The Debye Length

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Let's put a point charge q into a plasma. For simplicity, we'll make it an hydrogen plasma, with $n_e = n_i$ everywhere in the unperturbed state (before q is introduced.) (Remember– the plasma must be neutral in the unperturbed state.) Let's also suppose q is positive. When the charge is introduced, it attracts electrons toward itself and repels ions. However, because the ion mass is much greater than the electron mass, the electrons accelerate much faster. Thus to first order we may assume that the electrons move until equilibrium is established. The ions remain as they were. At this point we have:

- Electrostatic potential $\phi(r)$. (With only a single charge, we have spherical symmetry and thus ϕ is a function of r but not the angles θ and ϕ.)
- Undisturbed ions of density n_0 (constant)
- Electrons with density $n_e(r) = n_0 + n_1(r)$.
- The distribution function for the electrons is modified because the electron energy is affected by the electrostatic potential. The Boltzmann factor $e^{-E/kT}$ becomes

$$\exp\left\{-\frac{\frac{1}{2}mv^2 - e\phi}{kT}\right\} = \exp\left(-\frac{mv^2}{2kT}\right)\exp\left(\frac{e\phi}{kT}\right)$$

  Thus we can write the electron density as

$$n_e(r) = \int f(\vec{r}, \vec{v})\, d^3\vec{v} = n_0 \exp\left(\frac{e\phi}{kT}\right)$$

  where, as expected, $n_e \longrightarrow n_0$ at infinity, where $\phi \to 0$.

Now we proceed by writing Poisson's equation:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= -\nabla^2 \phi = \frac{\rho_q}{\varepsilon_0} = e\frac{n_i - n_e}{\varepsilon_0} + q\delta\left(\vec{r}\right) \\
-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) &= \frac{e}{\varepsilon_0} n_0 \left(1 - e^{e\phi/kT}\right) + q\delta\left(\vec{r}\right) \qquad (1)
\end{aligned}$$

where we used the fact that ϕ has spherical symmetry. Equation (1) is a messy, non-linear differential equation for ϕ. But we can simplify by noting that the potential is expected to get small very quickly as we move away from the point charge – much faster than the $1/r$ dependence we'd get in vacuum, because of the shielding by the plasma electrons. Thus we expand the exponential to get:

$$\begin{aligned}
-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) &= \frac{e}{\varepsilon_0} n_0 \left(1 - \left[1 + \frac{e\phi}{kT} + \frac{1}{2}\left(\frac{e\phi}{kT}\right)^2 + \cdots\right]\right) \\
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) &= \frac{en_0}{\varepsilon_0}\frac{e\phi}{kT} + \cdots
\end{aligned}$$

or, to first order:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = \frac{e^2 n_0}{\varepsilon_0 kT}\phi \qquad (2)$$

Looking at the physical dimensions in this equation, we see that the quantity

$$\frac{\varepsilon_0 kT}{e^2 n_0}$$

must have the dimensions of length squared, and the length so defined is expected to play an important role in the solution. Let's give it a name

$$\lambda = \sqrt{\frac{\varepsilon_0 kT}{e^2 n_0}} \qquad (3)$$

Equation (2) is expected to give the correct solution for the potential away from the point charge. At the position of the point charge the delta function in equation (1) dominates everything else. Thus very near the charge we expect the solution to be of the usual form:

$$\phi\left(\text{ near } r = 0\right) = \frac{q}{4\pi\varepsilon_0 r} \qquad (4)$$

The easiest way to proceed is to write the differential operator in its alternate form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)$$

(you should convince yourself of this identity) so that the differential equation becomes:

$$\frac{\partial^2}{\partial r^2}(r\phi) = \frac{r\phi}{\lambda^2},$$

an exponential equation for the function $r\phi$, with solution

$$r\phi = A\exp\left(-\frac{r}{\lambda}\right)$$

and thus

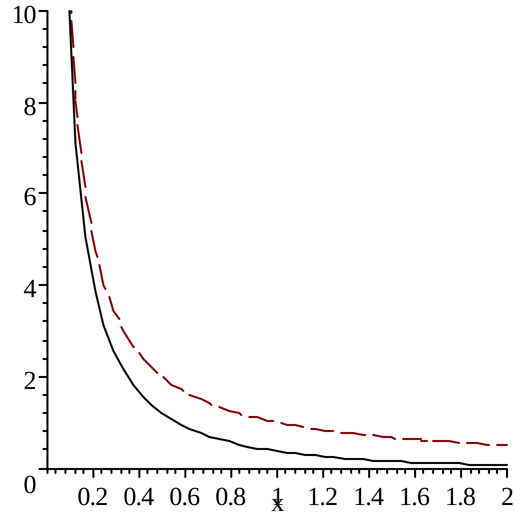
$$\phi(r) = \frac{A}{r}e^{-r/\lambda}$$

To obtain consistency with the solution (4) near the origin, we must take $A = \frac{q}{4\pi\varepsilon_0}$, giving the final solution for the potential:

$$\phi(r) = \frac{q}{4\pi\varepsilon_0 r}e^{-r/\lambda} \tag{5}$$

The potential drops to zero exponentially. The plasma beyond a distance λ is essentially *shielded* from the effects of the charge. We'll now give this length its usual name λ_D– the Debye length.

The plot shows the scaled potential ϕ/ϕ_0 versus the scaled length ρ/λ_D, where $\phi_0 = q/(4\pi\varepsilon_0\lambda_D)$ is the (vacuum) point charge potential at a distance of λ_D from the charge.



Numerically the *Debye length* for the plasma (equation 3) is,

$$\lambda_D \simeq \sqrt{\frac{kT \text{ in eV}}{n \text{ in cm}^{-3}}} 740 \text{ cm}$$

We should not expect to find appreciable charge imbalance or electric fields existing in a plasma over regions greater in extent than λ_D. Thus for a system to be *quasi-neutral* the typical dimension of the system must be much larger than λ_D. The Debye length decreases as n increases because there are more charges per unit volume to provide the shielding, and increases with T because the particles have greater energy and so can remain, on average, at greater distance from the charge.

In plasmas, Poisson's equation is not generally very useful for finding electric fields on scales greater than λ_D. Instead, we often find $\vec{E}$ by other means, and then use Poisson's equation to find the resulting small charge imbalance.

A second condition for a good plasma, one that can successfully be analyzed by computing averages over the particles, and one that exhibits collective behavior, is that a volume of order λ_D^3 contains a very large number of particles. The condition is usually stated in terms of the number of particles in a debye sphere, $N_D = \frac{4}{3}\pi n_0 \lambda_D^3$. We require

$$\ln N_D \gg 1$$

Numerically,

$$N_D = \frac{(kT \ \text{ in eV})^{3/2}}{\sqrt{n \text{ in cm}^{-3}}} 1.7 \times 10^9$$

showing once again that good plasmas have high temperatures and/or low densities.

Example. The x-ray emitting gas in clusters of galaxies has $n \sim 10^{-3}$ cm^{-3} and $T \sim 10$ keV. Thus for this plasma

$$\lambda_D \sim \sqrt{\frac{10^4}{10^{-3}}} 740 \text{ cm } = 2.3 \times 10^6 \text{ cm}$$

which should be compared with a typical scale length L in this system of about 10^{22} cm. Thus $\lambda_D \ll L$. The number of particles in a Debye sphere is

$$N_D = \frac{(10^4)^{3/2}}{\sqrt{10^{-3}}} 1.7 \times 10^9 = 5.4 \times 10^{16}$$

and thus

$$\ln N_D = \ln 5.4 \times 10^{16} = 38.5$$

which is indeed much greater than 1. Thus this system is a good plasma according to these two criteria. We will develop some additional criteria shortly.