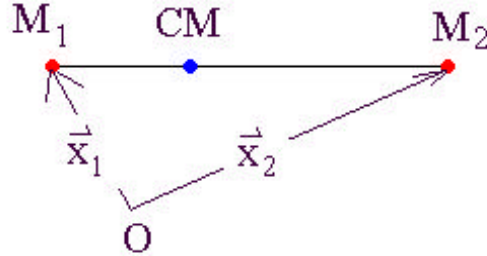


## 1 The two-body problem



Here we discuss the motion of two particles under their mutual gravitational influence. The forces acting on the two bodies are:

$$\vec{F}_i = -\frac{GM_1M_2}{R^3}(\vec{x}_i - \vec{x}_j)$$

where  $R$  is the separation of the masses,  $i, j = 1, 2$  and  $j \neq i$ . Since no external forces act on the system, the center of mass moves at constant velocity, so we may choose our origin at the CM. The position of the CM is at a distance  $s$  from the larger mass (call it  $M_1$ ) where

$$s = \frac{M_2}{M_1 + M_2}R$$

Thus the polar coordinates of the two bodies with respect to this origin are

$$r_1 = s, \quad r_2 = R - s = \frac{M_1}{M_1 + M_2}R = \frac{M_1}{M_2}r_1$$

To simplify, we introduce the reduced mass  $\mu$  :

$$\mu = \frac{M_1M_2}{M_1 + M_2}$$

so that

$$r_1 = \frac{\mu}{M_1}R \quad \text{and} \quad r_2 = \frac{\mu}{M_2}R$$

Now we write the equation of motion for each body:

$$M_i\vec{a}_i = -\frac{GM_1M_2}{R^2}\hat{r}$$

To solve these equations, first note that the torque about the CM is

$$\vec{\tau} = \vec{r} \times \vec{F} = 0$$

and thus the angular momentum of each body about the CM remains constant.

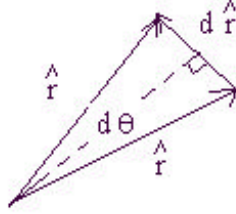
$$M_i \ell = M_i r_i^2 \dot{\theta}_i = L_i = \text{constant}$$

and we can solve for  $\dot{\theta}_i$

$$\frac{L_i}{M_i r_i^2} = \dot{\theta}_i \quad (1)$$

The velocity and acceleration in polar coordinates are

$$\begin{aligned} \vec{v}_i &= \frac{d}{dt} (r_i \hat{r}) = \frac{dr_i}{dt} \hat{r} + r_i \frac{d}{dt} \hat{r} = \frac{dr_i}{dt} \hat{r} + r_i \dot{\theta}_i \frac{d}{d\theta} \hat{r} = \dot{r}_i \hat{r} + r_i \dot{\theta}_i \hat{\theta} \\ \vec{a}_i &= \ddot{r}_i \hat{r} + \dot{r}_i \dot{\theta}_i \hat{\theta} + \dot{r}_i \dot{\theta}_i \hat{\theta} + r_i \ddot{\theta}_i \hat{\theta} - r_i \dot{\theta}_i^2 \hat{r} = (\ddot{r}_i - r_i \dot{\theta}_i^2) \hat{r} + (2\dot{r}_i \dot{\theta}_i + r_i \ddot{\theta}_i) \hat{\theta} \end{aligned} \quad (2)$$



The theta component is zero, which confirms our previous result that  $L$  is constant. The r-component gives

$$\ddot{r}_i - r_i \dot{\theta}_i^2 = -\frac{GM_j}{R^2} = -\frac{GM_j}{r_i^2} \frac{M_j^2}{(M_1 + M_2)^2}$$

Substituting in for  $\dot{\theta}_i$ , we have

$$\ddot{r}_i - r_i \left( \frac{L_i}{M_i r_i^2} \right)^2 = \ddot{r}_i - \frac{L_i^2}{M_i^2 r_i^3} = -\frac{GM_j}{r_i^2} \frac{M_j^2}{(M_1 + M_2)^2} = -\frac{G\mu^2}{r_i^2} \frac{M_j}{M_i^2}$$

Next note that we expect the path to be described by a relation  $r = r(\theta)$ , so we rewrite using

$$\dot{r}_i = \frac{dr}{d\theta} \dot{\theta}_i = \frac{dr}{d\theta} \frac{L_i}{M_i r_i^2}$$

and hence

$$\begin{aligned} \ddot{r}_i &= \frac{d\dot{r}_i}{d\theta} \dot{\theta}_i = \frac{d^2 r}{d\theta^2} \left( \frac{L_i}{M_i r_i^2} \right)^2 - \frac{2}{r_i^3} \left( \frac{dr}{d\theta} \right)^2 \frac{L_i}{M_i} \frac{L_i}{M_i} \frac{1}{r_i^2} \\ &= \frac{d^2 r}{d\theta^2} \left( \frac{L_i}{M_i r_i^2} \right)^2 - \frac{2}{r_i} \left( \frac{dr}{d\theta} \right)^2 \left( \frac{L_i}{M_i r_i^2} \right)^2 \end{aligned}$$

Then the equation of motion becomes:

$$\begin{aligned} \frac{d^2 r}{d\theta^2} \left( \frac{L_i}{M_i} \frac{1}{r_i^2} \right)^2 - \frac{2}{r_i} \left( \frac{dr}{d\theta} \right)^2 \left( \frac{L_i}{M_i} \frac{1}{r_i^2} \right)^2 - r_i \left( \frac{L_i}{M_i r_i^2} \right)^2 &= -\frac{G\mu^2 M_j}{r_i^2 M_i^2} \\ \frac{d^2 r}{d\theta^2} \frac{1}{r_i^2} - \frac{2}{r_i} \left( \frac{dr}{d\theta} \right)^2 \frac{1}{r_i^2} - \frac{1}{r_i} &= -G\mu^2 \frac{M_j}{L_i^2} \end{aligned}$$

Defining  $u_i = 1/r_i$ , we have

$$u_i' = -\frac{1}{r_i^2} r_i' = -u_i^2 r_i'$$

and

$$r_i'' = \frac{2}{u_i^3} (u_i')^2 - \frac{1}{u_i^2} u_i''$$

The equation of motion becomes:

$$\begin{aligned} u_i^2 \left[ \frac{2}{u_i^3} (u_i')^2 - \frac{1}{u_i^2} u_i'' \right] - 2u_i^3 \left( \frac{u_i'}{u_i^2} \right)^2 - u_i &= -G\mu^2 \frac{M_j}{L_i^2} \\ u_i'' &= -u_i + \frac{G\mu^2 M_j}{L_i^2} \end{aligned} \quad (3)$$

Now note the dimensions of the physical quantities are:

$$\left[ \frac{GM}{R^2} \right] = \frac{L}{T^2} \Rightarrow [GM] = \frac{L^3}{T^2}$$

and the angular momentum

$$[L] = M \frac{L^2}{T}$$

and so

$$\left[ \frac{G\mu^2 M_j}{L_i^2} \right] = [G\mu] \left[ \mu \frac{M_j}{L_i^2} \right] = \frac{L^3}{T^2} \cdot M^2 \cdot \left( \frac{T}{ML^2} \right)^2 = \frac{1}{L}$$

as required.

Equation (3) is easily solved:

$$\begin{aligned} \frac{d^2}{d\theta^2} \left( u_i - \frac{G\mu^2 M_j}{L_i^2} \right) &= - \left( u_i - \frac{G\mu^2 M_j}{L_i^2} \right) \\ u_i - \frac{G\mu^2 M_j}{L_i^2} &= A \cos(\theta - \theta_0) \end{aligned}$$

Thus the solution is

$$r_i = \frac{1}{u_i} = \frac{1}{\frac{G\mu^2 M_j}{L_i^2} + A \cos(\theta - \theta_0)} = \frac{L_i^2 / G\mu^2 M_j}{1 + B \cos(\theta - \theta_0)} \quad (4)$$

From this we see that  $r$  is a maximum when  $\theta - \theta_0 = \pi$  and a minimum when  $\theta = \theta_0$ . The path is a conic section with the major axis defined by  $\theta = \theta_0$ . We may choose  $\theta_0 = 0$  for convenience.

Since the center of mass remains fixed, we have  $\theta_2 = \pi + \theta_1$  and

$$\begin{aligned} M_1 r_1 &= M_2 r_2 \\ \frac{M_1}{M_2} \frac{L_1^2 / G \mu^2}{1 + B_1 \cos \theta_1} &= \frac{M_2}{M_1} \frac{L_2^2 / G \mu^2}{1 - B_2 \cos \theta_1} \end{aligned}$$

for all  $\theta_1$ . Thus, writing  $\alpha = \frac{M_1}{M_2}$ , the mass ratio, we have

$$\alpha^2 (1 - B_2 \cos \theta_1) = \frac{L_2^2}{L_1^2} (1 + B_1 \cos \theta_1)$$

This implies that

$$L_2 = \alpha L_1 \tag{5}$$

and  $B_2 = -B_1 = -B$ . We still need to find  $B$ .

The total energy is also conserved, and is most easily found at  $r_{\max}$  or  $r_{\min}$ , since at these points  $\dot{r} = 0$

$$r_{i,\max} = a(1 + e) = \frac{L_i^2}{(1 - B_i) G \mu^2 M_j} = \frac{L_1 L_2}{(1 - B_i) G \mu^2 M_i} \tag{6}$$

and

$$r_{i,\min} = a(1 - e) = \frac{L_i^2}{(1 + B_i) G \mu^2 M_j}$$

Thus

$$a_i = \frac{L_i^2}{(1 - B^2) G \mu^2 M_j} = \frac{L_1 L_2}{(1 - B^2) G \mu^2 M_i} \tag{7}$$

and

$$e = \frac{r_{i,\max} - r_{i,\min}}{2a_i} = B$$

Then when particle 1 is at  $r_{1,\max}$ ,  $r_2$  is at  $r_{2,\max}$

$$\begin{aligned}
E &= -\frac{GM_1M_2}{R} + \frac{1}{2}M_1 \left(r_1\dot{\theta}_1\right)^2 + \frac{1}{2}M_2 \left(r_2\dot{\theta}_2\right)^2 \\
&= -\frac{GM_1M_2}{M_1r_1}\mu + \frac{1}{2}M_1 \left(\frac{L_1}{M_1r_1}\right)^2 + \frac{1}{2}M_2 \left(\frac{L_2}{M_2r_2}\right)^2 \\
&= -\frac{GM_1M_2\mu}{L_1L_2} [(1-B)G\mu^2] + \frac{M_1}{2} \left(\frac{(1-B)G\mu^2}{L_2}\right)^2 + \frac{M_2}{2} \left(\frac{(1-B)G\mu^2}{L_1}\right)^2 \\
&= -\frac{G^2M_1M_2\mu^3}{L_1L_2} (1-B) + \frac{G^2\mu^4}{2} \frac{(1-B)^2M_1}{L_2^2} + \frac{G^2\mu^4}{2} \frac{(1-B)^2M_2}{L_1^2} \\
&= -\frac{G^2M_1M_2\mu^3}{L_1L_2} (1-B) + \frac{G^2\mu^4}{2} \frac{(1-B)^2M_2}{L_1L_2} + \frac{G^2\mu^4}{2} \frac{(1-B)^2M_1}{L_1L_2} \\
&= -\frac{G^2M_1M_2\mu^3}{L_1L_2} (1-B) + \frac{G^2\mu^4}{2} \frac{(1-B)^2}{L_1L_2} (M_1 + M_2) \\
&= \frac{G^2M_1M_2\mu^3}{2L_1L_2} [(1-B)^2 - 2(1-B)] = -\frac{G^2M_1M_2\mu^3}{2L_1L_2} (1-B^2) \tag{8}
\end{aligned}$$

Thus

$$1 - B^2 + \frac{2L_1L_2E}{G^2M_1M_2\mu^3} = 0$$

and so

$$B = e = \sqrt{1 + \frac{2L_1L_2E}{G^2M_1M_2\mu^3}} \tag{9}$$

If the system is bound ( $E < 0$ ) the eccentricity is less than 1 and the path is an ellipse.

In the limit  $M_2 \gg M_1$ ,  $L_2 = \alpha L_1 \ll L_1$  and  $\mu \simeq M_1$ , so

$$e \simeq \sqrt{1 + \frac{2L_1^2E}{G^2M_2^2M_1^3}}$$

which agrees with equation E3.4 in Lea and Burke.

We may summarize our results for the path as:

$$r_{1,2} = \frac{L_1L_2}{(1 \pm e \cos \theta_i) G\mu^2 M_i} \tag{10}$$

$$= \frac{a_{1,2} (1-e)^2}{(1 \pm e \cos \theta_i)} \tag{11}$$

Putting our result for  $e$  back into equation (7), we have

$$\begin{aligned}
a_i &= \frac{L_1L_2}{\left(-\frac{2L_1L_2E}{G^2M_1M_2\mu^3}\right) G\mu^2 M_i} \\
&= \frac{GM_i\mu}{-2E} \tag{12}
\end{aligned}$$

## 1.1 Period

The time needed to travel through an angle  $d\theta$  is  $dt = d\theta/\dot{\theta}$ . Thus the period is

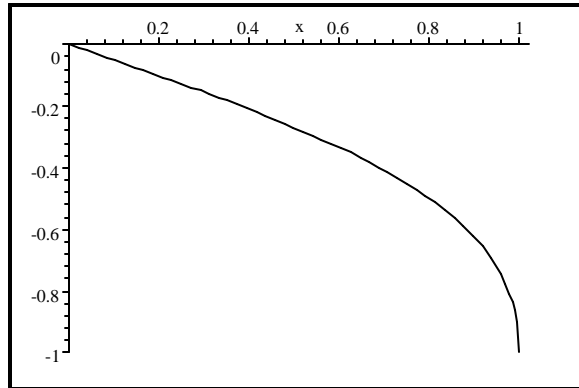
$$\begin{aligned}
 T &= \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} \\
 &= \frac{M_i}{L_i} \int_0^{2\pi} r_i^2 d\theta \\
 &= \frac{M_1}{L_1} \int_0^{2\pi} \left( \frac{a_1(1-e^2)}{1+e\cos\theta} \right)^2 d\theta \\
 &= \frac{M_1}{L_1} a_1^2 (1-e^2)^2 \int_0^{2\pi} \left( \frac{1}{1+e\cos\theta} \right)^2 d\theta
 \end{aligned}$$

We can do the integral by using the unit circle in the complex plane:

$$\begin{aligned}
 I &= \oint_{\text{unit circle}} \frac{1}{\left[1 + \frac{e}{2} \left(z + \frac{1}{z}\right)\right]^2} \frac{dz}{iz} \\
 &= \frac{4}{e^2 i} \oint_{\text{unit circle}} \frac{z}{[2z/e + z^2 + 1]^2} dz
 \end{aligned}$$

The poles are at

$$z_p = -\frac{1}{e} \pm \sqrt{\frac{1}{e^2} - 1}$$



zp versus e

Since  $e < 1$ , both roots are real, but only one is inside the circle. The

residue is

$$\begin{aligned}
\lim_{z \rightarrow z_{p+}} \frac{d}{dz} \frac{(z - z_{p+})^2 z}{[(z - z_{p+})(z - z_{p-})]^2} &= \lim_{z \rightarrow z_{p+}} \frac{d}{dz} \frac{z}{(z - z_{p-})^2} \\
&= \lim_{z \rightarrow z_{p+}} \frac{1}{(z - z_{p-})^2} - \frac{2z}{(z - z_{p-})^3} \\
&= \lim_{z \rightarrow z_{p+}} \frac{z - z_{p-} - 2z}{(z - z_{p-})^3} \\
&= \frac{-(z_{p-} + z_{p+})}{(z_{p+} - z_{p-})^3} \\
&= \frac{-2/e}{2^3 \left(\frac{1}{e^2} - 1\right)^{3/2}} = \frac{e^2}{4(1 - e^2)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
I &= \frac{4}{e^2 i} 2\pi i \frac{e^2}{4(1 - e^2)^{3/2}} \\
&= \frac{2\pi}{(1 - e^2)^{3/2}}
\end{aligned}$$

Thus the period is

$$\begin{aligned}
T &= \frac{M_1}{L_1} a_1^2 (1 - e^2)^2 \frac{2\pi}{(1 - e^2)^{3/2}} \\
&= 2\pi \frac{M_1}{L_1} a^2 (1 - e^2)^{1/2} \\
&= 2\pi \frac{M_1}{L_1} \left( \frac{GM_2\mu}{-2E} \right)^2 \left( -\frac{2L_1 L_2 E}{G^2 M_1 M_2 \mu^3} \right)^{1/2} \\
&= 2\pi \left( \frac{GM_2\mu}{-2E} \right)^{3/2} \left( \frac{L_2 M_1}{L_1 G \mu^2} \right)^{1/2} \\
T &= 2\pi \left( \frac{GM_2\mu}{-2E} \right)^{3/2} \left( \frac{M_1^2}{G \mu^2 M_2} \right)^{1/2} \tag{13}
\end{aligned}$$

$$= 2\pi a_1^{3/2} \left( \frac{M_1^2}{G \mu^2 M_2} \right)^{1/2} \tag{14}$$

In this form we can see the dependence on  $a$ . In the limit  $M_2 \gg M_1$ , we have  $\mu \simeq M_1$  and

$$T \simeq \frac{2\pi}{\sqrt{GM_2}} a^{3/2}$$

and we recover Kepler's Law.

Now let's show the symmetry in the expression for  $T$  :

$$\begin{aligned}
 T &= 2\pi \left( \frac{GM_2\mu}{-2E} \right)^{3/2} \left( \frac{M_1^2}{G\mu^2 M_2} \right)^{1/2} \\
 &= \pi \left( \frac{GM_2 M_1}{-2E} \right) \sqrt{-2\frac{\mu}{E}}
 \end{aligned} \tag{15}$$

## 1.2 Observations

We can observe the period of the system easily, but other quantities are harder. Let's rewrite the system energy in terms of the masses and observables, using equation 12:

$$E = -2\frac{GM_j\mu}{a_i}$$

Thus

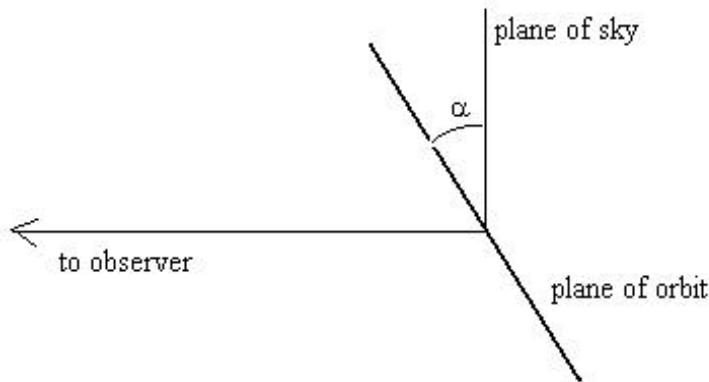
$$\begin{aligned}
 T &= \pi \left( \frac{GM_2 M_1}{GM_j\mu} a_i \right) \sqrt{4\frac{\mu a_i}{GM_j\mu}} \\
 &= 2\pi \frac{M_2 M_1}{M_j\mu} a_i^{3/2} \sqrt{\frac{1}{GM_j}} \\
 &= 2\pi \frac{M_1}{\mu} a_1^{3/2} \sqrt{\frac{1}{GM_2}}
 \end{aligned}$$

If we have a visual binary and we know the distance to the system, we will have a value for  $a_1$ , and so we will be able to evaluate the following function of the masses

$$\frac{M_1 + M_2}{M_2^{3/2}}$$

If we have both  $a_1$  and  $a_2$ , then we can evaluate both masses.

For spectroscopic binaries we have only the velocities of one or both stars along the line of sight.





$$\begin{aligned}
\vec{v}_i \cdot \hat{l} &= \dot{r}_i \hat{r} \cdot \hat{l} + r_i \dot{\theta}_i \hat{\theta}_i \cdot \hat{l} \\
&= \dot{r}_i \sin \theta \sin \alpha + r_i \dot{\theta}_i \cos \theta \sin \alpha
\end{aligned}$$

where

$$\begin{aligned}
\dot{r}_i &= \dot{\theta}_i \frac{d}{d\theta} \frac{L_1 L_2}{(1 \pm e \cos \theta_i) G \mu^2 M_i} \\
&= \frac{L_i}{M_i r_i^2} \frac{L_1 L_2 (\mp 2) e \sin \theta_i}{(1 \pm e \cos \theta_i)^2 G \mu^2 M_i} \\
&= \frac{L_i}{M_i} \frac{L_1 L_2 (\mp 2) e \sin \theta_i}{(1 \pm e \cos \theta_i)^2 G \mu^2 M_i} \frac{[(1 \pm e \cos \theta_i) G \mu^2 M_i]^2}{(L_1 L_2)^2} \\
\dot{r}_{1,2} &= \mp 2 \frac{G \mu^2}{L_{2,1}} e \sin \theta_i
\end{aligned}$$

and

$$r_i \dot{\theta}_i = \frac{L_i}{M_i r_i} = \frac{L_i}{M_i} \frac{(1 \pm e \cos \theta_i) G \mu^2 M_i}{L_1 L_2} = \frac{(1 \pm e \cos \theta_i) G \mu^2}{L_j}$$

Thus

$$\begin{aligned}
\vec{v}_i \cdot \hat{l} &= \left[ \mp 2 \frac{G \mu^2}{L_j} e \sin \theta_i \sin \theta + \frac{(1 \pm e \cos \theta_i) G \mu^2}{L_j} \cos \theta \right] \sin \alpha \\
&= \frac{G \mu^2}{L_j} [\mp 2 e \sin \theta_i \sin \theta + (1 \pm e \cos \theta_i) \cos \theta] \sin \alpha
\end{aligned}$$

If we can observe both stars we can obtain the mass ratio, since

$$\frac{\dot{r}_1}{\dot{r}_2} = \frac{L_1}{L_2} = \frac{M_2}{M_1}$$

If we can observe only one, say #1, then we observe the quantity

$$\frac{\mu^2}{L_2} e \sin \alpha \quad (16)$$

We can usually obtain the eccentricity by fitting a model to the observed curve of velocity versus time ( $\sin \theta_i(t)$ ). We still need to express  $L_2$  in terms of observables. So use equation (8) in equation (15) to get:

$$-\frac{G^2 M_1 M_2 \mu^3}{2 L_1 L_2} (1 - B^2)$$

$$\begin{aligned}
T &= \pi G M_2 M_1 \sqrt{\frac{\mu}{2}} \left[ \frac{2 L_1 L_2}{G^2 M_1 M_2 \mu^3 (1 - e^2)} \right]^{3/2} \\
&= 2\pi \frac{M_2 M_1}{G^2} \frac{1}{\mu^4} \left[ \frac{L_2^2}{M_1^2 (1 - e^2)} \right]^{3/2}
\end{aligned}$$

Thus

$$\begin{aligned} L_2 &= \left[ \frac{G^2 M_1^2 \mu^4 T}{2\pi M_2} (1 - e^2)^{3/2} \right]^{1/3} \\ &= \frac{G^{2/3} M_1 T^{1/3} \mu}{2\pi (M_1 + M_2)^{1/3}} (1 - e^2)^{1/2} \end{aligned}$$

and therefore we can measure (eqn 16)

$$\begin{aligned} \frac{\mu^2}{L_2} e \sin \alpha &= 2\pi \frac{\mu (M_1 + M_2)^{1/3}}{M_1 T^{1/3} (1 - e^2)} e \sin \alpha \\ &= \frac{M_2}{(M_1 + M_2)^{2/3}} \sin \alpha \left[ \frac{2\pi}{T} \frac{e}{\sqrt{1 - e^2}} \right] \end{aligned}$$

which is the same function of the masses that we found for visual binaries.

## 2 The three-body problem

We cannot solve this problem in general, so we look at the special case in which the third body is much less massive than the first two. We'll also specialize to the case in which the first two bodies orbit each other in a circular path, so that the angular velocity  $\dot{\theta}$  is constant. Then we can work in the rotating frame, and the first two bodies are at rest in this frame. In this frame there are two fictitious forces: the centrifugal force

$$\vec{F}_{\text{cent}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

and the Coriolis force

$$\vec{F}_{\text{coriolis}} = -2m\vec{\omega} \times \vec{v}$$

Expanding the triple cross product, the centrifugal force may be written as

$$\vec{F}_{\text{cent}} = m\omega^2 \vec{r} = -\vec{\nabla} \left( -\frac{m\omega^2 r^2}{2} \right)$$

where

$$V_{\text{eff}} = -\frac{m\omega^2 r^2}{2}$$

is the effective potential energy associated with this force. The Coriolis force may be neglected to first approximation if

$$v \ll \omega r$$

and we shall do so for now. Thus we may write the Lagrangian for a particle moving in the rotating frame as

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{GM_1 m}{r_1} + \frac{GM_2 m}{r_2} + \frac{m\omega^2 r^2}{2}$$

where  $r_1$  and  $r_2$  are distances from the two masses and  $r$  is the distance from the center of mass. We may simplify by putting the origin at the CM, and  $x$ -axis along the line joining the two bodies. The distance between the two massive bodies is  $R$ , Then the two bodies are at positions

$$x_1 = -R \frac{M_2}{M_1 + M_2} \quad \text{and} \quad x_2 = R \frac{M_1}{M_1 + M_2}$$

and

$$\mathcal{L} = \frac{1}{2} m v^2 + \frac{G M_1 m}{\sqrt{(x - x_1)^2 + y^2}} + \frac{G M_2 m}{\sqrt{(x - x_2)^2 + y^2}} + \frac{m \omega^2 (x^2 + y^2)}{2}$$

where

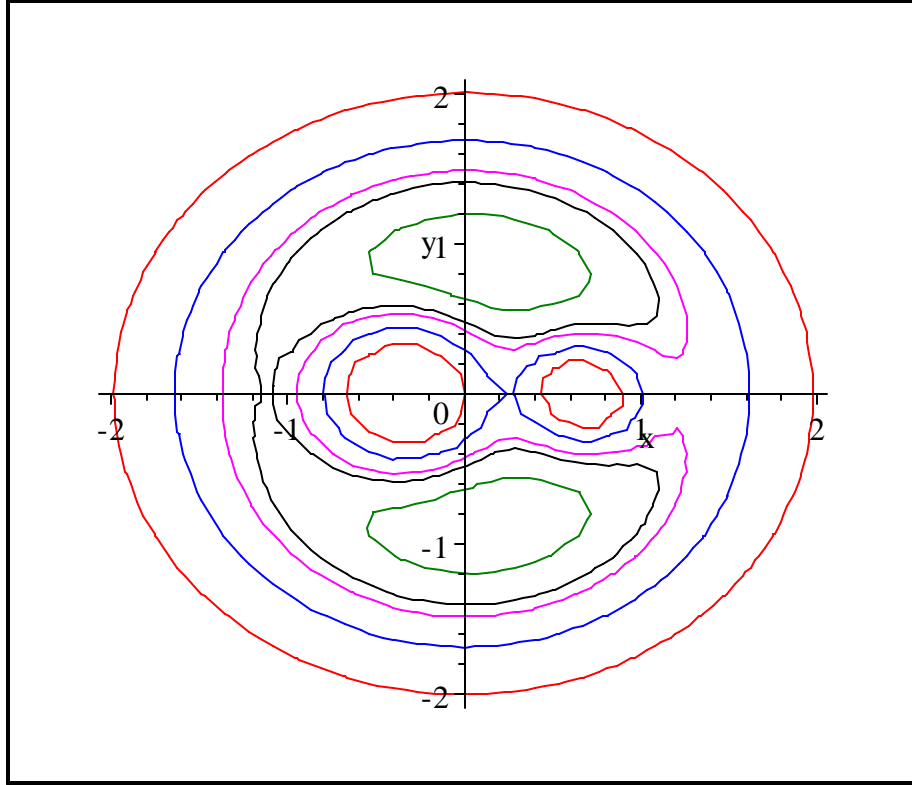
$$\begin{aligned} \omega &= \frac{2\pi}{T} = \frac{2\pi}{2\pi a_1^{3/2} \left( \frac{M^2}{G \mu^2 M_2} \right)^{1/2}} \\ &= \frac{\sqrt{G(M_1 + M_2)}}{R^{3/2}} \end{aligned}$$

Now write all coordinates in dimensionless form: i.e.  $x \rightarrow x/R$ . Then the potential is

$$-\frac{VR}{GmM_2} = \frac{\alpha}{\sqrt{\left(x + \frac{1}{1+\alpha}\right)^2 + y^2}} + \frac{1}{\sqrt{\left(x - \frac{\alpha}{1+\alpha}\right)^2 + y^2}} + \frac{(1+\alpha)(x^2 + y^2)}{2} \quad (17)$$

The small third body will be in equilibrium at points where  $\vec{\nabla}V = 0$

Plot of equipotentials with  $M_1/M_2 = \alpha = 2$



Green 4.5, Black- 5, magenta 5.15, blue 6, red 7.5,

$$-\frac{VR}{GmM_2} = \frac{\alpha}{\sqrt{\left(x + \frac{1}{1+\alpha}\right)^2 + y^2}} + \frac{1}{\sqrt{\left(x - \frac{\alpha}{1+\alpha}\right)^2 + y^2}} + \frac{(1+\alpha)(x^2 + y^2)}{2}$$

$$\frac{\partial V}{\partial x} = -\frac{\alpha \left(x + \frac{1}{1+\alpha}\right)}{\left[\left(x + \frac{1}{1+\alpha}\right)^2 + y^2\right]^{3/2}} - \frac{\left(x - \frac{\alpha}{1+\alpha}\right)}{\left[\left(x - \frac{\alpha}{1+\alpha}\right)^2 + y^2\right]^{3/2}} + x(1+\alpha) \quad (18)$$

and

$$\frac{\partial V}{\partial y} = -\frac{\alpha y}{\left[\left(x + \frac{1}{1+\alpha}\right)^2 + y^2\right]^{3/2}} - \frac{y}{\left[\left(x - \frac{\alpha}{1+\alpha}\right)^2 + y^2\right]^{3/2}} + y(1+\alpha) \quad (19)$$

So  $\partial V/\partial y = 0$  at  $y = 0$  and then we also have  $\partial V/\partial x = 0$  at:

For  $x > \alpha/(1+\alpha)$

$$-\frac{\alpha}{\left(x + \frac{1}{1+\alpha}\right)^2} - \frac{1}{\left(x - \frac{\alpha}{1+\alpha}\right)^2} + x(1+\alpha) = 0$$

For  $-\frac{1}{1+\alpha} < x < \frac{\alpha}{1+\alpha}$ ,

$$-\frac{\alpha}{\left(x + \frac{1}{1+\alpha}\right)^2} + \frac{1}{\left(x - \frac{\alpha}{1+\alpha}\right)^2} + x(1+\alpha) = 0$$

and for  $x < -\frac{1}{1+\alpha}$

$$+\frac{\alpha}{\left(x + \frac{1}{1+\alpha}\right)^2} + \frac{1}{\left(x - \frac{\alpha}{1+\alpha}\right)^2} + x(1+\alpha) = 0$$

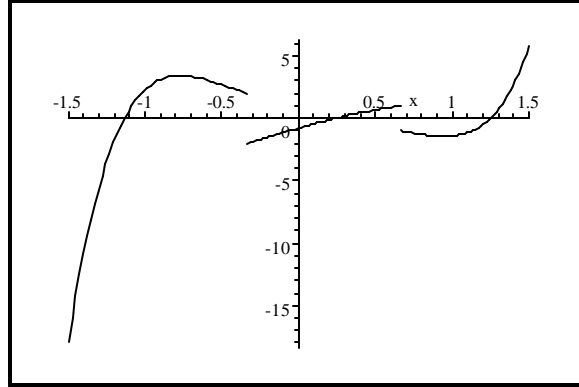
The different values arise from the fact that the numerators in equation (18) are distances and thus must be positive. Thus the points of stability are at:

$$-2\left(x - \frac{2}{3}\right)^2 - \left(x + \frac{1}{3}\right)^2 + 3x\left(x + \frac{1}{3}\right)^2 \left(x - \frac{2}{3}\right)^2 = 0, \text{ Solution is: } \{x = 1.249\}$$

$$-2\left(x - \frac{2}{3}\right)^2 + \left(x + \frac{1}{3}\right)^2 + 3x\left(x + \frac{1}{3}\right)^2 \left(x - \frac{2}{3}\right)^2 = 0, \text{ Solution is: } \{x = 0.23742\}$$

$$+2\left(x - \frac{2}{3}\right)^2 + \left(x + \frac{1}{3}\right)^2 + 3x\left(x + \frac{1}{3}\right)^2 \left(x - \frac{2}{3}\right)^2 = 0, \text{ Solution is: } \{x = -1.1364\}$$

These points agree with the values read from the plot of the equipotential surfaces.



For  $\alpha \ll 1$ :

$$-\frac{\alpha}{\left(x + \frac{1}{1+\alpha}\right)^2} - \frac{1}{\left(x - \frac{\alpha}{1+\alpha}\right)^2} + x(1+\alpha) = 0$$

$$-\frac{\alpha}{(x+1-\alpha)^2} - \frac{1}{(x-\alpha)^2} + x(1+\alpha) = 0$$

$$-\frac{\alpha}{(x+1)^2} - \frac{1}{x^2} \left(1 + \frac{2\alpha}{x}\right) + x + x\alpha = 0$$

To zeroth order in  $\alpha$  the solution is  $x = 1$ . Then let  $x = 1 + \varepsilon$ , where  $\varepsilon$  is of

order  $\alpha$ .

$$\begin{aligned}
-\frac{\alpha}{(2+\varepsilon)^2} - \frac{1}{(1+\varepsilon)^2} \left(1 + 2\frac{\alpha}{1+\varepsilon}\right) + 1 + \varepsilon + \alpha &= 0 \\
-\frac{\alpha}{4}(1-\varepsilon) - (1-2\varepsilon)(1+2\alpha) + 1 + \varepsilon + \alpha &= 0 \\
-\frac{5}{4}\alpha + 3\varepsilon &= 0 \\
\varepsilon &= \frac{5}{12}\alpha
\end{aligned}$$

Thus  $L_2$  is at  $1 + \frac{5}{12}\varepsilon$ . To find the next point, we change one sign.

$$-\frac{\alpha}{\left(x + \frac{1}{1+\alpha}\right)^2} + \frac{1}{\left(x - \frac{\alpha}{1+\alpha}\right)^2} + x(1+\alpha) = 0$$

The zeroth order solution ( $\alpha = 0$ ) is  $x = -1$ . Again we look for a solution  $x = -1 + \varepsilon$ . We have to be very careful with the first term, so let's go to second order in  $\alpha$  this time.

$$\begin{aligned}
-\alpha(x(1+\alpha) - \alpha)^2 + (x(1+\alpha) + 1)^2 + \frac{x}{(1+\alpha)}(x(1+\alpha) + 1)^2(x(1+\alpha) - \alpha)^2 &= 0 \\
-\alpha(1+\alpha)(x + \alpha(x-1))^2 + (1+\alpha)(x+1+x\alpha)^2 + x(x+1+x\alpha)^2(x + \alpha(x-1))^2 &= 0
\end{aligned}$$

$$\begin{aligned}
0 &= -\alpha(1+\alpha)(x^2 + 2\alpha x(x-1)) + (1+\alpha)\left((x+1)^2 + 2x(x+1)\alpha + x^2\alpha^2\right) \\
&\quad + x\left((x+1)^2 + 2x(x+1)\alpha + \alpha^2 x^2\right)\left(x^2 + 2\alpha x(x-1) + (x-1)^2\alpha^2\right) \\
&= -\alpha(1+\alpha)x^2 - 2\alpha^2 x(x-1) + (1+\alpha)(x+1)^2 + 2x(x+1)\alpha(1+\alpha) + x^2\alpha^2 \\
&\quad + 4x^4\alpha + 6x^5\alpha^2 + 4x^5\alpha + x^3 + 2x^4 + x^5 - 2\alpha x^3 + x\alpha^2 - 6x^3\alpha^2 - 2x^2\alpha
\end{aligned}$$

$$0 = +2x^4 + x^5 + x^3 + x^2 + 2x + 1 + 4x\alpha + 4x^5\alpha - 2\alpha x^3 + 4x^4\alpha^2 + 6x^5\alpha^2 - 6x^3\alpha^2 + 5x\alpha^2$$

Now set  $x = -1 + \varepsilon$

$$\begin{aligned}
0 &= 1 + 2(\varepsilon - 1) + (\varepsilon - 1)^2 + (\varepsilon - 1)^3 + 2(\varepsilon - 1)^4 + (\varepsilon - 1)^5 - 2\alpha\left((\varepsilon - 1)^3 - 2(\varepsilon - 1) - 2(\varepsilon - 1)^5\right) \\
&\quad + (\varepsilon - 1)\alpha^2\left(4(\varepsilon - 1)^3 + 6(\varepsilon - 1)^4 - 6(\varepsilon - 1)^2 + 5\right) \\
&= 3\varepsilon^3 - 3\varepsilon^4 + \varepsilon^5 - 2\alpha(3 - 9\varepsilon + 17\varepsilon^2 - 19\varepsilon^3 + 10\varepsilon^4 - 2\varepsilon^5) + \alpha^2(-1 + \varepsilon - 18\varepsilon^2 + 38\varepsilon^3 - 26\varepsilon^4 + 6\varepsilon^5)
\end{aligned}$$

As expected the terms in  $\varepsilon^0\alpha^0$  have vanished. Gathering terms in powers of  $\varepsilon$

$$0 = -\alpha^2 - 6\alpha + \alpha\varepsilon(18 + \alpha) + \dots$$

Thus

$$\begin{aligned}\varepsilon &= \frac{\alpha + 6}{18 + \alpha} = \frac{1}{18}(\alpha + 6) \left(1 - \frac{\alpha}{18}\right) \\ &= \frac{1}{3} + \frac{1}{27}\alpha\end{aligned}$$

and

$$x = -1 + \frac{1}{3} + \frac{1}{27}\alpha = -\frac{2}{3} + \frac{\alpha}{27}$$

Now for the last one:

$$\frac{\alpha}{\left(x + \frac{1}{1+\alpha}\right)^2} + \frac{1}{\left(x - \frac{\alpha}{1+\alpha}\right)^2} + x(1 + \alpha) = 0$$

As before, the zeroth order solution is  $-1$  and we have to be especially careful finding the correction. The only difference is the sign of the leading term. Thus

$$\begin{aligned}0 &= +\alpha(1 + \alpha)(x^2 + 2\alpha x(x - 1)) + (1 + \alpha)\left((x + 1)^2 + 2x(x + 1)\alpha + x^2\alpha^2\right) \\ &\quad + x\left((x + 1)^2 + 2x(x + 1)\alpha + \alpha^2 x^2\right)\left(x^2 + 2\alpha x(x - 1) + (x - 1)^2\alpha^2\right)\end{aligned}$$

and

These points are on either side of the small mass.  
The other values of  $y$  are given by

$$-\frac{\alpha}{\left[\left(x + \frac{1}{1+\alpha}\right)^2 + y^2\right]^{3/2}} - \frac{1}{\left[\left(x - \frac{\alpha}{1+\alpha}\right)^2 + y^2\right]^{3/2}} + (1 + \alpha) = 0$$

Together with the  $x$ -relation

$$-\frac{\alpha\left(x + \frac{1}{1+\alpha}\right)}{\left[\left(x + \frac{1}{1+\alpha}\right)^2 + y^2\right]^{3/2}} - \frac{\left(x - \frac{\alpha}{1+\alpha}\right)}{\left[\left(x - \frac{\alpha}{1+\alpha}\right)^2 + y^2\right]^{3/2}} + x(1 + \alpha) = 0$$

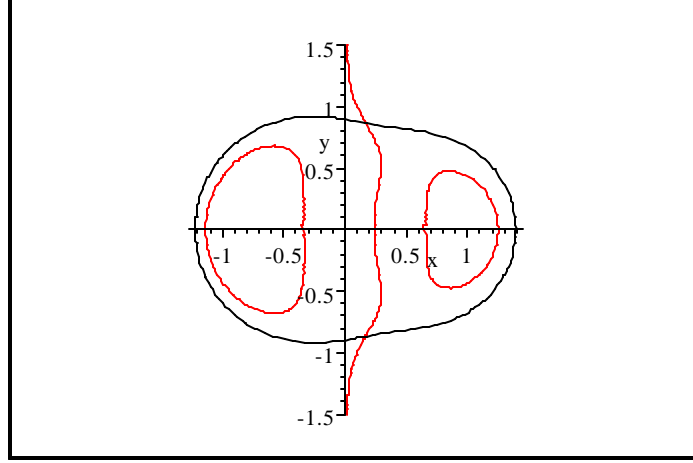
Let's look again at the special cases  $\alpha = 2$  and  $\alpha \ll 1$ . First  $\alpha = 2$ :

$$-\frac{2}{\left[\left(x + \frac{1}{3}\right)^2 + y^2\right]^{3/2}} - \frac{1}{\left[\left(x - \frac{2}{3}\right)^2 + y^2\right]^{3/2}} + 3 = 0$$

(black curve) and

$$-\frac{2\left(x + \frac{1}{3}\right)}{\left[\left(x + \frac{1}{3}\right)^2 + y^2\right]^{3/2}} - \frac{\left(x - \frac{2}{3}\right)}{\left[\left(x - \frac{2}{3}\right)^2 + y^2\right]^{3/2}} + 3x = 0$$

(red curve)



There are two intersections, symmetrically located about the  $x$ -axis.

The points that we have found are the five Lagrange points.

For  $\alpha \ll 1$ : The two equations are

$$-\frac{\alpha}{\left[(x+1-\alpha)^2 + y^2\right]^{3/2}} - \frac{1}{\left[(x-\alpha)^2 + y^2\right]^{3/2}} + (1+\alpha) = 0$$

and

$$-\frac{\alpha x}{\left[(x+1-\alpha)^2 + y^2\right]^{3/2}} - \frac{x-\alpha}{\left[(x-\alpha)^2 + y^2\right]^{3/2}} + x(1+\alpha) = 0$$

Keeping first order in  $\alpha$ :

$$-\frac{\alpha}{\left[(x+1)^2 - 2\alpha(x+1) + y^2\right]^{3/2}} - \frac{1}{\left[x^2 - 2\alpha x + y^2\right]^{3/2}} + (1+\alpha) = 0$$

$$-\frac{\alpha}{\left[(x+1)^2 + y^2\right]^{3/2}} - \frac{1}{(x^2 + y^2)^{3/2}} \left[1 + \frac{3\alpha x}{x^2 + y^2}\right] + (1+\alpha) = 0$$

$$1 - \frac{1}{(x^2 + y^2)^{3/2}} + \alpha \left\{ 1 - \frac{1}{\left[(x+1)^2 + y^2\right]^{3/2}} - \frac{3x}{(x^2 + y^2)^{5/2}} \right\} = \quad (20)$$



The zeroth order result is  $r = 1$ . For  $\alpha \ll 1$ , the second equation becomes:

$$\begin{aligned}
& -\frac{\alpha x}{[(x+1)^2 + y^2]^{3/2}} - \frac{x - \alpha}{[x^2 + y^2]^{3/2}} \left[ 1 + \frac{3\alpha x}{x^2 + y^2} \right] + x(1 + \alpha) = 0 \\
& -\frac{\alpha x}{[(x+1)^2 + y^2]^{3/2}} - \frac{x - \alpha}{[x^2 + y^2]^{3/2}} - \frac{3\alpha x^2}{(x^2 + y^2)^{5/2}} + x(1 + \alpha) = 0 \\
& x \left( 1 - \frac{1}{[x^2 + y^2]^{3/2}} \right) + \alpha \left( x - \frac{x}{[(x+1)^2 + y^2]^{3/2}} + \frac{1}{[x^2 + y^2]^{3/2}} - \frac{3x^2}{(x^2 + y^2)^{5/2}} \right) = (21)
\end{aligned}$$

which gives  $r = 1$  or  $x = 0$  as the zeroth order solution. Then let  $r = 1 + \varepsilon$  in equation (20) to get

$$\begin{aligned}
1 - \frac{1}{(1 + \varepsilon)^3} + \alpha \left( 1 - \frac{1}{[(1 + \varepsilon)^2 + 2(1 + \varepsilon)\cos\theta + 1]^{3/2}} - 3\frac{(1 + \varepsilon)\cos\theta}{(1 + \varepsilon)^5} \right) &= 0 \\
3\varepsilon + \alpha \left( 1 - \frac{1}{[1 + 2\cos\theta]^{3/2}} - 3\cos\theta \right) &= 0
\end{aligned}$$

and so

$$\varepsilon = -\frac{\alpha}{3} \left( 1 - \frac{1}{[2 + 2\cos\theta]^{3/2}} - 3\cos\theta \right) \quad (22)$$

Now we expand equation (21) to first order in  $\alpha$  and  $\varepsilon$ :

$$3\varepsilon \cos\theta + \alpha \left( \cos\theta - \frac{\cos\theta}{2^{3/2}[1 + \cos\theta]^{3/2}} + 1 - 3\cos^2\theta \right) = 0$$

and then insert relation (22) for  $\varepsilon$ :

$$\begin{aligned}
-\alpha \cos\theta \left( 1 - \frac{1}{[2 + 2\cos\theta]^{3/2}} - 3\cos\theta \right) + \alpha \left( \cos\theta - \frac{\cos\theta}{[2 + 2\cos\theta]^{3/2}} + 1 - 3\cos^2\theta \right) &= 0 \\
-\frac{\cos\theta}{\sqrt{2}(1 + \cos\theta)^{3/2}} + 1 &= 0
\end{aligned}$$

$$\cos\theta = \sqrt{2}(1 + \cos\theta)^{3/2}$$

$$\cos^2\theta = 2(1 + \cos\theta)^3 = 2 + 6\cos\theta + 6\cos^2\theta + 2\cos^3\theta$$

Thus  $\cos\theta$  satisfies the equation

$$\begin{aligned}
0 &= 2 + 6\cos\theta + 5\cos^2\theta + 2\cos^3\theta \\
&= (2\cos\theta + 1)(\cos^2\theta + 2\cos\theta + 2)
\end{aligned}$$

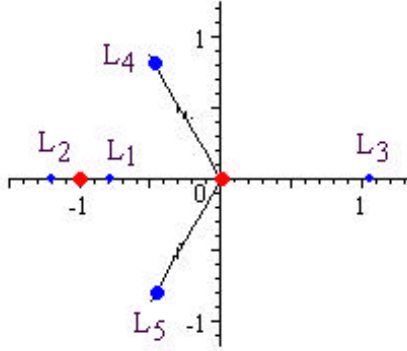
The real solution is  $\cos \theta = -1/2$ , or  $\theta = 120^\circ$ . Finally we put this result into equation (22) to get

$$\begin{aligned}\varepsilon &= -\frac{\alpha}{3} \left( 1 - \frac{1}{[2-1]^{3/2}} + \frac{3}{2} \right) \\ &= -\frac{\alpha}{2}\end{aligned}$$

Thus the last two Lagrange points are at

$$r = 1 - \frac{\alpha}{2}, \quad \theta = \pm 120^\circ$$

These points are  $60^\circ$  ahead of and behind the smaller mass.



## 2.1 Stability of the Lagrange points.

We can see from the figure on page 12 that  $L_4$  and  $L_5$  are potential minima, and thus stable, while  $L_1$ ,  $L_2$  and  $L_3$  are saddle points and thus unstable.

To analyze the stability, we need to include the Coriolis force. We will need to include a velocity dependent potential. Then Lagrange's equations take the form

$$\frac{d}{dt} \frac{\partial \mathcal{L}(\vec{x}, \vec{v})}{\partial v_i} - \frac{\partial \mathcal{L}(\vec{x}, \vec{v})}{\partial x_i} = 0$$

Let's recall what happens in E&M. The magnetic force is

$$\vec{F} = q\vec{v} \times \vec{B}$$

where

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

so

$$\begin{aligned}\vec{F} &= q\vec{v} \times (\vec{\nabla} \times \vec{A}) \\ &= q \left[ v_j \frac{\partial}{\partial x_i} A_j + (\vec{v} \cdot \vec{\nabla}) \vec{A} \right]\end{aligned}\tag{23}$$

We define the velocity-dependent potential as

$$-q\vec{A} \cdot \vec{v}$$

so that

$$\frac{\partial V}{\partial v_i} = -qA_i$$

and

$$\frac{d}{dt} \left( -\frac{\partial V}{\partial v_i} \right) = \left( \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j} \right) qA_i = v_j \frac{\partial}{\partial x_j} qA_i$$

then

$$\begin{aligned} \frac{d}{dt} \left( -\frac{\partial V}{\partial v_i} \right) + \frac{\partial V}{\partial x_i} &= v_j \frac{\partial}{\partial x_j} qA_i + q \frac{\partial}{\partial x_i} v_j A_j \\ &= v_j \frac{\partial}{\partial x_j} qA_i + qv_j \frac{\partial}{\partial x_i} A_j \end{aligned}$$

in agreement with equation (23).

Now in our case the Coriolis force is

$$\vec{F}_{\text{cor}} = -m\vec{\omega} \times \vec{v} = m\vec{v} \times \vec{\omega}$$

where

$$\vec{\omega} = \vec{\nabla} \times \left( \frac{\omega r}{2} \hat{\theta} \right) = \vec{\nabla} \times \vec{A}$$

(cf Lea problem 1.5). Thus our potential is now

$$-\frac{VR}{GmM_2} = \frac{\alpha}{\sqrt{\left(x + \frac{1}{1+\alpha}\right)^2 + y^2}} + \frac{1}{\sqrt{\left(x - \frac{\alpha}{1+\alpha}\right)^2 + y^2}} + \frac{(1+\alpha)(x^2 + y^2)}{2} - \frac{\sqrt{1+\alpha}r}{2\sqrt{GM_2}} \hat{\theta} \cdot \vec{v}$$