Diffusion

S.M.Lea

January 07

Now we are going to take a look at some of the consequences of collisions between the plasma particles. We'll start by looking at collisions between the charged particles and neutral particles (neutral atoms) in a partially-ionized plasma, because these collisions are easier to analyze than those between charged particles.

1 Effect of collisions on the equation of motion

Each collision is described by the *collision cross section* σ. Classically, σ is just the area of the particle we are colliding with. Electric and quantum effects provide corrections to the classical result.

We focus attention on a slab of area A and thickness dx. This slab contains

$$dN = n_0 dV = n_0 A dx$$

target particles, where n_0 is the number density of these target particles (neutral atoms, for now). The total area of the slab blocked to incoming charged particles is then

$$A_{\text{blocked}} = \sigma dN = \sigma n_0 A dx$$

Here we have assumed that dx is so small that target particles do not get in front of each other, thus reducing the total blocked area. The flux of outgoing particles is reduced because of the ones that get blocked. Then if the flux of incoming particles is F_{in} particles/m^2, the fraction that is blocked is just equal to A_{blocked}/A, and so

$$F_{\text{out}} = F_{\text{in}} (1 - \sigma n_0 dx)$$

and thus

$$dF = F_{\text{out}} - F_{\text{in}} = -F_{\text{in}} \sigma n_0 dx$$

and hence we form the differential equation

$$\frac{dF}{dx} = -\sigma n_0 F$$

which is easily integrated if n_0 and σ are independent of x :

$$F = F_0 \exp(-\sigma n_0 x) \tag{1}$$

The quantity

$$\lambda = \frac{1}{\sigma n_0} \tag{2}$$

is the *mean free path* between collisions.

The *mean time* between collisions is

$$\tau = \frac{\lambda}{v} = \frac{1}{\sigma n_0 v}$$

and the *collision frequency* is

$$\nu_{\text{coll}} = \frac{1}{\tau} = n_0 \sigma v \tag{3}$$

(Strictly, to obtain a more accurate result, we should average over the distribution function.) Now we can amend the fluid momentum equation to account for the momentum removed by the collisions.

$$\left.\frac{d\vec{p}}{dt}\right|_{\text{coll}} = -\frac{\vec{p}}{\tau} = -mn\nu_{\text{coll}}\vec{u}$$

Thus the momentum equation (plasfluid notes equation 11) becomes:

$$mn\frac{d\vec{u}}{dt} = mn\left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right] = qn\left(\vec{E} + \vec{u}\times\vec{B}\right) - \vec{\nabla}P - mn\nu_{\text{coll}}\vec{u} \tag{4}$$

Next let's specialize to the case $\vec{B} = 0$ (unmagnetized plasma), and look for a steady state solution, as we did before when looking at particle drifts. Then $\frac{\partial \vec{u}}{\partial t} = 0,$ and if the drift speed is small (very subsonic) we can also neglect the non-linear term $\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}.$ We are left with:

$$qn\vec{E} - \vec{\nabla}P - mn\nu_{\text{coll}}\vec{u} = 0$$

which we can solve to get

$$\vec{u} = \frac{qn\vec{E} - \vec{\nabla}P}{mn\nu_{\text{coll}}} = \frac{q}{m\nu_{\text{coll}}}\vec{E} - \frac{k_B T}{m\nu_{\text{coll}}}\frac{\vec{\nabla}n}{n}$$

The quantity

$$\frac{q}{m\nu_{\text{coll}}} \equiv \pm\mu \tag{5}$$

is called the *mobility,* (the $\pm$ being the sign of the charge q) and

$$\frac{k_B T}{m\nu_{\text{coll}}} \equiv D \tag{6}$$

is the *diffusion coefficient.* Notice that the diffusion coefficient has dinmensions of $L^2/T.$ The flux of particles is then:

$$\vec{F} = n\vec{u} = \pm\mu n\vec{E} - D\vec{\nabla}n \tag{7}$$

When the electric field is zero, the flux is proportional to the density gradient:

$$\vec{F} = -D\vec{\nabla}n \tag{8}$$

a result known as *Fick's Law.*

In a plasma it is unusual for the electric field to be exactly zero. Because the electrons have a higher thermal speed than the ions (at the same temperature), the diffusion coefficient is larger for the electrons than for the ions. The electrons diffuse down the density gradient faster, leading to an imbalance between the electron and ion densities, that in turn creates electric field. The electric field acts to slow the electrons and speed up the ions until both diffuse at the same rate. The resulting diffusion is called **ambipolar diffusion,** because both charges (*ambi* means both) diffuse together.

To determine the electric field and the diffusion rate, set the ion flux equal to the electron flux:

$$\vec{F}_i = \mu_i n\vec{E} - D_i\vec{\nabla}n = \vec{F}_e = -\mu_e n\vec{E} - D_e\vec{\nabla}n$$

(Here we are using the plasma approximation: $n_i \simeq n_e$ but $\vec{E} \neq 0$.) We can solve this relation for $\vec{E}$:

$$\vec{E} = \frac{(D_i - D_e)}{\mu_i + \mu_e}\frac{\vec{\nabla}n}{n} \tag{9}$$

and then the flux (of either species) is:

$$\begin{aligned}\vec{F} &= \mu_i n\frac{(D_i - D_e)}{\mu_i + \mu_e}\frac{\vec{\nabla}n}{n} - D_i\vec{\nabla}n \\ &= \left[\frac{\mu_i D_i - \mu_i D_e - \mu_i D_i - \mu_e D_i}{\mu_i + \mu_e}\right]\vec{\nabla}n \\ &= -\left(\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}\right)\vec{\nabla}n \end{aligned} \tag{10}$$

which is Fick's law with a new diffusion coefficient:

$$\begin{aligned} D_a &= \frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e} = \frac{\frac{e}{M\nu_{\text{coll}}}\frac{k_B T_e}{m\nu_{\text{coll}}} + \frac{e}{m\nu_{\text{coll}}}\frac{k_B T_i}{M\nu_{\text{coll}}}}{\frac{e}{M\nu_{\text{coll}}} + \frac{e}{m\nu_{\text{coll}}}} \\ &= \frac{k_B\left(T_e + T_i\right)}{(m + M)\nu_{\text{coll}}} \end{aligned} \tag{11}$$

The diffusion is dominated by the higher mass ions. Note the symmetry between properties of electrons and ions in this expressin for D_a.

2 Solutions to the diffusion equation

Inserting our expression for the particle flux into the continuity equation, we have:

$$\begin{aligned}\frac{\partial n}{\partial t} + \vec{\nabla}\cdot\vec{F} &= 0 \\ \frac{\partial n}{\partial t} - D\nabla^2 n &= 0\end{aligned} \tag{12}$$

We may amend this equation to include a source (or sink) of particles, S:

$$\frac{\partial n}{\partial t} - D\nabla^2 n = S$$

In a steady state, $\frac{\partial}{\partial t} \equiv 0$, and so the equation simplifies:

$$-D\nabla^2 n = S \tag{13}$$

(In the absence of a source, there is no steady state solution.)

Now let's consider what the source might be.

2.1 Ionization

The rate of collisional ionization (production of new ions) is proportional to the density of plasma particles (and also to the density of neutrals)

$$S = Zn$$

Then the diffusion equation (13) takes the form

$$\begin{aligned} -D\nabla^2 n &= Zn \\ \nabla^2 n &= -\frac{Z}{D}n \end{aligned}$$

In a one-dimensional problem, the solutions to this equation are sines and cosines:

$$n = A\cos\left(\sqrt{\frac{Z}{D}}x\right) + B\sin\left(\sqrt{\frac{Z}{D}}x\right)$$

The wavelength is fixed ($\lambda = 2\pi\sqrt{D/Z}$) and so we must fit the boundary conditions by adjusting the constants A and B. If $n = 0$ at $x = d$ then

$$A\cos\left(\sqrt{\frac{Z}{D}}d\right) + B\sin\left(\sqrt{\frac{Z}{D}}d\right) = 0$$

determines

$$A = -B\tan\left(\sqrt{\frac{Z}{D}}d\right)$$

and so

$$n = N\left\{\sin\left(\sqrt{\frac{Z}{D}}x\right) - \tan\left(\sqrt{\frac{Z}{D}}d\right)\cos\left(\sqrt{\frac{Z}{D}}x\right)\right\}$$

In cylindrical coordinates with azimuthal symmetry, the diffusion equation takes the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) + \frac{Z}{D}n = 0$$

Changing variables to $w = \sqrt{Z/D}r$, we have:

$$n'' + \frac{1}{w}n' + n = 0$$

This is Bessel's equation of order zero (Lea eqn 8.69 and 8.70), with solution

$$n = J_0\left(\sqrt{\frac{Z}{D}}r\right)$$

and $n \to 0$ as $r \to \infty$.

With a source term along the cylinder axis, we have:

$$\nabla^2 n = -\frac{S_0}{D}\delta(r)$$

which has the solution:

$$n = n_0 \ln a/r$$

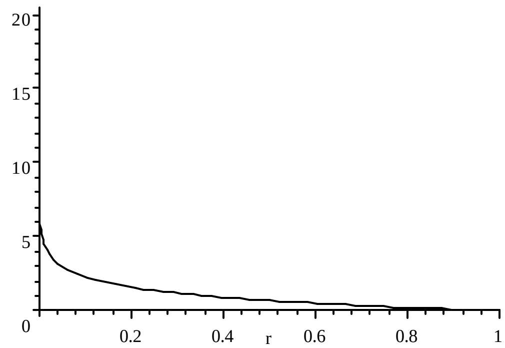
This solution gives $n = 0$ at the edge of the cylinder, $r = a$, and since (see Lea eqn 6.27)

$$\nabla^2 \ln\frac{1}{r} = -2\pi\delta(r)$$

then

$$n_0 = \frac{S_0}{2\pi D}$$

The solution is divergent at $r = 0$ because we chose the source to be a delta-function.



n versus r with a line source along the axis

2.2 Recombination

When an electron and an ion recombine to form a neutral atom, the plasma density decreases. We have a *sink*. Since the two particles must collide in order to recombine, the recombination rate is proportional to n^2 :

$$\frac{dn}{dt} = -\alpha n^2$$

and our diffusion equation takes the form

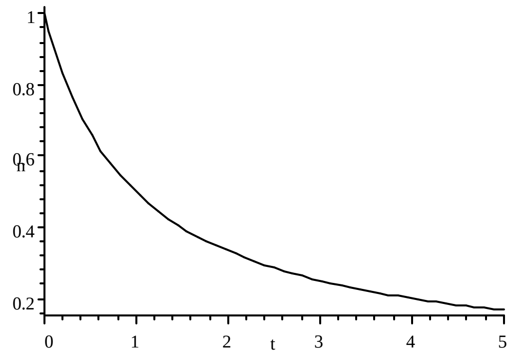
$$\frac{\partial n}{\partial t} - D\nabla^2 n = -\alpha n^2$$

In the absence of gradients, $\left(\nabla^2 n = 0\right)$ or when n is large, ($n \gg \frac{D}{\alpha L^2}$, where L is a relevant length scale) the equation becomes

$$\frac{\partial n}{\partial t} = -\alpha n^2$$

with solution

$$\begin{aligned} \frac{1}{n} &= \frac{1}{n_0} + \alpha t \\ n &= \frac{n_0}{1 + \alpha t n_0} \end{aligned} \tag{14}$$



Recombination: n/n_0 versus $\alpha n_0 t$

This solution will hold until n drops to a low enough value that the diffusion term becomes important. This happens when

$$\frac{n_0}{1+\alpha t n_0} \sim \frac{D}{\alpha L^2}$$

or

$$t \sim \frac{1}{\alpha n_0}\left(\frac{\alpha L^2 n_0}{D} - 1\right) = \frac{L^2}{D} - \frac{1}{\alpha n_0}$$

2.3 Time-dependent solutions

To obtain the full solution, we try separation of variables. In one dimension, look for a solution of the form

$$n(x,t) = X(x)\,T(t)$$

Then equation (12) takes the form

$$X\frac{dT}{dt} - DT\frac{d^2X}{dx^2} = 0$$

Dividing through by XT, we get:

$$\frac{1}{T}\frac{dT}{dt} - D\frac{1}{X}\frac{d^2X}{dx^2} = 0$$

The first terms depends only on t, and the second only on x, so we require that each term be separately equal to a constant.

$$\frac{dT}{dt} = -kT$$

We chose a negative separation constant because we expect n to decrease with time in the absence of sources. Thus

$$T = e^{-kt}$$

Then the equation for X is

$$\begin{aligned} -k - D\frac{1}{X}\frac{d^2X}{dx^2} &= 0 \\ \frac{d^2X}{dx^2} &= -\frac{k}{D}X \end{aligned}$$

and the solutions are sines and cosines

$$X = A\cos\sqrt{\frac{k}{D}}x + B\sin\sqrt{\frac{k}{D}}x$$

the complete solution is of the form

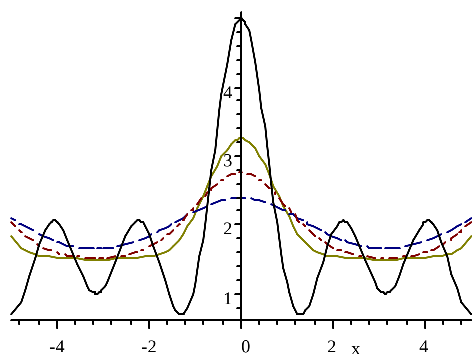
$$n = n_0 \exp\left(-\frac{t}{\tau}\right)\left[\cos\frac{x}{\sqrt{D\tau}} + B\sin\frac{x}{\sqrt{D\tau}}\right]$$

where we wrote $k = 1/\tau$.

Writing this in terms of the wavelength λ, with $\lambda = 2\pi\sqrt{D\tau}$, we get:

$$n = n_0 \exp\left(-\frac{(2\pi)^2 Dt}{\lambda^2}\right)\left[\cos\frac{2\pi x}{\lambda} + B\sin\frac{2\pi x}{\lambda}\right]$$

Here we can see that the shortest wavelengths decay the fastest. Thus the density distribution will be *smoothed* in time.



distribution at $t/t_0 = 0$, .25, .5 and 1, where $t_0 = \lambda_0^2/(2\pi)^2 D$

In a cylindrical system, the equation (12) is:

$$\frac{\partial n}{\partial t} - D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) = 0$$

and again we separate, with $n = R(r)\,T(t)$:

$$R\frac{dT}{dt} - DT\frac{1}{r}\frac{d}{dr}(rR') = 0$$

and dividing by RT we have

$$\frac{1}{T}\frac{dT}{dt} - \frac{D}{R}\left(R'' - \frac{R'}{r}\right) = 0$$

The solution for T is the same as before, and then the R equation is

$$D\left(R'' - \frac{R'}{r}\right) + \frac{1}{\tau}R = 0$$

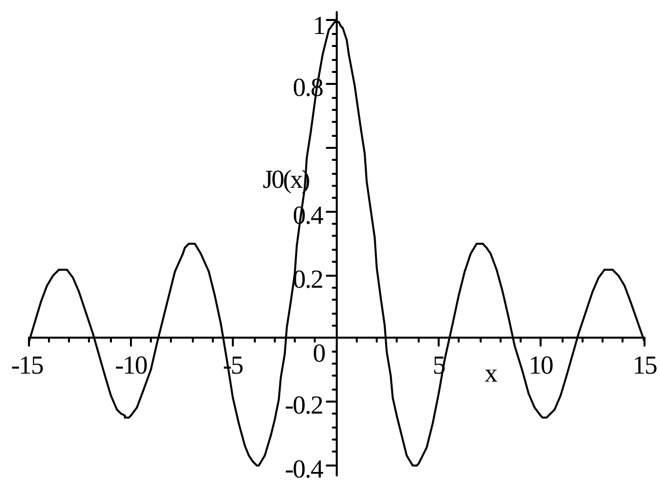
which is again the Bessel equation of order zero. The solution is:

$$R = J_0\left(\frac{r}{\sqrt{D\tau}}\right)$$

and

$$n\left(r,t\right) = n_0 J_0\left(\frac{r}{\sqrt{D\tau}}\right)e^{-t/\tau}$$

The function J_0 is like a sine function with decreasing amplitude and variable wavelength.



The first zero occurs at argument 2.4048. Thus n will become zero at

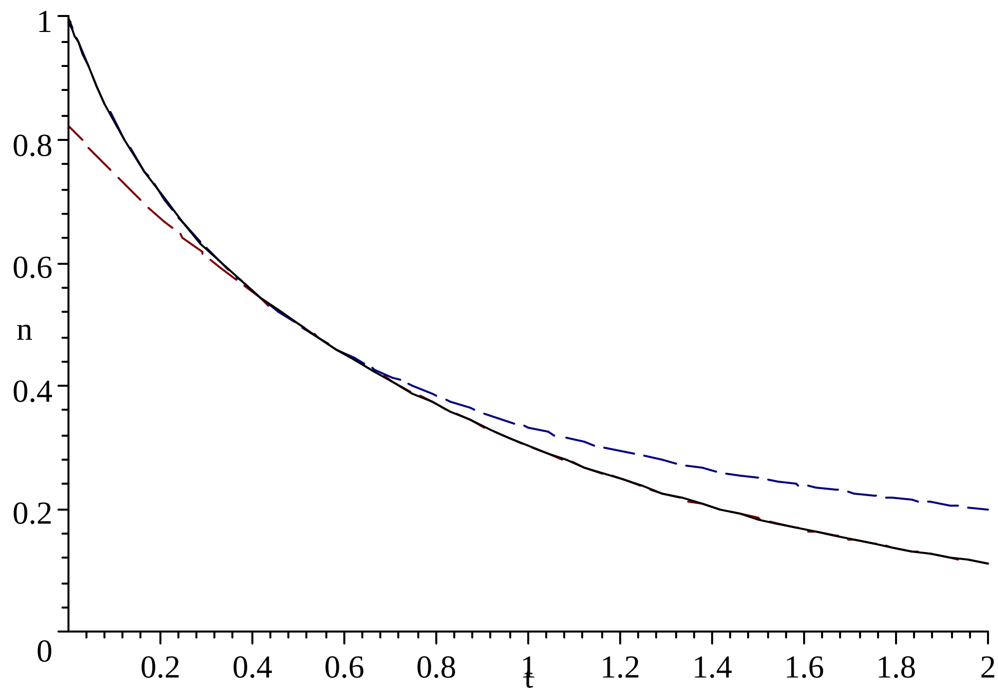
$$\frac{r}{\sqrt{D\tau}} = 2.4048$$

This value of r takes the place of the wavelength in the previous solution, and again we see that with $\tau \propto r^2$, the shortest length scales decay the fastest.

2.4 General solution

When a plasma is set up, it first decays by recombination, and later diffusion becomes dominant. When recombination dominates, n decreases relatively slowly, $n = \frac{n_0}{1+\alpha t n_0}$ (equation 14), but once diffusion sets in, the decay is exponential, $n \sim \exp(-t/\tau)$ with $\tau \sim L^2/D$.

In the plot below, t is in terms of the time constant L^2/D, and we chose $\alpha \frac{L^2}{D} n_0 = 2$.(Blue recombination, red diffusion, solid combination.)



2.5 The whole enchilada

Including everything, (ionization, recombination, diffusion, and external sources/sinks) the equation for evolution of the plasma density would be:

$$\frac{\partial n}{\partial t} - D\nabla^2 n = Q_0 n - \alpha n^2 + S$$

This is a big mess! We almost always approximate by including only the dominant effects. For example, we can find a uniform equilibrium solution ($\partial/\partial t \equiv 0$, $\nabla^2 \equiv 0$, $S = 0$) by balancing ionization and recombination:

$$n = \frac{Q_0}{\alpha}$$

This is often a good solution in astronomical situations, but is not so good in laboratory plasmas.

3 Diffusion as a random walk

The diffusion coefficient has dimensions

$$[D] = \frac{L^2}{T}$$

In fact, for the unmagnetized plasma that we have discussed so far, we found (eg 11)

$$D = \frac{kT}{m}\tau = \frac{kT\tau^2}{m\tau} = \frac{(v_{\text{th}}\tau)^2}{\tau} = \frac{\lambda^2}{\tau}$$

where λ is the mean free path and τ is the mean time between collisions. Thus

$$D \sim \frac{(\text{step})^2}{\text{time per step}} \tag{15}$$

Here's another way to look at this:

A particle has a displacement $\vec{s}_i$ from collision $i-1$ to collision i. Each displacement $\vec{s}_i$ is independent of all the others, in magnitude and direction, but the average (rms) length is

$$< |\vec{s}_i| >_{\text{rms}} \ = \lambda$$

the mean free path. i.e.

$$\lambda^2 = \frac{1}{N}\sum_{i=1}^{N} \vec{s}_i \cdot \vec{s}_i \tag{16}$$

Now after N collisions, the total displacement is:

$$\vec{s} = \sum_{i=1}^{N} \vec{s}_i$$

and its average value $< \vec{s} > = 0$. This happens because the direction of $\vec{s}$ is random. The length of the total displacement (i.e. the total distance travelled), s, is given by:

$$\begin{aligned} s^2 &= \vec{s} \cdot \vec{s} = \left(\sum_{i=1}^{N} \vec{s}_i\right) \cdot \left(\sum_{j=1}^{N} \vec{s}_j\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} \vec{s}_i \cdot \vec{s}_j \end{aligned}$$

and the mean value is

$$< s^2 > \ = \ < \sum_{i=1}^{N}\sum_{j=1}^{N} \vec{s}_i \cdot \vec{s}_j >$$

Now because the steps are independent,

$$< \vec{s}_i \cdot \vec{s}_j > \ = 0 \text{ for } i \neq j$$

Thus

$$< s^2 > \; = \; < \sum_{i=1}^{N} \vec{s}_i \cdot \vec{s}_i > \; = N\lambda^2$$

where we used equation (16). Thus

$$d_{\text{rms}} = \sqrt{< s^2 >} = \sqrt{N}\lambda$$

The mean distance travelled is $\sqrt{N}$ times the mean length of each step. Of course the direction travelled is random, so the average displacement $< \vec{s} > = 0$ as noted above.

We can now compute a diffusion speed of sorts:

$$\frac{\text{distance travelled}}{\text{time}} = \frac{\sqrt{N}\lambda}{N\tau} = \frac{\lambda}{\sqrt{N\tau}} = \frac{\lambda}{\sqrt{\tau t}}$$

Thus the longer the diffusion proceeds, the slower the speed.

The diffusion coefficient (15) is

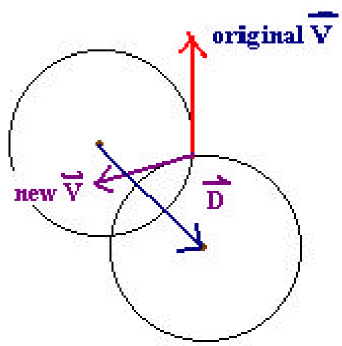
$$D = \frac{(\text{distance travelled})^2}{\text{time}}$$

4 Diffusion in magnetized plasmas

When the plasma is magnetized, the flux of particles along B is the same as in an unmagnetized plasma. Putting the $z-$axis along the magnetic field, we have:

$$F_z = \pm\mu n E_z - D\frac{\partial n}{\partial z}$$

The particles gyrate around the field lines, and collisions abruptly change the velocity, and cause abrupt jumps in the position of the guiding center:



This picture shows that the step size in the random walk is now roughly equal to the Larmor radius, and thus we should expect the diffusion coefficient for motion across B to be about

$$D = \frac{r_L^2}{\tau}$$

Here the ions diffuse faster, because the Larmor radius $r_L = mv/eB \sim \sqrt{kTm}/eB$ is greater for the ions than for the electrons.

Now for the equations. The equation of motion is:

$$mn\frac{d\vec{v}}{dt} = \pm en\left(\vec{E} + \vec{v}\times\vec{B}\right) - k_B T\vec{\nabla}n - mn\nu\vec{v}$$

Once again we look for a steady state solution:

$$0 = \pm en\left(\vec{E} + \vec{v}\times\vec{B}\right) - k_B T\,\vec{\nabla}n - mn\nu\vec{v}$$

Now $\vec{v}$ appears in two terms. Let's do it one component at a time:

$$\begin{aligned} mn\nu v_x &= \pm en\left(E_x + v_y B\right) - k_B T\frac{\partial n}{\partial x} \\ v_x &= \pm\mu\left(E_x + v_y B\right) - \frac{D}{n}\frac{\partial n}{\partial x} \end{aligned}$$

and similarly

$$v_y = \pm\mu\left(E_y - v_x B\right) - \frac{D}{n}\frac{\partial n}{\partial y}$$

Stuff this back into the equation for v_x :

$$\begin{aligned} v_x &= \pm\mu\left(E_x + \left[\pm\mu\left(E_y - v_x B\right) - \frac{D}{n}\frac{\partial n}{\partial y}\right]B_0\right) - \frac{D}{n}\frac{\partial n}{\partial x} \\ v_x\left(1 + \mu^2 B^2\right) &= \pm\mu E_x + \mu^2 E_y B - \frac{D}{n}\left(\frac{\partial n}{\partial x} \pm \mu B\frac{\partial n}{\partial y}\right) \qquad (17) \end{aligned}$$

But

$$\mu B = \frac{e}{m\nu}B = \omega_c\tau$$

Also recall the expression for diamagnetic drift (plasfluid eqn 15):

$$\vec{v}_D = \frac{k_B T}{qB}\frac{\vec{B}\times\vec{\nabla}n}{Bn} = \frac{k_B T}{qB}\frac{1}{n}\left(-\frac{\partial n}{\partial y}\hat{\mathbf{x}} + \frac{\partial n}{\partial x}\hat{\mathbf{y}}\right)$$

Thus

$$\begin{aligned} \mu B\frac{D}{n}\frac{\partial n}{\partial y} &= \omega_c\tau\frac{k_B T}{m\nu}\frac{1}{n}\frac{\partial n}{\partial y} = \omega_c\tau\frac{eB}{m\nu}\frac{k_B T}{eB}\frac{1}{n}\frac{\partial n}{\partial y} \\ &= -\left(\omega_c\tau\right)^2 v_{Dx} \end{aligned}$$

Thus equation (17) gives:

$$\begin{aligned} v_x &= \frac{1}{1 + \left(\omega_c\tau\right)^2}\left\{\pm\mu E_x + \left(\omega_c\tau\right)^2\frac{E_y}{B} + \left(\omega_c\tau\right)^2 v_{Dx} - \frac{D}{n}\frac{\partial n}{\partial x}\right\} \\ &= \pm\mu_\perp E_x + \frac{\left(\omega_c\tau\right)^2}{1 + \left(\omega_c\tau\right)^2}\left[\left(\vec{E}\times\vec{B}\right)_x + v_{Dx}\right] - \frac{D_\perp}{n}\frac{\partial n}{\partial x} \end{aligned}$$

with a similar expression for v_y. Here we wrote

$$D_\perp = \frac{D}{1 + \left(\omega_c\tau\right)^2} \qquad (18)$$

Thus

$$\vec{v}_\perp = \pm\mu_\perp \vec{E}_\perp - \frac{D_\perp}{n}\vec{\nabla}_\perp n + \frac{(\omega_c\tau)^2}{1+(\omega_c\tau)^2}\left[\vec{v}_E + \vec{v}_D\right] \tag{19}$$

The usual perpendicular drifts, $\vec{v}_E$ and $\vec{v}_D$, are slowed by collisions if $\omega_c\tau < 1$. The drifts parallel to the electric field and the density gradient are reduced by the magnetic field. When $\omega_c\tau \ll 1$ the field has little effect on the plasma confinement, (diffusion is almost the same as in the absence of $\vec{B}$) but when $\omega_c\tau \gg 1$ the magnetic field significantly retards diffusion and aids plasma confinement.

Now

$$\omega_c\tau = \frac{eB}{m}\frac{\lambda_{\text{mfp}}}{v} = \frac{\lambda_{\text{mfp}}}{r_L}$$

and if $\omega_c\tau \gg 1$, $(\lambda_{\text{mfp}} \gg r_L)$ the perpendicular diffusion coefficient (18) is approximately:

$$D_\perp \simeq \frac{D}{(\omega_c\tau)^2} = \frac{k_BT}{m\nu\,(\omega_c\tau)^2} = \frac{k_BT}{m\omega_c^2}\nu$$

In this form we can see that $D_\perp \propto \nu$: this happens because collisions are necessary to move the guiding center. In contrast, with $\vec{B} = 0$, $D \propto 1/\nu$: collisions slow the drift. Finally we express D in terms of the Larmor radius, in the strong field limit ($\omega_c\tau \gg 1$).

$$D_\perp \simeq v_{\text{th}}^2\left(\frac{r_L}{v_{\text{th}}}\right)^2\frac{1}{\tau} = \frac{r_L^2}{\tau}$$

In this form we can see that the step size in the random walk is indeed the Larmor radius.

4.1 Ambipolar diffusion across $\tilde{\mathrm{B}}$

With a magnetic field, there are many different ways for the plasma to maintain quasi-neutrality. Currents across $\vec{B}$ can be balanced by currents along $\vec{B}$. Here we'll consider one simple example to show some of the features that can arise. A second example is in the problems.

Let the plasma be infinite in the $z-$direction (along $\vec{B}$) and let the thickness perpendicular to $\vec{B}$ be much greater than the Debye length. Because the plasma is infinite in z, motions parallel to the field cannot help to maintain the quasi-neutrality, and we can set the ion and electron fluxes perpendicular to $\vec{B}$ equal. Then we have:

$$-n\mu_{\perp,e}\vec{E}_\perp - D_{\perp,e}\vec{\nabla}_\perp n = +n\mu_{\perp,i}\vec{E}_\perp - D_{\perp,i}\vec{\nabla}_\perp n$$

Solve for $\vec{E}$:

$$\vec{E} = \frac{D_{\perp,i} - D_{\perp,e}}{\mu_{\perp,e} + \mu_{\perp,i}}\frac{\vec{\nabla}_\perp n}{n}$$

and then the flux is

$$
\begin{aligned}
\vec{F} &= n\vec{v} = n\mu_{\perp,i}\frac{D_{\perp,i}-D_{\perp,e}}{\mu_{\perp,e}+\mu_{\perp,i}}\frac{\vec{\nabla}_\perp n}{n} - D_{\perp,i}\vec{\nabla}_\perp n \\
&= \frac{\left(D_{\perp,i}-D_{\perp,e}\right)\mu_{\perp,i} - D_{\perp,i}\left(\mu_{\perp,e}+\mu_{\perp,i}\right)}{\mu_{\perp,e}+\mu_{\perp,i}}\vec{\nabla}_\perp n \\
&= -\frac{\mu_{\perp,i}D_{\perp,e}+D_{\perp,i}\mu_{\perp,e}}{\mu_{\perp,e}+\mu_{\perp,i}}\vec{\nabla}_\perp n = -D_{\perp,a}\vec{\nabla}_\perp n
\end{aligned}
$$

which is Fick's law again. The ambipolar diffusion coefficient *for this particular situation* is

$$
D_{\perp,a} = \frac{\mu_{\perp,i}D_{\perp,e}+D_{\perp,i}\mu_{\perp,e}}{\mu_{\perp,e}+\mu_{\perp,i}}
$$

and in the strong field case ($\omega_c\tau \gg 1$),

$$
\begin{aligned}
D_{\perp,i}\mu_{\perp,e} &= \frac{k_BT_i}{M\;{}_c^2}\nu_i\frac{e}{m\omega_c^2}\nu_e = \frac{k_BT_iM^2}{Me^2B^2}\nu_i\frac{em^2}{me^2B^2}\nu_e \\
&= \frac{k_BT_iMm}{e^3B^4}\nu_i\nu_e = \frac{T_i}{T_e}\left(D_{\perp,e}\mu_{\perp,i}\right)
\end{aligned}
$$

and

$$
\mu_{\perp,e} = \frac{em^2}{me^2B^2}\nu_e = \frac{m}{eB^2}\nu_e
$$

But

$$
\nu_e = n\sigma v \simeq n\sigma\sqrt{\frac{k_BT_e}{m}}
$$

So

$$
\mu_{\perp,e} \simeq \frac{m}{eB^2}n\sigma\sqrt{\frac{k_BT_e}{m}} = \frac{n\sigma}{eB^2}\sqrt{k_BT_em} = \sqrt{\frac{T_e}{T_i}\frac{m}{M}}\mu_{\perp,i}
$$

Then the ambipolar diffusion coefficient becomes

$$
\begin{aligned}
D_{\perp,a} &= \frac{D_{\perp,e}\mu_{\perp,i}}{\mu_{\perp,i}\left(1+\sqrt{\frac{T_e}{T_i}\frac{m}{M}}\right)}\left(1+\frac{T_i}{T_e}\right) \\
&= \frac{D_{\perp,e}}{1+\sqrt{\frac{T_e}{T_i}\frac{m}{M}}}\left(1+\frac{T_i}{T_e}\right) \simeq 2D_{\perp,e}
\end{aligned}
$$

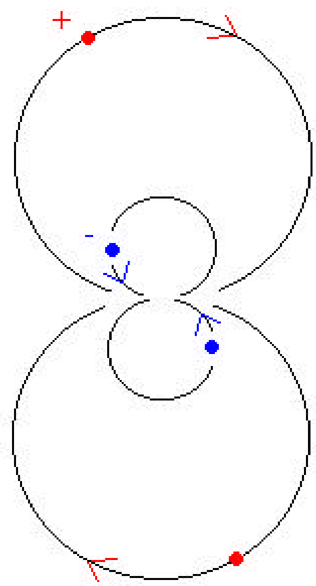
where in the last step we assumed $T_e \simeq T_i$. Again the diffusion is dominated by the slower species– this time the electrons.

Another example of ambipolar diffusion in a magnetized plasma is explored in the problem set.

5 Diffusion in fully ionized plasmas

Collisions between like particles (electron-electron or ion-ion collisons) give rise to little diffusion, since the center of mass of the system remains fixed. The two guiding centers have equal and opposite displacements. (These collisions are important in establishing a Maxwellian distribution, however. See Spitzer's book "Physics of Fully ionized gases" for more details.)

When particles of opposite charge collide, things get more interesting. Both guiding centers move in the *same* direction, leading to diffusion. In the diagram below, both particles reverse directions, and both guiding centers move downward. In a more realistic situation, the ion guiding center moves little, while the electron guiding center moves a lot. The big idea is the same.



5.1 Coulomb collisions and resistivity

We are interested in the electron-ion collisions, and in particular the term $P_{ie} = -P_{ei}$, the momentum transfer term that appears in the equation of motion. We may write this term (momentum transferred to electrons by ions) P_{ei} as

$$P_{ei} = mn_e\,(\vec{v}_i - \vec{v}_e)\,\nu_{ei} \tag{20}$$

We expect the collision frequency ν_{ei} to be proprtional to n_i and also to the square of the charge, because the Couolmb force is involved. So we may write

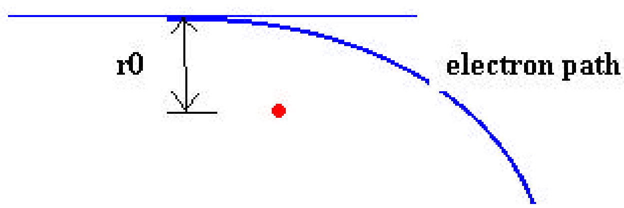
$$\vec{P}_{ei} = e^2 n^2\,(\vec{v}_i - \vec{v}_e)\,\eta \tag{21}$$

where η is the *specific resistivity.* By analyzing the collisons, we hope to obtain an expression for η. Comparing equations (20) and (21), we see how ν_{ei} and η are related:

$$\nu_{ei} = \frac{e^2 n}{m}\eta = \varepsilon_0 \omega_p^2 \eta \tag{22}$$

5.2 Coulomb collisions

If an electron collides with an ion, $M \gg m$, we may assume that the ion remains fixed during the collision. The ion exerts an impulsive force on the electron.



The force acting on the electron is

$$F = \frac{Z_i e^2}{4\pi\varepsilon_0 r^2}$$

and it acts for a time approximately r_0/v where r_0 is the *impact parameter* (see figure). (This approximation gets better the faster the electron is travelling.) Then the impulse delivered is

$$I = \Delta(mv) = \frac{Z_i e^2}{4\pi\varepsilon_0 r_0^2}\frac{r_0}{v} = Z_i \frac{e^2}{4\pi\varepsilon_0 r_0 v}$$

For a 90° collision, $\Delta(mv) \sim mv$, and so the impact parameter for a 90° collision is approximately:

$$r_0 = Z_i \frac{e^2}{4\pi\varepsilon_0 m v^2}$$

(Note: you can also get this result by setting the initial KE approximately equal to the electric PE at closest approach.) Then the collision cross section is

$$\sigma = \pi r_0^2 = \pi\left(Z_i \frac{e^2}{4\pi\varepsilon_0 m v^2}\right)^2$$

and the collison frequency is

$$\nu_{ei} = n_i \sigma v = n_i v \pi \left(Z_i \frac{e^2}{4\pi\varepsilon_0 m v^2}\right)^2 = \frac{n_i}{v^3}\frac{Z_i^2 e^4}{16\pi\varepsilon_0^2 m^2} \tag{23}$$

and with $v^2 \simeq kT/m$,

$$\nu_{ei} = \frac{n_i}{(kT_e)^{3/2}}\frac{Z_i^2 e^4}{16\pi\varepsilon_0^2 \sqrt{m}} \tag{24}$$

and (equation 22)

$$\eta = \frac{\nu_{ei}}{\varepsilon_0 \omega_p^2} = \frac{m}{n_e e^2} \frac{n_i}{(kT_e)^{3/2}} \frac{Z_i^2 e^4}{16\pi\varepsilon_0^2 \sqrt{m}}$$
$$= \frac{\sqrt{m}}{16\pi\varepsilon_0^2 (kT_e)^{3/2}} Z_i e^2 \tag{25}$$

where we used the fact that $n_e = Z_i n_i$.

Although this calculation is very rough, it captures the major features of the exact result. (See Jackson Chapter 13). In fact, many small angle collisions prove to be more important than the large angle collisions we considered. To account for these, we multiply by a factor $\ln\Lambda$, where

$$\Lambda = \frac{\lambda_D}{r_0} \simeq \sqrt{\frac{\varepsilon_0 kT}{e^2 n_e}} \frac{8\pi\varepsilon_0 kT}{Z_i e^2} = \sqrt{\frac{(\varepsilon_0 kT)^3}{n_e}} \frac{8\pi}{Z_i e^3}$$

is the ratio of the Debye length to the impact parameter for a 90° collision. (The reason for this is that since the electric fields are shielded at distances greater than λ_D, that is an effective upper limit to the impact parameter.) Then the resistivity is:

$$\eta = \frac{Z_i e^2 m^{1/2}}{16\pi\varepsilon_0^2 (kT_e)^{3/2}} \ln\Lambda \tag{26}$$

Ignoring the weak dependence in $\ln\Lambda$, the resistivity is independent of the plasma density.

The exact expression for η is

$$\eta = \frac{16\sqrt{\pi} Z_i e^2 m^{1/2}}{3 (4\pi\varepsilon_0)^2 (2kT_e)^{3/2}} \ln\Lambda \tag{27}$$

which differs from our approximate value by a factor

$$\frac{16}{3\sqrt{2\pi}} = 2.13$$

At a temperature $kT \simeq 100$ eV, equation (27) gives

$$\eta = \frac{16\sqrt{\pi}\left(1.6\times10^{-19}\text{ C}\right)^2 \left(9\times10^{-31}\text{ kg}\right)^{1/2} \left(9\times10^{9}\text{ N}\cdot\text{m}^2/\text{C}^2\right)^2}{3\left(2\times100\times1.6\times10^{-19}\text{ J}\right)^{3/2}} \ln\Lambda$$
$$= 1.\times10^{-7}\sqrt{\text{kg}}\frac{\text{N}^2}{\text{C}^2}\frac{\text{m}^4}{\text{J}^{3/2}} \ln\Lambda$$

Now since 1 J = 1 N·m = 1 kg·m^2/s^2, the units are:

$$\frac{\text{kg}^{1/2}\cdot\text{kg}^{1/2}\cdot\text{m}^3/\text{s}}{\text{C}^2} = \frac{\text{kg}\cdot\text{m}^3}{\text{C}^2\cdot\text{s}}$$

We expect resistivity to be ·m.

$$\cdot\,\text{m} = \frac{\text{V}}{\text{A}}\cdot\text{m} = \frac{\text{J}}{\text{C}}\frac{\text{s}}{\text{C}}\cdot\text{m}. = \frac{\text{kg}\cdot\text{m}^3}{\text{C}^2\cdot\text{s}}$$

and so it checks. The value 10^{-7} ·m is comparable to that of some metals, e.g. steel (see LB pg 851).

5.3 Ohm's law

The equation of motion for an electron in an unmagnetized plasma, taking collisions into account, becomes:

$$mn\frac{\partial \vec{v}_e}{\partial t} = -en\vec{E} + \vec{P}_{ei}$$

Assuming the ions remain immobile, the current density is

$$\vec{j} = -ne\vec{v}_e$$

In a steady state, $\frac{\partial}{\partial t} \equiv 0$, using equation (21) with $\vec{v}_i = 0$,

$$\begin{aligned} en\vec{E} &= \vec{P}_{ei} = e^2 n^2 \left(-\vec{v}_e\right) \eta \\ &= \eta \vec{j} en \end{aligned}$$

and thus

$$\vec{E} = \eta \vec{j} \tag{28}$$

which is Ohm's law with the usual definition of resistivity.

Since η is independent of the plasma density, so is the current. This might seem like an odd result. The current should increase as n_e increases, but the drag increases as n_i increases. Thus higher n_i decreases v_e. And because of quasi-neutrality, $n_i = n_e$.

Compare the result for a weakly ionized plasma, where (equation 7)

$$\vec{j} = -e\mu n\vec{E}$$

and the current density is proportional to n_e but the drag is proportional to n_0.

Since the resistivity is strongly temperature dependent, $\eta \propto T^{-3/2}$, a cold plasma is very resistive. As the plasma is heated, its resistance decreases. This is the opposite of the effect we expect in most metals. This means that we cannot effectively use ohmic heating to raise the plasma temperature. As the temperature rises, the heating becomes less effective. Practically, the temperature limit we can achieve by this process is about 1 keV.

Since the collision frequency is proportional to $1/v^3$, (equation 23), fast electrons make fewer collisions. Thus the fastest electrons effectively carry the current. This dependence also leads to an effect called electron runaway. When the electric field is turned on, some electrons in the tail of the Maxwellian happen to be moving fast in the direction opposite $\vec{E}$. These electrons are now accelerated to even higher speeds, and consequently make very few collisions, and thus continue to accelerate to even higher speeds. If $\vec{E}$ is large enough, there are electrons that never make another collision. They form a beam of runaway electrons.

How large an electric field do we need? We want to accelerate to a speed of order $\sqrt{kT/m}$ in a time less than one collision time. Thus

$$\begin{aligned} \frac{e}{m}E\tau_c &> \sqrt{\frac{kT}{m}} \\ E &> \frac{\sqrt{kTm}}{e}\nu_{ei} \\ &= \frac{\sqrt{kTm}}{e}\frac{n_i}{\left(kT_e\right)^{3/2}}\frac{Z_i^2 e^4}{16\pi\varepsilon_0^2\sqrt{m}} \end{aligned}$$

$$E_{\text{crit}} = \frac{n_e}{kT_e} \frac{Z_i e^3}{16\pi\varepsilon_0^2}$$

Since $E_{\text{crit}} \propto 1/T$, and current heats the plasma, (up to a point), electron runaway occurs relatively easily as E_{crit} decreases to meet the applied E.