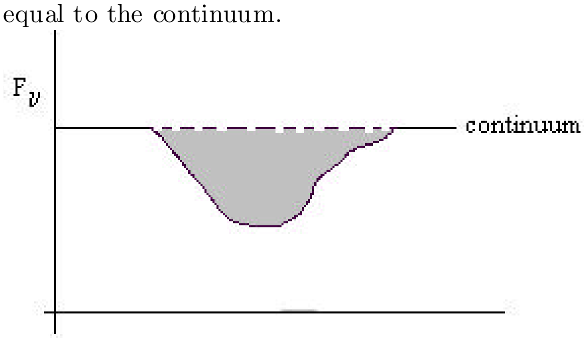
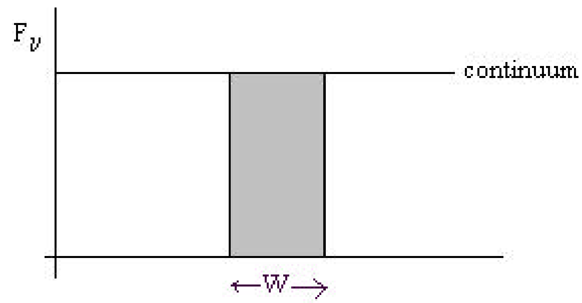
Curve of Growth

The curve of growth describes how the line strength increases as the optical depth increases. An appropriate measure of line strength is the *equivalent width:* the area of a rectangle with the same area as the line profile and height equal to the continuum.





We begin with a purely absorbing, simple slab model with radiation incident on one side. Then from the solution to the transfer equation, we have the intensity emerging:

$$I_\nu = I_0 e^{-\tau_n}$$

where the optical depth is

$$\tau_n = \int n_x \kappa_0 \phi_v \; d\ell \simeq n_x L \kappa_0 \phi_v$$

where n_x is the density of atoms of element x, L is the thickness of the slab, κ_0 is the absorption coefficent per atom at line center, and ϕ_ν is the line profile function. The factor $\int n_x \; d\ell \equiv N_x \simeq n_x L$ is called the column density The area under the curve $I_\nu(\nu)$ is $\int I_\nu \; d\nu$ and the area of the line is $\int (I_0 - I_\nu) \; d\nu$ The equivalent width of the line is defined to be

$$W = \int \left(1 - \frac{I_\nu}{I_0}\right) \; d\nu = \int_0^\infty \left(1 - e^{-N_x \kappa_0 \phi_v}\right) d\nu$$

Of course this is an idealization. In practice the limits are frequencies far enough from the line center that we are clearly in the continuum.

In the classical oscillator model,

$$\kappa_0 = \frac{\pi e^2}{mc} f \left(1 - e^{-h\nu_0/kT}\right)$$

where the oscillator strength f takes quantum effects into account and ν_0 is the frequency at line center.

The line profile function depends on the broadening mechanisms in effect.

1. We always have Doppler broadening due to the velocity distribution of the atoms. In LTE,

$$\phi_{\nu,\text{doppler}} = \frac{1}{\sqrt{\pi}\Delta\nu_D} \exp\left[-\left(\frac{\nu - \nu_0}{\Delta\nu_D}\right)^2\right]$$

and the line breadth factor is

$$\Delta\nu_D = \frac{\nu_0}{c}\left[\frac{2kT}{M} + (\text{turbulent velocity})^2\right]^{1/2}$$

2. Natural broadening is due to radiation reaction (my notes, R&L pg 287, Shore pg 197-8)

$$\phi_\nu = \frac{\gamma}{4\pi^2\left(\nu-\nu_0\right)^2 + \left(\gamma/2\right)^2}$$

   This is sometimes called the Lorentz profile. We can include the effect of collisions by adding a term γ_C to the natural broadening damping factor γ.

0.1 Weak lines

$$N_x\kappa_0\phi_0 \ll 1$$

Then

$$\begin{aligned} W &= \int_0^\infty \left(1-e^{-N_x\kappa_0\phi_v}\right)d\nu \simeq \int_0^\infty N_x\kappa_0\phi_v \; d\nu \\ &= N_x\kappa_0\int_0^\infty \phi_v \; d\nu = N_x\kappa_0 \end{aligned}$$

The result is independent of the shape of ϕ_ν and thus of the type of broadening. The equivalent width is linearly proportional to the column density.

0.2 Strong lines

$$\tau_0 = N_x\kappa_0\phi_0 \gg 1$$

Notice that $N_x\kappa_0\phi_\nu$ cannot be $\gg 1$ everywhere, since $\phi_\nu \to 0$ far from the line center. Let the frequency where $\tau_\nu \simeq 1$ be $\nu_0 \pm \Delta$. Then

$$W \simeq \int_0^{\nu_0-\Delta}\left(1-e^{-N_x\kappa_0\phi_v}\right)d\nu + \int_{\nu_0-\Delta}^{\nu_0+\Delta}\left(1-e^{-N_x\kappa_0\phi_v}\right)d\nu + \int_{\nu_0+\Delta}^{\infty}\left(1-e^{-N_x\kappa_0\phi_v}\right)d\nu$$

In the middle factor, $e^{-\tau_\nu}$ is $\ll 1$. The two other terms are each of order

$$N_x\kappa_0\int_{\nu_0+\Delta}^{\infty}\phi_v \; d\nu \ll N_x\kappa_0$$

and we neglect them. Thus

$$W \simeq 2\Delta$$

0.2.1 Moderately strong lines

In these lines $N_x \kappa_0 \phi_\nu$ becomes <1 while still in the Doppler core. So

$$\begin{aligned} N_x \kappa_0 \phi_v (\nu_0 + \Delta) &= 1 \\ N_x \kappa_0 \frac{1}{\sqrt{\pi}\Delta\nu_D} \exp\left[-\left(\frac{\Delta}{\Delta\nu_D}\right)^2\right] &= 1 \\ \Delta &= \Delta\nu_D \sqrt{\ln\left(\frac{N_x\kappa_0}{\sqrt{\pi}\Delta\nu_D}\right)} \end{aligned}$$

where the square root is $\lesssim 1$ for self consistency, and it follows that

$$W \simeq 2\Delta\nu_D \sqrt{\ln\left(\frac{N_x\kappa_0}{\sqrt{\pi}\Delta\nu_D}\right)}$$

Thus the equivalent width increases logarithmically with N.

Check that the terms we neglected are small:

$$\frac{W_{\text{neglected}}}{W_{\text{kept}}} \ll \frac{N_x\kappa_0}{2\Delta\nu_D\sqrt{\ln\left(\frac{N_x\kappa_0}{\sqrt{\pi}\Delta\nu_D}\right)}} \lesssim \frac{e}{2} \sim 1$$

So the approximations are consistent.

0.2.2 Very strong lines

Here $N_x\kappa_0\phi_\nu$ does not become <1 until the wings of the line. Thus the transition occurs at Δ where

$$\frac{N_x\kappa_0\gamma}{4\pi^2\Delta^2 + (\gamma/2)^2} = 1$$

Uusally $\gamma \ll 2$, so we may approximate:

$$\Delta = \frac{\sqrt{N_x\kappa_0\gamma}}{2\pi}$$

and then

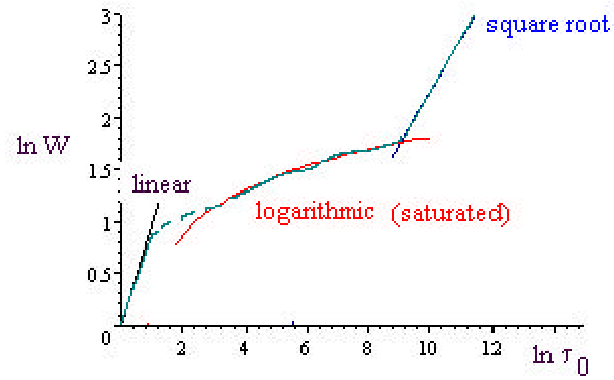
$$W \simeq \frac{\sqrt{N_x\kappa_0\gamma}}{\pi}$$

and increases as the square root of N.

Check as in the previous case. The neglected term is:

$$\begin{aligned} \int_\Delta^\infty \frac{N_x\kappa_0\gamma}{4\pi^2 x^2 + (\gamma/2)^2} dx &= \frac{N_x\kappa_0\gamma}{4\pi^2} \int_\Delta^\infty \frac{dx}{x^2 + \gamma^2/16\pi^2} \\ &= \frac{N_x\kappa_0\gamma}{4\pi^2}\frac{4\pi}{\gamma}\left(\frac{\pi}{2} - \tan^{-1}\frac{4\pi\Delta}{\gamma}\right) \\ &\sim \frac{N_x\kappa_0\gamma}{4\pi^2\Delta} \sim \frac{\sqrt{N_x\kappa_0\gamma}}{2\pi} \end{aligned}$$

Thus we get the curve of growth:
let $x = \frac{N_x \kappa_0}{\Delta\nu_D}$ and $y = W/\Delta\nu_D$then



The transition to the saturated regime occurs sooner (lower W) for lower temperatures because $\Delta\nu_D$ is smaller. The damped (square root) portion is higher is the damping paramater γ is larger. The line profile becomes squarer in the saturated portion abd devlops broad wings in the damping portion. (See figure on next page.)

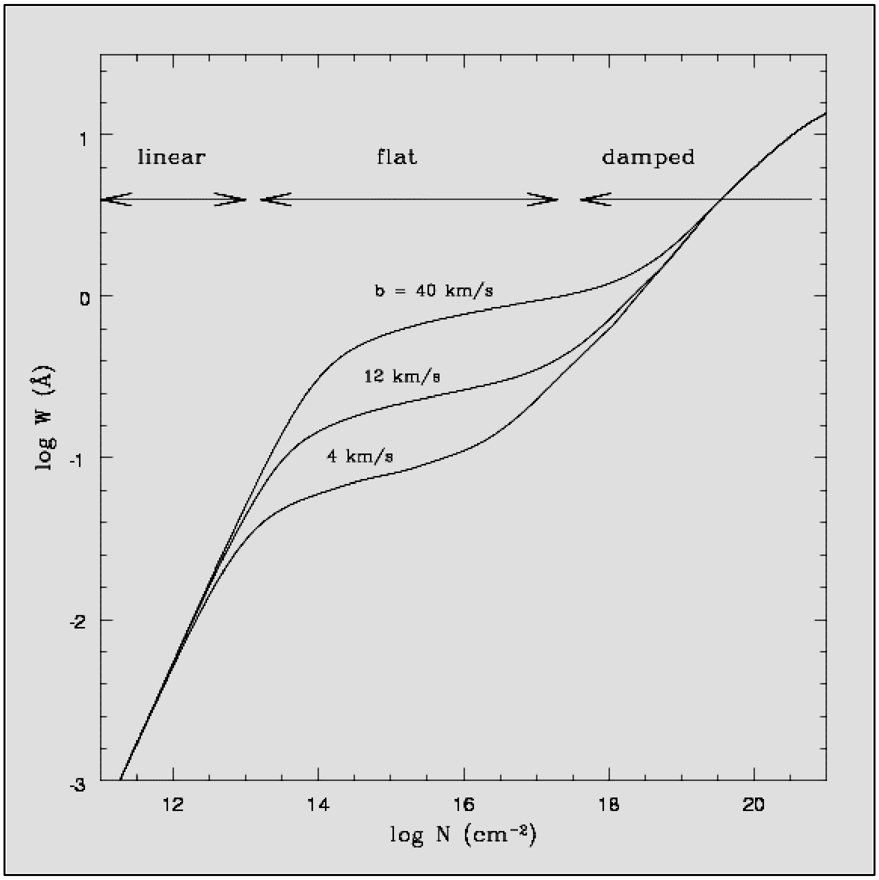
To use the curve of growth one compares the theoretical curve with an emipirical curve contructed from the data. The column density and temperature can then be deduced. One can use line multiplets of the same atom, or different trancitions (but beware here if γ is dominated by natural rather than collisional broadening.)

Here's some data for hydrogen from

http://nedwww.ipac.caltech.edu/level5/Charlton/Charlton1_1.html

b is the Doppler parameter $\Delta\nu_D$ in velocity units: $b = \sqrt{2kT/m}$ and m is the atomic mass.

Here's another one from http://www-int.stsci.edu/~leonidas/voss03/


linear
flat
damped
b = 40 km/s
12 km/s
4 km/s
log W (Å)
log N (cm⁻²)
1
0
-1
-2
-3
12
14
16
18
20

Q0420-388 z=3.12