Equilibrium

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1 Equilibrium and stability

It seems easy to confine a collisionless plasma: we arrange the geometry so that the particle drifts are harmless. But collisions give rise to diffusion which cause the plasma to escape rapidly. Even ignoring collisions, things may not be as good as they seem. Random clumps of charge may form ("charge bunching") creating E-fields and the resulting E-cross-B drifts. Random motions of charges create fluctuating currents producing magnetic fields, grad-B and curvature drifts. And on and on. We need to analyze the stabilty of each plasma configuration carefully.

1. First find an equilibrium configuration.

2. Check to make sure it's a *stable* equilibrium.

Stable means that small perturbations of the system away from the equilibrium state are damped and the system returns to its original state.

Some instabilities are worse than others. "Explosive" instabilities arise when one disturbance is able to gain energy at the expense of another, and grow rapidly as a result. (eg plasma oscillations in a drifting plasma where the fast and slow waves may grow together.) Absolute instability (which involves a perturbation gowing at one spot) are worse than convective instabilities. Hydromagnetic (low-frequency) instabilities are very destructive to confinement.

It is not always easy to find an equilibrium state.

2 General considerations.

In an equilibrium we may set the explicit time derivative $(\partial/\partial t)$ terms to zero. Let's begin with $\vec{v} = 0$ and $\vec{g} = 0$ too. Then the equation of motion (MHD eqn 7) simplifies.

$$\vec{\nabla} P = \vec{j} \times \vec{B} \tag{1}$$

The pressure gradient force is balanced by Lorentz force on the plasma currents. Where does the current come from? We can see this by crossing equation (1) with $\vec{B}$.

$$\frac{\vec{\nabla} P \times \vec{B}}{B^2} = \frac{\left(\vec{j} \times \vec{B}\right) \times \vec{B}}{B^2} = \vec{j}$$

Thus $\vec{j}$ is just the diamagnetic current (fluid notes eqn 16) induced by the pressure gradients themselves.

In the form I labelled Astrophysical MHD, $\vec{j}$ has been eliminated and we have

$$\vec{\nabla}\left(P+\frac{B^2}{2\mu_0}\right)=\frac{1}{\mu_0}\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B} \tag{2}$$

If right hand side is zero, magnetic pressure balances gas pressure. Magnetic field confines the plasma. When the RHS is not zero, field line tension also contributes to confinement.

These equations show that both $\vec{j}$ and $\vec{B}$ are perpendicular to $\vec{\nabla}P$: current flow and field lines lie on constant pressure surfaces, or, put another way, if T is constant n is constant along field lines.

Examples of equilibrium states:

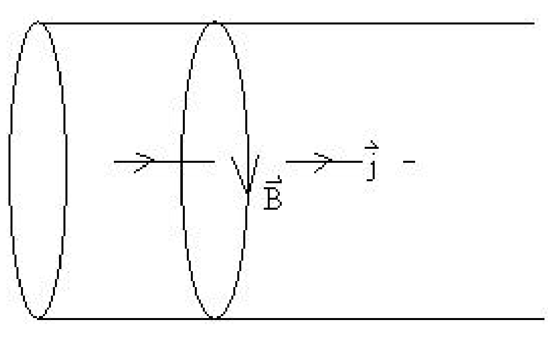
Earth's or neutron star's magnetosphere, solar corona, CNF experiments.

From equation (2), we see that the sum of particle plus magnetic pressures is constant: as one decreases the other increases. Decrease of B is caused by diamagnetic currents. The ratio

$$\frac{P}{P_{\rm mag}}\equiv\beta$$

is a significant indicator of plasma behavior. Most of what we've done so far applies to low-β plasmas (any system with uniform $\vec{B}$ for example). High-β plasmas occur often in astronomy where $\beta>1$ (eg a cluster of galaxies) and in CNF ($\beta<1$ but not $\ll 1$). But pulsars are very low β because B is so enormous! We can have regions of very high β locally, but averaged over a finite region, β is always <1 for confinement. .

3 An example of equilibrium: The $z-$pinch.



In this configuration, we have a current along the axis of the cylinder, and magnetic field lines wrap around in the θ direction. The plasma extends from $r = 0$ to $r = a$.

$\overrightarrow{\mathbf{j}} = j\widehat{\mathbf{z}}$, $(r < a)$; $\qquad \overrightarrow{\mathbf{j}} = 0$, $(r > a)$, with no dependence on z, θ.

Assume a steady state. Then the MHD equations give:

MHD eqn 8

$$\overrightarrow{\nabla} \cdot (n\overrightarrow{\mathbf{v}}) = 0 \Longrightarrow \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) = 0 \tag{3}$$

Equation (1)

$$\overrightarrow{\nabla} P = \overrightarrow{\mathbf{j}} \times \overrightarrow{\mathbf{B}} \Longrightarrow -j_z B_\theta = \frac{\partial P}{\partial r} \tag{4}$$

Ohm's law (MHD eqn 14):

$$\overrightarrow{\mathbf{E}} + \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}} = \eta \overrightarrow{\mathbf{j}} + \frac{1}{en}\left(\overrightarrow{\mathbf{j}} \times \overrightarrow{\mathbf{B}} - \overrightarrow{\nabla} P_e\right) = \eta \overrightarrow{\mathbf{j}} + \frac{1}{en}\left(\overrightarrow{\nabla} P_i\right) \tag{5}$$

This has components:

$$E_r - v_z B_\theta = \frac{1}{en}\left(\frac{\partial P_i}{\partial r}\right) \tag{6}$$

$$E_\theta = 0 \tag{7}$$

$$E_z = \eta j_z \tag{8}$$

The $z-$component of electric field is needed to drive the current along the axis.

Finally we add Ampere's law:

$$\overrightarrow{\nabla} \times \overrightarrow{\mathbf{B}} = \mu_0 \overrightarrow{\mathbf{j}} \Longrightarrow \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \mu_0 j_z \tag{9}$$

We can integrate this equation if j_z is constant for $r < a$:

$$B_\theta = \frac{\mu_0}{2} j_z r \qquad r < a \tag{10}$$

For $r > a$, the current is zero, so $rB_\theta =$ constant. Requiring the field to be continuous at $r = a$, we have

$$B_\theta = \frac{\mu_0}{2} j_z \frac{a^2}{r} \qquad r > a \tag{11}$$

Next we integrate equation (4) to find P :

$$\begin{aligned} -j_z B_\theta &= \frac{\partial P}{\partial r} = -j_z\left(\frac{\mu_0}{2} j_z r\right) \qquad r < a \\ &= -\frac{1}{2} j_z^2 \mu_0 r \end{aligned}$$

So:

$$P = -\frac{1}{4} j_z^2 \mu_0 r^2 + C \qquad r < a \tag{12}$$

For $r > a$, since $j = 0$, and there is no plasma, $P = 0$. Thus the constant $C = \frac{1}{4} j_z^2 \mu_0 a^2$ and:

$$P = \frac{1}{4} j_z^2 \mu_0 \left(a^2 - r^2 \right) \qquad r < a \tag{13}$$

Thus we can determine the size of the column in terms of the central pressure P_0 and the current density it carries:

$$a = \sqrt{\frac{P_0}{\mu_0} \frac{2}{j_z}} \tag{14}$$

The total current is:

$$I = \pi a^2 j_z = \pi \left(\sqrt{\frac{P_0}{\mu_0} \frac{2}{j_z}} \right)^2 j_z = 4\pi \frac{P_0}{\mu_0 j_z} = 2\pi a \sqrt{\frac{P_0}{\mu_0}} \tag{15}$$

Note: for $r < a$ (equations 13 and 10)

$$P + \frac{B^2}{\mu_0} = \frac{1}{4} j_z^2 \mu_0 \left(a^2 - r^2 \right) + \frac{1}{\mu_0} \left(\frac{\mu_0}{2} j_z r \right)^2 = \frac{1}{4} j_z^2 \mu_0 \left(a^2 - r^2 \right) + \frac{1}{4} \mu_0 j_z^2 r^2 = \frac{1}{4} j_z^2 \mu_0 a^2 \tag{16}$$

which is a constant.

Using "Astrophysical MHD" , equation (30) in MHD notes, we would find the gradient of the total (gas plus magnetic) pressure to be:

$$\vec{\nabla} \left(P + \frac{B^2}{2\mu_0} \right) = \frac{1}{\mu_0} \left(\vec{\mathbf{B}} \cdot \vec{\nabla} \right) \vec{\mathbf{B}} \tag{17}$$

Since the only variations are in the radial direction, we have:

$$\frac{\partial}{\partial r} \left(P + \frac{B^2}{2\mu_0} \right) \hat{\mathbf{r}} = \frac{1}{\mu_0} \frac{B_\theta}{r} \frac{\partial}{\partial \theta} \left(B_\theta \hat{\theta} \right) = \frac{1}{\mu_0} \frac{B_\theta^2}{r} \frac{\partial}{\partial \theta} \left(\hat{\theta} \right) = -\frac{1}{\mu_0} \frac{B_\theta^2}{r} \hat{\mathbf{r}} \tag{18}$$

Thus:

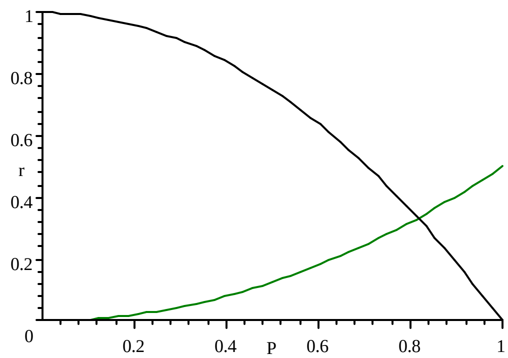
$$P + \frac{B^2}{2\mu_0} = -\frac{1}{\mu_0} \int \left(\frac{\mu_0}{2} j_z \right)^2 r\,dr = -\frac{1}{4} j_z^2 \mu_0 \frac{r^2}{2} + \text{ constant} \tag{19}$$

$$= -\frac{1}{2\mu_0} \left(\frac{\mu_0}{2} j_z r \right)^2 + \text{ constant} = -\frac{1}{2\mu_0} B_\theta^2 + \text{ constant} \tag{20}$$

Thus

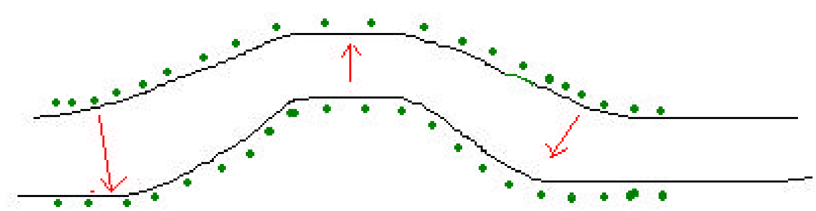
$$P + \frac{B^2}{\mu_0} = \text{ constant} \tag{21}$$

as we obtained above. Thus we see that field line tension helps to confine the plasma.

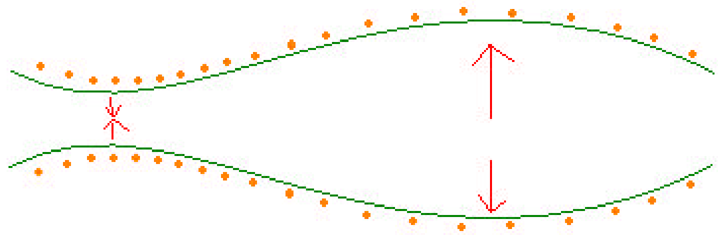


Black - gas pressure. Green- magnetic pressure

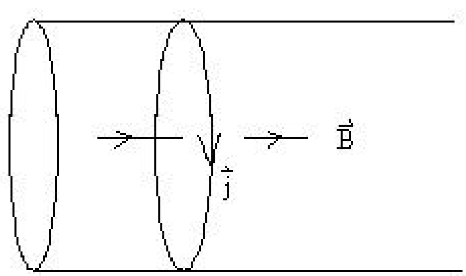
The problem with this configuration is that it is unstable. Suppose we put a kink in the cylinder. Then at the outside of the kink, the field lines are pushed apart, which means that the field is weaker. On the inside of the kink the field lines are closer together, which means the field is stronger. Thus the plasma drifts outward where there is less confining field, and the kink grows.



Another instability, called the sausage instability, also occurs. Imagine perturbing the cylinder as shown below. The same kind of field perturbations occur, and the sausage gets fatter where it is already fat, and skinnier where it is skinny.



Another kind of pinch, called the $\theta-$pinch , may be constructed. In this configuration the magnetic field is along the cylinder and the current is azimuthal. One of these models may be relevant to astrophysical jets.



A combination of these two ideas, plus rolling the cylinder up into a torus, gave rise to the tokamak.

4 Accreting neutron star magnetospheres

The accreting plasma exerts a ram pressure that is balanced by magnetic pressure at the magnetosphere boundary:

$$\rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = \frac{1}{2}\vec{\nabla}\left(\rho v^2\right) - \frac{v^2}{2}\vec{\nabla}\rho - \rho\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)$$

so our equation of motion, in equilibrium, is:

$$0 = \vec{\nabla}\left(P + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0}\right) - \frac{v^2}{2}\vec{\nabla}\rho - \frac{1}{\mu_0}\left(\vec{\mathbf{B}}\cdot\vec{\nabla}\right)\vec{\mathbf{B}} - \rho\vec{v}\times\left(\vec{\nabla}\times\vec{v}\right)$$

If the flow is irrotational, the last term is zero. $\vec{\nabla}\rho$ is ≈ 0 except at the boundary, where $v \rightarrow 0$, and a similar result holds for B. Thus,integrating across the magnetopause boundary, we have

$$P + \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} = \text{constant}$$

On one side the ram pressure dominates, and on the other the magnetic pressure. Thus, for a dipole field (recall that the dipole dominates at large distance from the source)

$$\left.\frac{\rho v^2}{2}\right|_{\text{outside}} = \left.\frac{B^2}{2\mu_0}\right|_{\text{inside}} = \frac{B_*^2}{2\mu_0}\left(\frac{r_*}{r}\right)^6 \tag{22}$$

The velocity is almost free-fall

$$v^2 \approx \frac{2GM_*}{r}$$

and the mass accretion rate is

$$\dot{M} = 4\pi r^2 \rho v$$

thus

$$\rho v^2 = \frac{\dot{M}}{4\pi r^2}\sqrt{\frac{2GM_*}{r}}$$

The gravitational potential energy of the infalling matter is converted to radiation, so the luminosity is

$$L = \dot{M}\frac{GM_*}{r_*} \Longrightarrow \dot{M} = \frac{Lr_*}{GM_*}$$

Thus equation (22) becomes

$$\frac{Lr_*\sqrt{2}}{4\pi r^{5/2}\sqrt{GM_*}} = \frac{B_*^2}{\mu_0}\left(\frac{r_*}{r}\right)^6$$

and thus the magnetopause radius is given by

$$r_M^{7/2} = \frac{B_*^2 r_*^5}{\mu_0 L}\sqrt{GM_*}$$

or

$$r_M = \left(\frac{B_*^2 r_*^5}{\mu_0 L}\right)^{2/7} (GM_*)^{1/7}$$

Using typical values, we get

$$\begin{aligned}
r_M &= \frac{\left(10^8\ \mathrm{T}\right)^{4/7}\left(10^4\ \mathrm{m}\right)^{10/7}}{\left(4\pi \times 10^{-7}\ \mathrm{N/A^2} \times 10^{30}\ \mathrm{J/s}\right)^{2/7}}\left(6.7 \times 10^{-11}\ \mathrm{m^3/kg \cdot s^2} \times 2 \times 10^{30}\ \mathrm{kg}\right)^{1/7} \\
&= 1.8859 \times 10^6 \frac{\mathrm{T}^{4/7} \cdot \mathrm{m}^{10/7}}{\left(\mathrm{N/A^2}\right)^{2/7} (\mathrm{J/s})^{2/7}} \frac{\mathrm{m}^{3/7}}{\mathrm{s}^{2/7}} \\
&= 2 \times 10^6 \frac{(\mathrm{N/A \cdot m})^{4/7} \cdot \mathrm{m}^{13/7}}{\left(\mathrm{N/A^2}\right)^{2/7} (\mathrm{J})^{2/7}} = 2 \times 10^6 \frac{(\mathrm{N})^{4/7} \cdot \mathrm{m}^{9/7}}{(\mathrm{N})^{2/7} (\mathrm{N \cdot m})^{2/7}} \\
&= 2 \times 10^6\ \mathrm{m} = 2000\ \mathrm{km}
\end{aligned}$$

The stabilty of the magnetosphere was investigated by Arons and Lea. 1976 Astrophysical Journal, 207,914. and 1976 Astrophysical Journal, 210,792