Stellar atmospheres

1 The Eddington approximation and the two-stream model

A stellar atmosphere is a thin region with total depth much less than the stellar radius. Thus we usually model it as a flat slab. The opical depth is measured vertically downward from the surface.

Let's start by looking at a "grey" atmosphere in which α has no frequency dependence. Then we can drop the subscript ν.

For a ray travelling at angle θ to the normal,

$$\frac{dI}{ds} = -\alpha I + j$$

where $ds = -dz/\cos\theta$ and $d\tau = \alpha dz$. Thus

$$\cos\theta \frac{dI}{d\tau} = I - S$$

We now define the moments:

$$\text{mean intensity: } J = \frac{1}{4\pi}\int I d$$

$$\text{Flux: } H = \frac{1}{4\pi}\int I\mu d$$

$$\text{radiation pressure } K = \frac{1}{4\pi}\int I\mu^2 d$$

Now we take moments of the radiative transfer equation. First, integrate the equation over solid angle to obtain:

$$\frac{dH}{d\tau} = J - S$$

Next multiply by μ before integrating, to get

$$\frac{dK}{d\tau} = H$$

Now in an atmosphere we do not expect any energy generation, and thus $H =$constant (and $J = S$). . Thus from the last equation,

$$K = H\tau + \text{constant} \tag{1}$$

To make further progress, we need to assume something about the angular distribution of radiation. The simplest model is the two-stream approximation, in which

$$I = \begin{cases} I_1 & if \quad 0 < \theta < \pi/2 \ \text{(outgoing radiation)} \\ I_2 & if \quad \pi/2 < \theta < \pi \ \text{(incoming radiation)} \end{cases}$$

Then the moments are

$$\begin{aligned} J &= \frac{1}{2}\left\{\int_0^1 I_1 d\mu + \int_{-1}^0 I_2 d\mu\right\} = \frac{1}{2}\left(I_1 + I_2\right) \\ H &= \frac{1}{2}\left\{\int_0^1 I_1 \mu d\mu + \int_{-1}^0 I_2 \mu d\mu\right\} = \frac{1}{4}\left(I_1 - I_2\right) \\ K &= \frac{1}{2}\left\{\int_0^1 I_1 \mu^2 d\mu + \int_{-1}^0 I_2 \mu^2 d\mu\right\} = \frac{1}{3} J \end{aligned}$$

The third relation is not unexpected! This is the Eddington approximation.

At the surface, we have

$$I_2 = 0 \ \text{ at } \tau = 0$$

Thus, at $\tau = 0$, $J = 2H = 3K$. Thus the constant in equation (1) equals $2H/3$. Thus

$$K = H\left(\tau + 2/3\right)$$

and

$$J = H\left(3\tau + 2\right)$$

With $J = \sigma T^4$, this becomes a relation for temperature versus depth:

$$T^4 = \frac{T_e^4}{2}\left(1 + \frac{3}{2}\tau\right) \tag{2}$$

where the effective temperature T_e is defined to be the temperature of a surface emitting an amount of radiation equal to that from the star. The effective temperature equals the actual temperature at an optical depth $\tau = \frac{2}{3}$. Thus we may interpret "the surface" of the star to be at $\tau = 2/3$.

See R&L p 42-45 for an alternative treatment of this topic.

2 Limb darkening

Using the formal solution of the transfer equation, with $\tau \to \infty$ at the bottom of the atmosphere, and assuming LTE ($S_\nu = B_\nu(T)$), we have:

$$I_v\left(\theta, 0\right) = \int_0^\infty B_\nu\left(T\left(\tau\right)\right) e^{-\tau_\nu \sec\theta} d\tau_\nu \sec\theta$$

Now we expand B_ν in a Taylor series about the optical depth τ^* (unspecified for the moment):

$$B_\nu\left(T\left(\tau\right)\right) = B_\nu\left(T\left(\tau^*\right)\right) + \left.\frac{dB_\nu}{d\tau}\right|_{\tau^*}\left(\tau_\nu - \tau^*\right) + \cdots$$

Putting this into the integral, the first term is

$$B_\nu\left(T\left(\tau^*\right)\right)\left[-e^{-\tau_\nu \sec\theta}\right]_0^\infty = B_\nu\left(T\left(\tau^*\right)\right)$$

In the second term, we integrate by parts:

$$\begin{aligned}
\int_0^\infty (\tau_\nu - \tau^*)\, e^{-\tau_\nu \sec\theta} d\tau_\nu \sec\theta &= \cos\theta \int_0^\infty (x - x^*)\, e^{-x} dx \quad \text{where } x = \tau\sec\theta \\
&= \cos\theta \left[-\left(x - x^*\right) e^{-x} \Big|_0^\infty + \int_0^\infty e^{-x} dx \right] \\
&= \cos\theta \left[-\tau^* \sec\theta - \, e^{-x} \Big|_0^\infty \right] \\
&= \cos\theta - \tau^*
\end{aligned}$$

Thus

$$I_v(\theta, 0) = B_\nu(T(\tau^*)) + \left.\frac{dB_\nu}{d\tau}\right|_{\tau^*} (\cos\theta - \tau^*) + \cdots$$

If we now choose $\tau^* = \cos\theta$, we obtain, correct to 2nd order:

$$I_v(\theta, 0) = B_\nu(T(\tau = \cos\theta))$$

Then for $\theta = 0$ (ray from the center of the stellar disk)

$$I_\nu(0, 0) \simeq B_\nu(T(\tau = 1))$$

while from the limb ($\theta = \pi/2$)

$$I_\nu\left(\frac{\pi}{2}, 0\right) \simeq B_\nu(T(\tau = 0))$$

Since temperature increases inward (equation 2), the intensity is less at the limb. This is the phenomenon of "limb darkening".