned} -ik\vec{B}_{\mathrm{t}} + \vec{\nabla}_{\mathrm{t}} B_z &= -i\omega\varepsilon\mu\hat{z} \times \left(-\frac{\omega}{k}\hat{z} \times \vec{B}_{\mathrm{t}}\right) \\ \vec{\nabla}_{\mathrm{t}} B_z &= i\left(k - \frac{\omega^2\varepsilon\mu}{k}\right)\vec{B}_{\mathrm{t}} \end{aligned} \tag{24}$$

Then finally from equation (13) we have:

$$\begin{aligned} ikB_z + \vec{\nabla}_{\mathrm{t}} \cdot \frac{\vec{\nabla}_{\mathrm{t}} B_z}{i\left(k - \frac{\omega^2\varepsilon\mu}{k}\right)} &= 0 \\ \nabla_{\mathrm{t}}^2 B_z + \gamma^2 B_z &= 0 \end{aligned}$$

which is the same differential equation that we found for E_z in the TM modes. The solutions are different because the boundary conditions are different. Thus the solution for the two modes proceeds as follows:

	TM modes	**TE modes**	
assumed	$B_z \equiv 0$	$E_z \equiv 0$	
differential equation	$\nabla_{\mathrm{t}}^2 E_z + \gamma^2 E_z = 0$	$\nabla_{\mathrm{t}}^2 B_z + \gamma^2 B_z = 0$	This is an eigenvalue/
boundary condition	$E_z = 0$ on S	$\frac{\partial B_z}{\partial n} = 0$ on S	eigenfunction problem.
next find	$\vec{E}_t = \frac{ik}{\gamma^2}\vec{\nabla}_{\mathrm{t}} E_z$	$\vec{B}_t = \frac{ik}{\gamma^2}\vec{\nabla}_{\mathrm{t}} B_z$	
then find	$\vec{B}_{\mathrm{t}} = \frac{\omega}{k}\varepsilon\mu\hat{z} \times \vec{E}_{\mathrm{t}}$	$\vec{E}_{\mathrm{t}} = -\frac{\omega}{k}\hat{z} \times \vec{B}_{\mathrm{t}}$	

The differential equation plus boundary condition is an eigenvalue problem that produces a set of eigenfunctions $E_{z,n}$ (or $B_{z,n}$) and a set of eigenvalues γ_n. The wave number k_n is then determined from equation (22):

$$k_n^2 = \omega^2\varepsilon\mu - \gamma_n^2 \tag{25}$$

[3] This equation is the Helmholtz equation.

Clearly if γ_n is greater than $\omega\sqrt{\varepsilon\mu} = \omega/v$, where v is the wave phase speed in unbounded space, k_n becomes imaginary and the wave does not propagate. There is a cut-off frequency for each mode, given by

$$\omega_n = \gamma_n v$$

If γ_c is the *lowest* eigenvalue for *any* mode, the corresponding frequency ω_c is *the* cutoff frequency for the guide, and waves at lower frequencies cannot propagate in the guide.

A few things to note: the wave number k_n is always less than the free-space value ω/v, and thus the wavelength is always greater than the free-space wavelength. The phase speed in the waveguide is

$$v_\phi = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{1}{\sqrt{1-\omega_n^2/\omega^2}} > \frac{1}{\sqrt{\mu\varepsilon}} = v$$

and we can differentiate eqn (25) to get

$$2\omega\frac{d\omega}{dk}\varepsilon\mu = 2k$$

Then the group speed in the guide is

$$v_g = \frac{d\omega}{dk} = \frac{2k}{2\omega\varepsilon\mu} = \frac{1}{\mu\varepsilon v_\phi} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1-\frac{\omega_n^2}{\omega^2}} < \frac{1}{\sqrt{\mu\varepsilon}} = v$$

Thus information travels more slowly than if the wave were to propagate in free space.

TM modes in a rectangular wave guide

We use Cartesian coordinates with origin at one corner of the guide. Let the guide have dimensions a in the x-direction by b in the $y-$direction. Let the interior be full of air so $\varepsilon/\varepsilon_0 = \mu/\mu_0 \simeq 1$. Then the differential equation for E_z is (21)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) E_z = 0$$

As usual we look for a separated solution, choosing $E_z = X(x)\,Y(y)$ to obtain:

$$\frac{X''}{X} + \frac{Y''}{Y} + \gamma^2 = 0$$

Each term must separately be constant, so we have:

$$\frac{X''}{X} = -\alpha^2$$

$$\frac{Y''}{Y} = -\beta^2$$

and

$$-\alpha^2 - \beta^2 + \gamma^2 = 0$$

The boundary condition is $E_z = 0$ on S, so:

$$X = 0 \text{ at } x = 0 \text{ and } x = a$$

and

$$Y = 0 \text{ at } y = 0 \text{ and } y = b$$

Thus the appropriate solutions are $X = \sin \alpha x$ and $Y = \sin \beta y$ with eigenvalues chosen to fit the second boundary condition in each coordinate:

$$\alpha = \frac{n\pi}{a} \text{ and } \beta = \frac{m\pi}{b}$$

Thus, putting back the dependence on z and t, we have

$$E_z = E_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{ikz-i\omega t} \tag{26}$$

and

$$\gamma_{nm}^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \tag{27}$$

Notice that the lowest possible values of n and m are 1 in each case, since taking m or $n = 0$ would render E_z identically zero. Thus the lowest eigenvalue is

$$\gamma_{11} = \pi \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

and the cutoff frequency for the TM modes is:

$$\omega_{\text{c,TM}} = c\gamma_{11} = c\pi \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{c\pi}{a}\sqrt{1 + \frac{a^2}{b^2}}$$

Jackson solves for the TE modes (pg 361). The eigenvalues are the same, but in this case it is possible for one (but not both) of m and n to be zero, leading to a lower cutoff frequency for the TE modes:

$$\omega_{c,\text{TE}} = \frac{c\pi}{\max(a, b)}$$

This would be *the* cutoff frequency for the guide.

In the TM mode, the remaining fields are (eqns 20 and 26):

$$\begin{aligned} \vec{E}_{\text{t}} &= \frac{ik}{\gamma^2} \vec{\nabla}_{\text{t}} E_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{ikz-i\omega t} \\ &= \frac{ik\pi}{\gamma^2} E_{nm} \left(\frac{n}{a} \hat{x} \cos \frac{n\pi x}{a} \sin \frac{m\pi y}{b} + \frac{m}{b} \hat{y} \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \right) e^{ikz-i\omega t} \end{aligned} \tag{28}$$

and (eqn 19)

$$\begin{aligned} \vec{B}_{\text{t}} &= \frac{\omega}{kc^2} \hat{z} \times \frac{ik}{\gamma^2} E_{nm} \left(\frac{n\pi}{a} \hat{x} \cos \frac{n\pi x}{a} \sin \frac{m\pi y}{b} + \frac{m\pi}{b} \hat{y} \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \right) e^{ikz-i\omega t} \\ &= i \frac{\omega}{c^2\gamma^2} E_{nm} \left(\frac{n\pi}{a} \hat{y} \cos \frac{n\pi x}{a} \sin \frac{m\pi y}{b} - \frac{m\pi}{b} \hat{x} \sin \frac{n\pi x}{a} \cos \frac{m\pi y}{b} \right) e^{ikz-i\omega t} \end{aligned} \tag{29}$$

where (eqns 25 and 27)

$$k = \sqrt{\omega^2 \varepsilon \mu - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}$$

As usual, the physical fields are given by the real part of each mathematical expression, so that, for example, $\vec{E}_{\rm t} \propto \sin(kz - \omega t)$. You should verify that these fields satisfy the boundary conditions at $x = 0, a$ and at $y = 0, b$.

Power

The power transmitted by the waves in the guide is:

$$\vec{S}(t) = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

where here we must take the real, physical fields. Usually we are interested in the time-averaged Poynting flux, which is given by

$$< \vec{S} > \; = \text{Re}\, \frac{1}{2\mu_0} \vec{E} \times \vec{B}^* \tag{30}$$

where the fields on the right are the complex functions we have just found.

Proof of this result:

If $\vec{E} = \vec{E}_0 e^{-i\omega t} = \hat{e} E_0 e^{i\phi_E} e^{-i\omega t}$ and similarly for $\vec{B}$, where E_0 and B_0 are real, then

$$\vec{S} = \frac{1}{\mu_0} \text{Re}\left(\vec{E}\right) \times \text{Re}\left(\vec{B}\right)$$

and the time average is

$$\begin{aligned}
< \quad \vec{S} > \; &= <\hat{e} \times \hat{b} \frac{E_0 B_0}{\mu_0} \cos(\phi_E - \omega t) \cos(\phi_B - \omega t) > \\
&= <\hat{e} \times \hat{b} \frac{E_0 B_0}{\mu_0} (\cos\phi_E \cos\omega t + \sin\phi_E \sin\omega t)(\cos\phi_B \cos\omega t + \sin\phi_B \sin\omega t) > \\
&= \hat{e} \times \hat{b} \frac{E_0 B_0}{\mu_0} \big\langle \big[\cos\phi_E \cos\phi_B \cos^2\omega t + \\
&\qquad (\cos\phi_E \sin\phi_B + \cos\phi_B \sin\phi_E) \cos\omega t \sin\omega t + \sin\phi_E \sin\phi_B \sin^2\omega t\big] \big\rangle \\
&= \hat{e} \times \hat{b} \frac{E_0 B_0}{2\mu_0} (\cos\phi_E \cos\phi_B + \sin\phi_E \sin\phi_B) = \hat{e} \times \hat{b} \frac{E_0 B_0}{2\mu_0} \cos(\phi_E - \phi_B) \\
&= \text{Re}\, \frac{1}{2\mu_0} \vec{E} \times \vec{B}^*
\end{aligned}$$

so the two results are the same. (See also Notes 2 page 11.)

Using the solution (28, 29), the components of $\vec{S}$ in the rectangular guide are:

$$
\begin{aligned}
< \quad S_z > \quad &= \mathrm{Re}\,\frac{1}{2\mu_0}\left(E_x B_y^* - E_y B_x^*\right) \\
= \quad & \mathrm{Re}\,\frac{1}{2\mu_0}\frac{ik}{\gamma^2}E_{nm}\,(-i)\,\frac{\omega}{c^2\gamma^2}E_{nm}\left[\left(\frac{n\pi}{a}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\right)^2 + \left(\frac{m\pi}{b}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\right)^2\right] \\
= \quad & \frac{E_{nm}^2}{2\mu_0}\frac{k}{\gamma^4}\frac{\omega}{c^2}\left[\left(\frac{n\pi}{a}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\right)^2 + \left(\frac{m\pi}{b}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\right)^2\right] \\
= \quad & \frac{E_{nm}^2}{2\mu_0}\frac{\sqrt{\omega^2/c^2-\gamma^2}}{\gamma^4}\frac{\omega}{c^2}\left[\left(\frac{n\pi}{a}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\right)^2 + \left(\frac{m\pi}{b}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\right)^2\right] \\
= \quad & \frac{\varepsilon_0 E_{nm}^2\omega}{2}\frac{\sqrt{\frac{\omega^2}{c^2}-\left(\frac{n\pi}{a}\right)^2-\left(\frac{m\pi}{b}\right)^2}}{\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]^2}\left[\left(\frac{n\pi}{a}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\right)^2 + \left(\frac{m\pi}{b}\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\right)^2\right]
\end{aligned}
$$

Check the dimensions! $< S_z >$ is positive for all values of x and y, showing that power is propagating continuously along the guide in the positive z-direction.

The transverse component S_x is:

$$
\begin{aligned}
< \quad S_x > \quad &= \mathrm{Re}\,\frac{1}{2\mu_0}\left(E_y B_z^* - E_z B_y^*\right) = \mathrm{Re}\,\frac{1}{2\mu_0}\left(-E_z B_y^*\right) \\
= \quad & \mathrm{Re}\,\frac{1}{2\mu_0}\left(-E_0\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,(-i)\,\frac{\omega}{c^2\gamma^2}E_0\frac{n\pi}{a}\cos\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\right) \\
= \quad & 0
\end{aligned}
$$

Because there is no real part, the time averaged power flowing across the guide is zero. Power sloshes back and forth, but there is no net energy transfer.

Fields in a parallel plate wave guide

By simplifying the shape of the guide even more, we can demonstrate how the wave modes are formed by reflection of waves at the guide walls. Let this guide exist in the region $0 \le y \le a$, with infinite extent in x. For the TM modes, the equation to be satisfied is:

$$\left(\nabla_{\mathrm{t}}^2 + \gamma^2\right)E_z = 0$$

with

$$E_z = 0 \text{ at } y = 0 \text{ and } y = a$$

Because the region is infinite in the $x-$direction, the appropriate solution has no $x-$dependence:

$$E_z = E_n \sin\frac{n\pi y}{a}e^{ikz-i\omega t} \tag{31}$$

with

$$\gamma_n = \frac{n\pi}{a}$$

Thus the cutoff frequency for these modes is

$$\omega_{c,\mathrm{TM}} = \gamma_1 c = \frac{\pi c}{a} \tag{32}$$

Then the other components of the fields are:

$$\begin{aligned}\vec{E}_t &= \frac{ik}{\gamma^2}\vec{\nabla}_t E_z = \frac{ik}{\gamma^2}\frac{n\pi}{a}E_n \cos\frac{n\pi y}{a}e^{ikz-i\omega t}\hat{y} \\ &= \frac{ika}{n\pi}E_n \cos\frac{n\pi y}{a}e^{ikz-i\omega t}\hat{y} \qquad (33)\end{aligned}$$

and

$$\begin{aligned}\vec{B}_t &= \frac{\omega}{kc^2}\hat{z}\times\vec{E}_t = -\frac{\omega}{kc^2}\frac{ika}{n\pi}E_n \cos\frac{n\pi y}{a}e^{ikz-i\omega t}\hat{x} \\ &= -\frac{\omega}{c^2}\frac{ia}{n\pi}E_n \cos\frac{n\pi y}{a}e^{ikz-i\omega t}\hat{x} \qquad (34)\end{aligned}$$

Let's look at the electric field first. We write the sine and cosine as combinations of complex exponentials. From (31),

$$E_z = E_n\left(\frac{e^{i\gamma y}-e^{-i\gamma y}}{2i}\right)e^{ikz-i\omega t} = i\frac{\gamma}{\gamma}E_n\left(\frac{-e^{i\gamma y}+e^{-i\gamma y}}{2}\right)e^{ikz-i\omega t}$$

and from (33)

$$E_y = \frac{ik}{\gamma}E_n\left(\frac{e^{i\gamma y}+e^{-i\gamma y}}{2}\right)e^{ikz-i\omega t}$$

Thus we can write the electric field as a superposition

$$\vec{E}_{n,\text{total}} = \frac{1}{2}\left(\vec{E}_{n1}+\vec{E}_{n2}\right)$$

where the two superposed fields are

$$\vec{E}_{n1} = i\left(k\hat{y}-\gamma\hat{z}\right)\frac{E_n}{\gamma}\exp\left(ikz+i\gamma y\right)e^{-i\omega t}$$

and

$$\vec{E}_{n2} = i\left(k\hat{y}+\gamma\hat{z}\right)\frac{E_n}{\gamma}\exp\left(ikz-i\gamma y\right)e^{-i\omega t}$$

Similarly:

$$\vec{B}_{\text{t}} = -i\frac{\omega}{c^2\gamma}E_n\hat{x}\left(\frac{e^{i\gamma y}+e^{-i\gamma y}}{2}\right)e^{ikz-i\omega t} = \frac{1}{2}\left(\vec{B}_{n1}+\vec{B}_{n2}\right)$$

with

$$\vec{B}_{n1,2} = -i\frac{\omega}{\gamma c^2}E_n\hat{x}\exp\left[i\left(kz\pm\gamma y\right)\right]e^{-i\omega t}$$

Now define the four vectors

$$\vec{u}_1 = k\hat{y}-\gamma\hat{z}; \qquad \vec{u}_2 = k\hat{y}+\gamma\hat{z}$$

and

$$\vec{v}_1 = \gamma\hat{y}+k\hat{z}; \qquad \vec{v}_2 = -\gamma\hat{y}+k\hat{z}$$

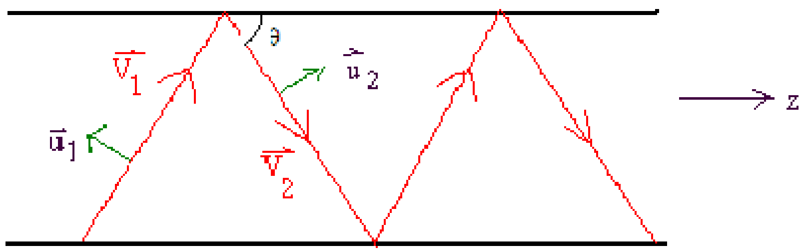
Then for $i = 1, 2$, the vectors are perpendicular:

$$\vec{u}_i \cdot \vec{v}_i = 0$$

and

$$\vec{v}_1 \times \vec{u}_1 = (k\hat{z} + \gamma\hat{y}) \times (k\hat{y} - \gamma\hat{z}) = -\hat{x}\left(\gamma^2 + k^2\right) = -\hat{x}\frac{\omega^2}{c^2} = \vec{v}_2 \times \vec{u}_2$$

where we used equation (25). The vectors are shown in the diagram below.



The two electric field components are then:

$$\vec{E}_{n1} = i\vec{u}_1 \frac{E_n}{\gamma} \exp\left(i\vec{v}_1 \cdot \vec{x}\right) e^{-i\omega t} \quad \text{and} \quad \vec{E}_{n2} = i\vec{u}_2 \frac{E_n}{\gamma} \exp\left(i\vec{v}_2 \cdot \vec{x}\right) e^{-i\omega t}$$

while

$$\frac{\vec{v}_1 \times \vec{E}_{n1}}{\omega} = -i\frac{\omega}{\gamma c^2}\hat{x}E_n \exp\left[i\left(kz + \gamma y\right)\right] = \vec{B}_{n1}$$

consistent with (2) and (34).

Each of these sets of fields ($\vec{E}_{n1}$ and $\vec{B}_{n1}$, $\vec{E}_{n2}$ and $\vec{B}_{n2}$) has the form of a free-space wave propagating in the direction given by the vectors $\vec{v}_1$ and $\vec{v}_2$ respectively and with wave number

$$|\vec{v}_1| = |\vec{v}_2| = \sqrt{\gamma^2 + k^2} = \frac{\omega}{c}$$

. These waves are moving across the guide at an angle given by

$$\tan\theta = \frac{v_y}{v_z} = \pm\frac{\gamma}{k} = \pm\frac{\gamma}{\sqrt{\frac{\omega^2}{c^2} - \gamma^2}}$$

that is, the waves are reflecting off the plates at $y = 0$, a, as shown in the diagram. When the angle θ becomes $\pi/2$, the wave ceases to propagate along the guide, but just bounces back and forth, perpendicular to the walls. This happens when $\tan\theta \to \infty$, or

$$\gamma = \frac{\omega}{c}$$

This gives the cut-off frequency (32) we found before.

See also Lea and Burke pages 1058-1060.