Fluids in astrophysics

S.M.Lea

March 1998 revised 2000

1 The equations of fluid dynamics

Fluid dynamics describes the behavior of liquids and gases when the *mean free path* for particles is very small compared with the other length scales of interest in the problem. In astrophysics, densities are often low and mfp correspondingly large, but the relevant length scales are also very large. In addition, interactions between the particles of gas and the magnetic field serve to reduce the mfp. Thus in most astrophysical situations we may safely use the fluid equations to describe the motion of matter.

1.1 The continuity equation

The density in a region is due to flow of matter into or out of that region. Thus:

$$\frac{\partial}{\partial t}\int \rho dV = -\int \rho\vec{v}\cdot d\vec{s}$$

Using the divergence theorem, we may write this relation as:

$$\int \left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})\right) dV = 0$$

Since this relation must be true for *any* volume V, we may conclude:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{1}$$

Expanding the divergence, we get:

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho + \rho\vec{\nabla}\cdot\vec{v} &= 0 \qquad (2)\\ \frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{v} &= 0\end{aligned}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} \tag{3}$$

is the Lagrangian derivative, the rate of change as seen by an observer moving with the fluid. This is often the best derivative to use since it refers to fixed particles, as Newton's law do.

1.2 The equation of motion

We apply Newton's 2nd law, $\vec{F} = m\vec{a}$ to a fluid "particle" of mass ρdV. The forces acting on the particle are pressure forces and gravity. Thus:

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla} P + \rho \vec{g} \tag{4}$$

Here we have ignored the effect of magnetic fields, which can be very important in astrophysics. (These effects are studied in Physics 712.) We can expand out the Lagrangian derivative to get:

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} P + \rho \vec{g} \tag{5}$$

1.3 The energy equation

So far we have two equations (one vector, one scalar) for the varaiables $\vec{v}$, ρ, and P. So we need another equation. (We also need $\vec{g}$. This may be specified independent of the fluid under consideration, or it may be derived from Poisson's equation in the case of self gravity.) The third fluid equation is the energy equation. Sometimes we can use a short cut by using an equation of state:

$$P = nkT \tag{6}$$

for an ideal gas at constant temperature, or

$$P = K\rho^{\gamma} \tag{7}$$

for an adiabatic process. When neither idealization is valid we need to use the first law of thermodynamics:

$$dU = Q - W$$

We apply this to the specific internal energy (energy per particle) $U_{specific} = \frac{3}{2}kT$ with Γ and Λ representing the heating and cooling rates (per unit volume) for the fluid. Then the first law of thermodynamics may be written:

$$\begin{aligned} \frac{D}{Dt}\left(\frac{3}{2}kT\right) &= \frac{\Gamma - \Lambda}{n} - nkT\frac{D}{Dt}\left(\frac{1}{n}\right) \\ &= \frac{\Gamma - \Lambda}{n} + \frac{kT}{n}\frac{Dn}{Dt} \end{aligned} \tag{8}$$

or

$$\begin{aligned} n\frac{D}{Dt}\left(\frac{3}{2}kT\right) - kT\frac{Dn}{Dt} &= \Gamma - \Lambda \\ \frac{D}{Dt}\left(\frac{3}{2}nkT\right) - \frac{5}{2}kT\frac{Dn}{Dt} &= \Gamma - \Lambda \end{aligned} \tag{9}$$

Setting $\Gamma - \Lambda = 0$, we retrieve the adiabatic equation of state with $\gamma = 5/3$ for standard astrophysical situations where the fluid is a monatomic gas. (The situation could be different in molecular clouds, however.)

Note that we have neglected thermal conductivity in this derivation.

2 Applications of the fluid equations

2.1 The virial theorem

The virial theorem applies to a system enclosed within a surface S. Take the momentum equation and dot with the position vector $\vec{r}$:

$$\rho\vec{r}\cdot\frac{\partial\vec{v}}{\partial t}+\rho\vec{r}\cdot\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}=-\vec{r}\cdot\vec{\nabla}P+\rho\vec{r}\cdot\left(-\vec{\nabla}\Phi\right)$$

Integrate over the volume V containing the system. Consider first the steady state case, so that the first term with the time derivative is zero. Then the orther terms are:

$$\begin{aligned}
\int_V \rho\vec{r}\cdot\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}dV &= \int_V \rho r_i v_j \partial_j v_i dV \\
&= \int_V \left[\partial_j\left(\rho r_i v_j v_i\right)-v_i\partial_j\left(\rho r_i v_j\right)\right]dV \\
&= \int_S \left(\rho r_i v_j v_i\right)n_j dS-\int_V \left[v_i r_i\partial_j\left(\rho v_j\right)+\rho v_i v_j \partial_j r_i\right]dV \\
&= \int_S \rho\left(\vec{r}\cdot\vec{v}\right)\vec{v}\cdot\hat{n}\,dS-(0)-\int_V \rho v_i v_j \delta_{ij}dV \\
&= -2T+\int_S \rho\left(\vec{r}\cdot\vec{v}\right)\vec{v}\cdot\hat{n}\,dS
\end{aligned}$$

where

$$T=\int_V \frac{1}{2}\rho v^2 dV$$

is the kinetic energy of the system.

On the other side of the equation we have:

$$\begin{aligned}
-\int_V \tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\nabla}}PdV &= -\int_V \left[\tilde{\boldsymbol{\nabla}}\left(\tilde{\mathbf{r}}P\right)-P\tilde{\boldsymbol{\nabla}}\cdot\tilde{\mathbf{r}}\right]dV \\
&= -\int_S P\tilde{\mathbf{r}}\cdot\hat{\mathbf{n}}dS+3\int_V PdV \\
&= 3\Pi-\int_S P\tilde{\mathbf{r}}\cdot\hat{\mathbf{n}}dS
\end{aligned}$$

where

$$\Pi=\int_V PdV=\int_V nkTdV=\frac{2}{3}\int_V \frac{3}{2}nkTdV=\frac{2}{3}U$$

and

$$-\int_V \rho\vec{r}\cdot\vec{\nabla}\Phi\equiv W$$

So the volume-integrated equation of motion is:

$$2T+3\Pi+W+\text{surface intregrals }=0$$

If the system is finite, then we can put the surface outside the system and thus ensure that the surface integrals are all zero. Then we have the steady state *virial theorem:*

$$2T+3\Pi+W=0=2T+2U+W \tag{10}$$

Note that T is always positive, as is Π. Thus W is negative and is necessary for the system to be in a steady state. This is a *gravity-bounded* system. Writing the kinetic plus internal energy of the system as $E = T + U$, equation 10 becomes:

$$2E + W = 0$$

or

$$E = -\frac{W}{2}$$

which is often the most useful form of the virial theorem.

W is related to the gravitational energy $\quad = \frac{1}{2}\int \rho\Phi dV$ for the system:

$$W = \quad - \int_S \left(\frac{g^2}{8\pi G}\vec{r} + \frac{\Phi}{8\pi G}\vec{g} - \frac{\vec{r}\cdot\vec{g}}{4\pi G}\vec{g} \right) \cdot d\vec{S}$$

To see this, we start with the answer and work backwards:

$$\begin{aligned}
& \frac{1}{4\pi G}\int_S \left(\frac{g^2}{2}\vec{r} + \frac{\Phi}{2}\vec{g} - (\vec{r}\cdot\vec{g})\,\vec{g} \right) \cdot d\vec{S} \\
= & \; -\frac{1}{4\pi G}\int_V \partial_i \left(r_j g_j g_i - \frac{g_j g_j}{2} r_i - \frac{\Phi}{2} g_i \right) dV \\
= & \; -\frac{1}{4\pi G}\int_V \left(r_j g_j \partial_i g_i + \delta_{ij} g_j g_i + r_j g_i \partial_i g_j - \frac{g_j g_j}{2}\partial_i r_i - g_j r_i \partial_i g_j - \frac{\partial_i \Phi}{2} g_i - \frac{\Phi}{2}\partial_i g_i \right) dV
\end{aligned}$$

Now we use Poisson's equation, $\partial_i g_i = -4\pi G\rho$, and simplify:

$$\begin{aligned}
\text{surface integral} \quad = & \int_V \left(r_j g_j - \frac{\Phi}{2} \right)(-\rho)\, dV - \frac{1}{4\pi G}\int_V \left(g^2 - r_j g_i \partial_i \partial_j \Phi - \frac{3}{2} g^2 + g_j r_i \partial_i \partial_j \Phi - \frac{-g_i}{2} g_i \right) dV \\
= & \; + \int_V r_j \partial_j \Phi \rho dV - \frac{1}{4\pi G}\int_V (-r_j g_i + g_j r_i)\, \partial_i \partial_j \Phi dV \\
= & \; - W
\end{aligned}$$

The remaining volume integral is zero because the integrand is an antisymmetric tensor times a symmetric tensor, and is thus identically zero.

The surface integral terms are often zero. For a bounded system, we know that $\Phi \sim 1/R$ at large distances, and similarly $g \sim 1/R^2$. Thus $g^2 R \sim 1/R^3$, which goes to zero faster than the surface area goes to infinity (as R^2). Thus the whole integral goes to zero as $R \to \infty$. In fact for any spherically symmetric system, the surface terms sum to zero. For then the surface S is a sphere of radius a, $\vec{g} = -g\hat{\mathbf{r}}$, $\Phi(a) = -GM(a)/a = g(a)\,a$, and so:

$$\int_S \left(\frac{g^2}{2}\vec{r} + \frac{\Phi}{2}\vec{g} - (\vec{r}\cdot\vec{g})\,\vec{g} \right) \cdot d\vec{S} = 4\pi a^2 \left(\frac{g^2}{2} a + \frac{ag}{2} g - g^2 a \right) = 0$$

In such cases $W = \quad$. For a system of mass M and radius R, $\quad \sim -M^2/R$. For example, for a spherical system with $\rho = \rho_0 R/r$, we have

$$M(r) = \int_0^r \rho(r)\, 4\pi r^2 dr = 4\pi\rho_0 R \int_0^r r\, dr = 2\pi\rho_0 R r^2$$

$$\Phi(r) = -G\frac{M(r)}{r} = -2G\pi\rho_0 R r$$

and

$$\begin{aligned} = \quad & \frac{1}{2}\int \rho\Phi dV = \frac{1}{2}\rho_0 R\left(-2G\pi\rho_0 R\right)\int_0^R \frac{1}{r} r 4\pi r^2 dr \\ = \quad & -G\left(2\pi\rho_0 R\right)^2 \frac{R^3}{3} = -\frac{G}{3}\frac{M^2}{R} \end{aligned}$$

is negative, as expected.

2.1.1 Example: a slowly contracting star

If the star constracts very slowly, then $T \approx 0$,and so the virial theorem reduces to:

$$= -2U$$

As the star contracts, $| \quad | \propto M^2/R$ increases, so the internal energy $U = \frac{1}{2}| \quad |$ increases also. Thus the star heats up. Notice that to remain in equilibrium, only half of the gravitational energy released is converted to thermal energy: the other half must be radiated away.

2.1.2 Example: a spherical interstellar cloud

The cloud is embedded in the surrounding medium, so we cannot ignore all the surface terms. However, if the cloud is spherical, the gravitational surface terms sum to zero, and if $\vec{v} = 0$ at the surface, then only the pressure term remains. So:

$$2\left(T + U\right) + \quad = \int_S P\vec{r}\cdot\hat{n}\; dS = 4\pi a^3 P\left(a\right) \qquad (11)$$

This system can be *pressure bounded* even if $-$ is small. Notice that $4\pi a^3 = 3V$, and $2U = 3\Pi = 3\int PdV$, so equation 11 becomes:

$$2T + \quad + 3\int\left(P - P\left(a\right)\right)dV = 0$$

If the kinetic energy $T \approx 0$, and ≈ 0, the pressure at the surface must be:

$$P\left(a\right) = \frac{1}{V}\int PdV = < P >$$

the average pressure inside the cloud. If gravity is negligible, there can be no pressure gradients: pressure is constant inside the cloud.

Magnetic fields can be included in this analysis. Magnetic field inside the cloud always tends to expand the cloud, but external fields can help to confine the cloud.

2.1.3 Example: stellar systems

The virial theorem is traditionally applied to stellar systems such as clusters of galaxies or globular clusters. For such systems, which are "pressureless", $U = 0$ but $T \neq 0$. The virial theorem becomes:

$$2T + \quad = 0$$

Thus by measuring the system's kinetic energy T, we can measure its mass (through).

2.2 The linearized equations

The fluid equations are, in general, non-linear (see, for example, the $\rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}$ terms in equation 5). This means that the solutions are interesting and complex, but also difficult to find. In many cases the fluid has a steady equilibrium state ($\partial/\partial t \equiv 0$) and we can look for small perturbations to that state. We shall label the steady state variables with a substcript 0 and the perturbed quantities with the subscript 1. Then we have:

Equation 2:

$$\frac{\partial\left(\rho_0+\rho_1\right)}{\partial t}+\left(\vec{v}_0+\vec{v}_1\right)\cdot\vec{\nabla}\left(\rho_0+\rho_1\right)+\left(\rho_0+\rho_1\right)\vec{\nabla}\cdot\left(\vec{v}_0+\vec{v}_1\right)=0$$

Use the fact that $\partial\rho_0/\partial t \equiv 0$ and ignore products of two perturbed quantities:

$$\frac{\partial\rho_1}{\partial t}+\underline{\vec{v}_0\cdot\vec{\nabla}\rho_0}+\vec{v}_0\cdot\vec{\nabla}\rho_1+\vec{v}_1\cdot\vec{\nabla}\rho_0+\underline{\rho_0\vec{\nabla}\cdot\vec{v}_0}+\rho_0\vec{\nabla}\cdot\vec{v}_1+\rho_1\vec{\nabla}\cdot\vec{v}_0=0$$

The underlined terms cancel: this is the equation describing the original steady state. Thus we are left with:

$$\frac{\partial\rho_1}{\partial t}+\vec{v}_0\cdot\vec{\nabla}\rho_1+\vec{v}_1\cdot\vec{\nabla}\rho_0+\rho_0\vec{\nabla}\cdot\vec{v}_1+\rho_1\vec{\nabla}\cdot\vec{v}_0=0 \tag{12}$$

Now we do the same thing with the momentum equation (5):

$$\left(\rho_0+\rho_1\right)\frac{\partial\left(\vec{v}_0+\vec{v}_1\right)}{\partial t}+\left(\rho_0+\rho_1\right)\left(\vec{v}_0+\vec{v}_1\right)\cdot\vec{\nabla}\left(\vec{v}_0+\vec{v}_1\right)=-\vec{\nabla}\left(P_0+P_1\right)+\left(\rho_0+\rho_1\right)\left(\vec{g}_0+\vec{g}_1\right)$$

Linearize:

$$\rho_0\frac{\partial\vec{v}_1}{\partial t}+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_1+\rho_0\left(\vec{v}_1\cdot\vec{\nabla}\right)\vec{v}_0+\rho_1\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0=-\vec{\nabla}P_1+\rho_0\vec{g}_1+\rho_1\vec{g}_0 \tag{13}$$

Before we perturb the energy equation, let's look at some examples.

2.2.1 Special case: uniform, static equilibrium state with $\vec{g}=0$

With $\vec{v}_0=0$ and gradients of all equilibrium quantities zero, equations 12 and 13 reduce to:

$$\frac{\partial\rho_1}{\partial t}+\rho_0\vec{\nabla}\cdot\vec{v}_1=0 \tag{14}$$

and

$$\rho_0\frac{\partial\vec{v}_1}{\partial t}=-\vec{\nabla}P_1 \tag{15}$$

Now we look for a solution in which all the perturbed quantities are proportional to $\exp\left(i\vec{k}\cdot\vec{x}-i\omega t\right)$ Equations 14 and 15 become:

$$-i\omega\rho_1+i\rho_0\vec{k}\cdot\vec{v}_1=0 \tag{16}$$

and

$$-i\omega\rho_0\vec{v}_1=-i\vec{k}P_1 \tag{17}$$

Now we add the linearized version of the adiabatic equation of state 7:

$$P_1=\frac{dP}{d\rho}\rho_1=\gamma K\rho_1\rho_0^{\gamma-1}=\gamma\frac{P_0}{\rho_0^{\gamma}}\rho_1\rho_0^{\gamma-1}=\gamma\frac{P_0}{\rho_0}\rho_1 \tag{18}$$

Combining equations 16, 17 and 18, we get:

$$
\begin{aligned}
-i\omega\rho_1 + i\rho_0\vec{k}\cdot\left(\frac{-i\vec{k}P_1}{-i\omega\rho_0}\right) &= 0 \\
-\omega\rho_1 + \vec{k}\cdot\left(\frac{\vec{k}\gamma\frac{P_0}{\rho_0}\rho_1}{\omega}\right) &= 0
\end{aligned}
$$

Now this equation can be satisifed with a non-zero perturbation ρ_1 only if:

$$\omega^2 - k^2\gamma\frac{P_0}{\rho_0} = 0$$

or

$$\frac{\omega}{k} = \sqrt{\gamma\frac{P_0}{\rho_0}}$$

So we have sound waves with speed $\sqrt{\gamma\frac{P_0}{\rho_0}}$.

2.2.2 The Jeans instability.

Now we take $\vec{g}_1 \neq 0$, but keep ρ_0 constant and $\vec{v}_0 = 0$. Thus we still have equation 14, but the momentum equation becomes:

$$\rho_0\frac{\partial\vec{v}_1}{\partial t} = -\vec{\nabla}P_1 + \rho_0\vec{g}_1$$

We find $\vec{g}_1$ from Poisson's equation:

$$\nabla^2\Phi_1 = -\vec{\nabla}\cdot\vec{g}_1 = 4\pi G\rho_1 \tag{19}$$

Again we assume the wave form for all the perturbed quantities.

$$-i\vec{k}\cdot\vec{g}_1 = 4\pi G\rho_1 \tag{20}$$

and

$$-i\omega\rho_0\vec{v}_1 = -i\vec{k}P_1 + \rho_0\vec{g}_1 \tag{21}$$

We dot equation 21 with $\vec{k}$, and use equations 20, 16 and 18:

$$
\begin{aligned}
-i\omega\rho_0\vec{k}\cdot\vec{v}_1 &= -ik^2P_1 + \rho_0\vec{k}\cdot\vec{g}_1 \qquad (22)\\
&= -ik^2\gamma\frac{P_0}{\rho_0}\rho_1 + \rho_0\frac{4\pi G\rho_1}{-i} \\
&= i\left(-k^2\gamma\frac{P_0}{\rho_0} + \rho_0 4\pi G\right)\frac{\rho_0\vec{k}\cdot\vec{v}_1}{\omega}
\end{aligned}
$$

Again we want a solution with a non-zero $\vec{k}\cdot\vec{v}_1$, and this is only possible if

$$
\begin{aligned}
\omega^2 &= k^2\gamma\frac{P_0}{\rho_0} - \rho_0 4\pi G \\
&= \gamma\frac{P_0}{\rho_0}\left(k^2 - k_J^2\right)
\end{aligned}
$$

where

$$k_J^2 = \frac{4\pi G\rho_0}{\gamma P_0/\rho_0} \tag{23}$$

Thus the frequency is real (the perturbation oscillates) if $k > k_J$, but ω is imaginary (the perturbation grows) if $k < k_J$, or equivalently, the wavelength of the perturbation λ is greater than the Jeans wavelength

$$\lambda_J = \frac{2\pi}{k_J} = 2\pi\sqrt{\frac{\gamma P_0/\rho_0}{4\pi G\rho_0}} = \sqrt{\frac{\pi\gamma P_0/\rho_0}{G\rho_0}}$$

We can also express this in terms of the minimum mass of the perturbation:

$$M_J \sim \rho_0\lambda_J^3 = \sqrt{\left(\frac{\pi\gamma P_0}{\rho_0 G}\right)^3 \frac{1}{\rho_0}}$$

When the wavelength is short, the increased pressure in the compressed regions is able to overcome the increased gravitational attraction, and the system oscillates. But for large wavelengths, the increased density causes sufficient gravitational attraction to overcome the pressure forces, and the system collapses.

This result is generally interpreted to mean that an interstellar cloud needs to have a mass greater than M_J in order to collapse. The limiting mass may be expressed in terms of the cloud temperature, density, and mean molecular weight μ as:

$$\begin{aligned}
M_J &\sim \sqrt{\left(\frac{\pi\gamma kT}{\mu nG}\right)^3 \frac{1}{n\mu}} \\
&= \sqrt{\left(\frac{\pi\frac{5}{3}\,(1.38\times10^{-14}\text{ erg/K})\,(100\text{ K})\,T_{100}}{m\,(1.66\times10^{-24}\text{ g})\,n\,(6.7\times10^{-8}\text{ cm}^3/\text{g}\cdot\text{s}^2)}\right)^3 \frac{1}{nm\,(1.66\times10^{-24}\text{ g})}} \\
&= 4.1\times10^{41}\frac{T_{100}^{3/2}}{m^2n^2}\text{ g/cm}^6 \\
&= \frac{4.1\times10^{41}}{2\times10^{33}}\frac{T_{100}^{3/2}}{m^2n^2}\text{ M}_\odot/\text{cm}^6 = 2\times10^8\frac{T_{100}^{3/2}}{m^2n^2}\text{ M}_\odot/\text{cm}^6
\end{aligned}$$

which is large! Cold, dense clouds collapse the most easily. For a temperature of 20 K, a mean molecular weight of 2 (molecular hydrogen) and a density of 1000 cm^{-3}, we find the minimum mass for collapse to be

$$M_J \sim 2\times10^8\frac{0.2^{3/2}}{2^2\,(1000)^2}\text{ M}_\odot = 4.5\text{ M}_\odot$$

However these parameters are not typical of a cloud before it collapses.

Notice that Jean's original analysis, as presented here, has a serious flaw: the initial equilibrium is not an equilibrium! We cannot have a cloud with uniform ρ_0 and P_0 since it will not have $\vec{g}_0 = 0$, and without pressure gradients it will collapse! However, the calculation does give the salient features of a more accurate (and complicated!) calculation, which leads to a critcal mass about $\sqrt{2}$ times larger than we found here. (See Spitzer for details.)

2.2.3 Thermal instability

This instability arises because, in general, heating processes are proportional to n while cooling processes go like n^2. When a system is perturned from equilibrium, heating and colling no longer balance. In particular, if $\delta n > 0$, cooling increases faster than heating, T and hence P are reduced, and the density increases yet more.

To investigate this instability, we consider the energy equation in the form (equation 8):

$$n\frac{D}{Dt}\left(\frac{3}{2}kT\right) = \Gamma - \Lambda + kT\frac{Dn}{Dt}$$

In equilibrium, $\Gamma = \Lambda$. Let

$$\Gamma - \Lambda = -n\pounds\left(n, T\right)$$

where £ is the net energy loss rate, and

$$\pounds\left(n_0, T_0\right) = 0 \tag{24}$$

Now we perturb, to get

$$\begin{aligned} & \left(n_0 + n_1\right)\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(\frac{3}{2}k\left(T_0 + T_1\right)\right) - k\left(T_0 + T_1\right)\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\left(n_0 + n_1\right) \\ = & -\left(n_0 + n_1\right)\left[\pounds\left(n_0, T_0\right) + \frac{\partial\pounds}{\partial n}n_1 + \frac{\partial\pounds}{\partial T}T_1\right] \end{aligned}$$

Setting $\vec{v}_0 = 0$, and assuming all perturbed quantities go like $\exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right)$, we have

$$n_0\left(\frac{\partial}{\partial t} + \vec{v}_1\cdot\vec{\nabla}\right)\left(\frac{3}{2}kT_1\right) - kT_0\left(\frac{\partial}{\partial t} + \vec{v}_1\cdot\vec{\nabla}\right)n_1 = -\left(n_0 + n_1\right)\left[\pounds\left(n_0, T_0\right) + \frac{\partial\pounds}{\partial n}n_1 + \frac{\partial\pounds}{\partial T}T_1\right]$$

Now linearize, and use equation 24 to get:

$$n_0\left(-i\omega\right)\left(\frac{3}{2}kT_1\right) - kT_0\left(-i\omega\right)n_1 = -n_0\left[\frac{\partial\pounds}{\partial n}n_1 + \frac{\partial\pounds}{\partial T}T_1\right]$$

Next we use the perturbed ideal gas equation of state to relate P_1 to n_1 and T_1 :

$$P_1 = n_1 k_B T_0 + n_0 k_B T_1 \tag{25}$$

Substituting this into equation 22, we have:

$$-i\omega\rho_0\vec{k}\cdot\vec{v}_1 = -ik^2\left(n_1 k_B T_0 + n_0 k_B T_1\right) + \rho_0\vec{k}\cdot\vec{g}_1$$

Again we have dotted with $\vec{k}$ so we can use equation 20

$$\vec{k}\cdot\vec{g}_1 = \frac{4\pi G\rho_1}{-i} = i4\pi G\rho_1$$

So

$$-i\omega\rho_0\vec{k}\cdot\vec{v}_1 = -ik^2\left(n_1 k_B T_0 + n_0 k_B T_1\right) + i4\pi G\rho_0\rho_1$$

and, as before, the continuity equation 10 gives:

$$\rho_1 = \frac{\rho_0\vec{k}\cdot\vec{v}_1}{\omega}$$

So

$$-i\omega\rho_0\vec{k}\cdot\vec{v}_1 = -ik^2\left(\frac{\rho_0\vec{k}\cdot\vec{v}_1}{\omega}\frac{k_BT_0}{m} + n_0k_BT_1\right) + i4\pi G\rho_0\frac{\rho_0\vec{k}\cdot\vec{v}_1}{\omega} \tag{26}$$

and finally the energy equation gives

$$-in_0\omega\left(\frac{3}{2}k_BT_1\right) + n_0\frac{\partial \pounds}{\partial T}T_1 = -ik_BT_0\omega n_1 - n_0\frac{\partial \pounds}{\partial n}n_1$$

So

$$T_1 = n_1\frac{-\frac{\partial \pounds}{\partial n} - i\omega\frac{k_BT_0}{n_0}}{\frac{\partial \pounds}{\partial T} - \frac{3}{2}i\omega k_B}$$

Now write

$$k_T = \frac{2}{3}\frac{\partial \pounds}{\partial T}\frac{1}{k_Bc_s} \approx \frac{1}{\text{cooling length}}$$

where the isothermal sound speed is

$$c_s = \sqrt{\frac{k_BT_0}{m}}$$

and the "cooling length" = sound speed times (cooling time at constant density). Similarly, we define

$$k_\rho = \frac{2}{3}\frac{n_0\partial\pounds/\partial n}{kT_0c_s}$$

and then

$$\begin{aligned}\frac{T_1}{T_0} &= \frac{n_1}{n_0}\left(\frac{-k_\rho c_s - \frac{2}{3}i\omega}{k_Tc_s - i\omega}\right)\\ &= \frac{\tilde{\mathbf{k}}\cdot\tilde{\mathbf{v}}_1}{\omega}\left(\frac{-k_\rho c_s - \frac{2}{3}i\omega}{k_Tc_s - i\omega}\right)\end{aligned}$$

Now putting this into the momentum equation 26, and cancelling the factor $\tilde{\mathbf{k}}\cdot\tilde{\mathbf{v}}_1$ that appears in each term, we have:

$$\begin{aligned}-i\omega\rho_0 &= -ik^2\left(\frac{\rho_0}{\omega}\frac{k_BT_0}{m} + n_0k_BT_0\frac{1}{\omega}\left(\frac{-k_\rho c_s - \frac{2}{3}i\omega}{k_Tc_s - i\omega}\right)\right) + i\frac{4\pi G\rho_0^2}{\omega}\\ \omega^2 &= k^2\left(\frac{k_BT_0}{m} + \frac{k_BT_0}{m}\left(\frac{-k_\rho c_s - \frac{2}{3}i\omega}{k_Tc_s - i\omega}\right)\right) - 4\pi G\rho_0\end{aligned}$$

Then we introduce k_J (equation 23 with $\gamma = 1$),to get

$$\omega^2 = k^2c_s^2\left(1 + \frac{-k_\rho c_s - \frac{2}{3}i\omega}{k_Tc_s - i\omega}\right) - k_J^2c_s^2$$

which is, in general, a messy cubic equation for ω.

$$\begin{aligned}
\left(\omega^2-\left(k^2-k_J^2\right)c_s^2\right)\left(k_T c_s - i\omega\right) &= k^2 c_s^2\left(-k_\rho c_s - \frac{2}{3}i\omega\right) \\
\omega^2 k_T c_s - i\omega^3 - c_s^3 k^2 k_T + i c_s^2 k^2\omega + c_s^3 k_J^2 k_T - i c_s^2 k_J^2\omega &= -c_s^3 k^2 k_\rho - \frac{2}{3} i c_s^2 k^2 \omega \\
-i\omega^3 + \omega^2 k_T c_s + i c_s^2\omega\left(\frac{5}{3}k^2 - k_J^2\right) - c_s^3\left(k^2\left(k_T - k_\rho\right) - k_J^2 k_T\right) &= 0 \qquad (27)
\end{aligned}$$

First note that setting k_T and $k_\rho = 0$ gives back our previous result for the Jeans instability.

$$-\frac{1}{3}i\left(3\omega^2 - c_s^2\left(5k^2 - 3k_J^2\right)\right)\omega = 0$$

which has solutions $\omega = 0$ (which is a spurious root here) and

$$\omega = \pm\sqrt{\frac{5}{3}}c_s\sqrt{k^2 - \frac{3}{5}k_J^2}$$

(remember that our new definition of k_J differs from the old one by a factor of $\gamma = 5/3$).

To get an idea of the thermal effects, let both k_T and $k_\rho \ll k$, and expand ω in a series

$$\omega = \omega_0 + \omega_1 + \cdots$$

where ω_1 is 1st order in $k_T,\ k_\rho$ and so on. The to first order, equation 27 becomes:

$$-i3\omega_0^2\omega_1 + \omega_0^2 k_T c_s + i c_s^2 \omega_1\left(\frac{5}{3}k^2 - k_J^2\right) - c_s^3\left(k^2\left(k_T - k_\rho\right) - k_J^2 k_T\right) = 0$$

So

$$i\omega_1 = \frac{\omega_0^2 k_T c_s - c_s^3\left(k^2\left(k_T - k_\rho\right) - k_J^2 k_T\right)}{3\omega_0^2 - \left(\frac{5}{3}k^2 - k_J^2\right)c_s^2} \qquad (28)$$

The new result has $\omega_0 = 0$, and then:

$$\begin{aligned}
i\omega_1 &= \frac{-c_s^3\left(k^2\left(k_T - k_\rho\right) - k_J^2 k_T\right)}{-\left(\frac{5}{3}k^2 - k_J^2\right)c_s^2} \\
&= \frac{c_s\left(k_T\left(k^2 - k_J^2\right) - k^2 k_\rho\right)}{\left(\frac{5}{3}k^2 - k_J^2\right)}
\end{aligned}$$

Remembering that our functions have time dependence $\exp\left(-i\omega t\right)$, this shows that we will have growth ($i\omega < 0$) if $k < \sqrt{\frac{3}{5}}k_J$, as before, and

$$\begin{aligned}
k_T\left(k^2 - k_J^2\right) - k^2 k_\rho &> 0 \\
k^2\left(k_T - k_\rho\right) &> k_T k_J^2
\end{aligned}$$

So we would need

$$k_J\sqrt{\frac{k_T}{k_T - k_\rho}} < k < \sqrt{\frac{3}{5}}k_J$$

or if $k > \frac{\sqrt{3}}{5} k_J$ and

$$\sqrt{\frac{3}{5}} k_J < k < k_J \sqrt{\frac{k_T}{k_T - k_\rho}}$$

Since the frequency has no real part, this mode has pure exponential growth.

The previous Jeans result is modified. With:

$\omega_0 = \pm\sqrt{\frac{5}{3}} c_s \sqrt{k^2 - \frac{3}{5} k_J^2}$ we find:

$$\begin{aligned} i\omega_1 &= \frac{\frac{5}{3} c_s^2 \left(k^2 - \frac{3}{5} k_J^2\right) k_T c_s - c_s^3 \left(k^2 \left(k_T - k_\rho\right) - k_J^2 k_T\right)}{5 c_s^2 \left(k^2 - \frac{3}{5} k_J^2\right) - \left(\frac{5}{3} k^2 - k_J^2\right) c_s^2} \\ &= c_s \frac{\frac{2}{3} k^2 k_T + k^2 k_\rho}{\frac{10}{3} k^2 - 2 k_J^2} = c_s \frac{\frac{1}{3} k^2 \left(2k_T + 3k_\rho\right)}{\frac{10}{3} \left(k^2 - \frac{3}{5} k_J^2\right)} \end{aligned}$$

Again we have growth for $k < k_J$ if k_T and k_ρ are positive, but also if $k > k_J$ and k_T or k_ρ is sufficiently negative. In both these cases we have a growing wave, sometimes called *overstability*.

To see what's going on in these modes, look at the first solution (pure instability) in the case $k_J = 0$ (i.e. gravity is not important). Then

$$i\omega_1 = \frac{c_s \left(k_T k^2 - k^2 k_\rho\right)}{\frac{5}{3} k^2} = \frac{3}{5} c_s \left(k_T - k_\rho\right)$$

independent of k. Putting back the definitions of k_T and k_ρ, we have:

$$\begin{aligned} i\omega_1 &= \frac{3}{5} c_s \left(\frac{2}{3} \frac{\partial \pounds}{\partial T} \frac{1}{k_B c_s} - \frac{2}{3} \frac{n_0 \partial \pounds / \partial n}{k T_0 c_s}\right) \\ &= \frac{\gamma - 1}{\gamma k_B} \left[\frac{\partial \pounds}{\partial T} - \frac{n_0}{T_0} \frac{\partial \pounds}{\partial n}\right] \end{aligned}$$

Now

$$\frac{\gamma - 1}{\gamma} \frac{m}{k_B} = \frac{1}{c_P}$$

where c_P is the specific heat at constant pressure (see, e.g, Lea and Burke equation 19.18), and

$$\begin{aligned} \left.\frac{\partial \pounds}{\partial T}\right|_P &= \left.\frac{\partial \pounds}{\partial T}\right|_\rho + \left.\frac{\partial \rho}{\partial T}\right|_P \frac{\partial \pounds}{\partial \rho} \\ &= \left.\frac{\partial \pounds}{\partial T}\right|_\rho + \left(-\frac{\rho_0}{T_0}\right) \frac{\partial \pounds}{\partial \rho} \end{aligned}$$

Thus

$$i\omega_1 = \frac{1}{c_P} \left.\frac{\partial \pounds}{\partial T}\right|_P$$

Since $\omega_0 = 0$, accelerations are zero to 1st order, so this is a constant pressure perturbation. But if $\partial \pounds / \partial T$ at constant pressure is < 0, the the perurbation is unstable. The temperature and density perturbations are out of phase (see equation 25 with $P_1 = 0$), so if $\delta n > 0$, then

$\delta T < 0$. Then cold regions have

$$\delta \pounds = \left.\frac{\partial \pounds}{\partial T}\right|_P \delta T > 0$$

(both terms negative) and so cool faster. Hot regions expand and get hotter while cool regions contract and get colder.

What about the other mode? Again with $k_J = 0$, we have:

$$i\omega_1 = c_s \frac{\frac{1}{3}k^2\left(2k_T + 3k_\rho\right)}{\frac{10}{3}k^2} = c_s\left(\frac{1}{5}k_T + \frac{3}{10}k_\rho\right)$$

which is also independent of k. Now

$$\left.\frac{\partial \pounds}{\partial T}\right|_S = \left.\frac{\partial \pounds}{\partial T}\right|_\rho + \left.\frac{\partial \rho}{\partial T}\right|_S \frac{\partial \pounds}{\partial \rho}$$

where S is the entropy. Now if entropy is constant (no heat flow) then $P \propto \rho^\gamma$, $T \propto \rho^{\gamma-1}$, and so $\rho \propto T^{1/(\gamma-1)}$, and thus

$$\left.\frac{\partial \rho}{\partial T}\right|_S = \frac{1}{\gamma - 1}\frac{\rho_0}{T_0}$$

So

$$\left.\frac{\partial \pounds}{\partial T}\right|_S = \left.\frac{\partial \pounds}{\partial T}\right|_\rho + \frac{1}{2/3}\frac{\rho_0}{T_0}\frac{\partial \pounds}{\partial \rho}$$

and so

$$\begin{aligned} i\omega_1 &= \frac{c_s}{5}\left(\frac{2}{3}\frac{\partial \pounds}{\partial T}\frac{1}{k_B c_s} + \frac{3}{2}\frac{2}{3}\frac{n_0 \partial \pounds/\partial n}{k_B T_0 c_s}\right) \\ &= \frac{2}{15 k_B}\left(\frac{\partial \pounds}{\partial T} + \frac{3}{2}\frac{n_0}{T_0}\frac{\partial \pounds}{\partial n}\right) \\ &= \frac{2}{15 k_B}\left.\frac{\partial \pounds}{\partial T}\right|_S \end{aligned}$$

This mode is a sound wave, and we get instability if $\frac{\partial \pounds}{\partial T}$ along an adiabat is negative. Then the dense parts of the wave heat up, and the wave grows.

2.3 Discontinuities

Sharp boundaries often exist where two distinct fluid phases exist in pressure equilibrium with each other. (Processes such as thermal conductivity broaden the boundary to a finite width.) There is no velocity of material across such a boundary. Pressure must balance across the boundary or the boundary will accelerate. To see this mathematically, integrate the momentum equation across the boundary. Let x be a coordinate perpendicular to the boundary. Then:

$$\begin{aligned}
\int_{x_b-\varepsilon}^{x_b+\varepsilon} \rho \frac{D\tilde{\mathbf{v}}}{Dt} dx &= \int_{x_b-\varepsilon}^{x_b+\varepsilon} \left(-\vec{\nabla} P - \rho \vec{\nabla} \Phi_g \right) dx \\
&= \int_{x_b-\varepsilon}^{x_b+\varepsilon} \left(-\vec{\nabla} P - \vec{\nabla} \left(\rho \Phi_g \right) + \Phi_g \vec{\nabla} \rho \right) dx \\
&= -\left(P + \rho \Phi_g \right) \Big|_{x_b-\varepsilon}^{x_b+\varepsilon} + \Phi_g \int_{x_b-\varepsilon}^{x_b+\varepsilon} \vec{\nabla} \rho dx \\
&= -\left(P + \rho \Phi_g \right) \Big|_{x_b-\varepsilon}^{x_b+\varepsilon} + \Phi_g \; \rho \Big|_{x_b-\varepsilon}^{x_b+\varepsilon} \\
&= P \left(x_b - \varepsilon \right) - P \left(x_b + \varepsilon \right)
\end{aligned}$$

where we have used the fact that Φ_g changes negligibly across the thin boundary. Thus to make the left hand side zero (no acceleration) we need P to be the same on both sides of the boundary.

2.3.1 Stability of contact discontinuities

Ref: p 222 Spitzer

Imagine a cold cloud in a hot surrounding medium in the galaxy. Gravity pulls the cloud toward the galactic plane, and the motion is opposed by pressure gradients in the surrounding inter-cloud medium. Let's examine the stability of this situation.

We begin by writing linearized fluid equations for the material on both sides of the boundary. For simplicity, we consider a plane interface. As usual we choose $\vec{v}_0 = 0$, and in the unperturbed state we have:

$$\vec{\nabla} P = \rho \vec{g}$$

. Now I'm going to use the labels 1 and 2 to denote the two sides of the boundary, so I'll use a prime to denote the small perurbation. Here g is not due to self gravity, so it does not change when we perturb the fluid.

Continuity equation:

$$-i\omega \rho_1' + i\rho_1 \vec{k} \cdot \vec{v}_1' = 0 \tag{29}$$

Momentum equation:

$$-i\omega \rho_1 \vec{v}_1' = -i\vec{k} P_1' + \rho_1' \vec{g} \tag{30}$$

with similar equations for medium 2. Now we add the boundary conditions. Let's choose a coordinate system with x running along the boundary and z upward, perpendicular to the boundary. The boundary is at $z = 0$. Then $\vec{g} = -g\hat{\mathbf{z}}$, and for each fluid $v_z' = v_b$, the boundary velocity. We can expect to find a solution with no dependence on y.

Next we look for incompressible perturbations in which there is no density change, $\rho' \equiv 0$. Then equation 29 becomes:

$$\vec{k} \cdot \vec{v}_1' = 0 \tag{31}$$

Solving equation 30 for P', we have:

$$P_1' = \frac{\omega \rho_1 v_b}{k_z} \tag{32}$$

Next we dot the momentum equation with $\vec{k}$, using 31, to get:

$$k^2 P_1' = 0$$

and since we expect $P_1' \neq 0$, then $k^2 = 0$, or

$$k_x^2 + k_z^2 = 0$$

and so either k_x or k_z must be imaginary. We expect the perturbations to get smaller as we go away from the boundary, so we take $k_z = i\kappa$, with solutions of the form $e^{-\kappa z}$ for $z > 0$ and $e^{+\kappa z}$ for $z < 0$. Then $k_x = \pm i\kappa$. Then we have pressure balance at the perturbed boundary:

$$\frac{DP}{Dt} = -i\omega P' + \vec{v}_b \cdot \vec{\nabla} P = -i\omega P' + v_b(-\rho g)$$

So the pressure change at the perturbed boundary is

$$\begin{aligned} \delta P &= P' + \frac{v_b \rho g}{i\omega} \\ &= \frac{\omega \rho_1 v_b}{i\kappa} + \frac{v_b \rho_1 g}{i\omega} = \frac{\omega \rho_2 v_b}{-i\kappa} + \frac{v_b \rho_2 g}{i\omega} \end{aligned}$$

Thus

$$\begin{aligned} \omega^2 \left(\frac{\rho_1 + \rho_2}{\kappa} \right) &= g(\rho_2 - \rho_1) \\ \omega^2 &= g\kappa \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \end{aligned} \tag{33}$$

Thus ω^2 is real (boundary is stable) if $\rho_2 > \rho_1$ (denser fluid on the bottom) but is unstable ($\omega^2 < 0$) if $\rho_2 < \rho_1$ (denser fluid on the top). This is the *Rayleigh-Taylor instability*. Note that in the linear regime, $|\omega| \propto \sqrt{gk}$, and so the shortest wavelengths have the fastest growth rates.

Thus we conclude that cold interstellar clouds should "drip" towards the galactic plane with a timescale

$$t \sim \frac{1}{|\omega|} \sim \sqrt{\frac{\rho_2 + \rho_1}{\rho_2 - \rho_1} \frac{1}{gk}}$$

For a disk, we find $g \sim G\sigma$,where σ is the surface density in the disk.

$$\sigma \sim \frac{10^{12}\ \mathrm{M}_\odot}{\pi\,(10\ \mathrm{kpc})^2} = \frac{10^{12}\ \left(2 \times 10^{31}\ \mathrm{Kg}\right)}{\pi\,(10\ \times 3 \times 10^{19}\ \mathrm{m})^2} = 71\ \mathrm{kg/m^2}$$

If $\delta\rho/\rho \sim 10$, for a cloud of radius 1 pc this time is less than or of the order of

$$\begin{aligned} t &\sim \sqrt{\frac{3 \times 10^{16}\ \mathrm{m}}{10\,(6.7 \times 10^{-11}\ \mathrm{m^3/kg \cdot s^2})\left(71\ \mathrm{kg/m^2}\right) 2\pi}} = 3 \times 10^{11}\ \mathrm{s} \\ &\sim 10^4\ \mathrm{y} \end{aligned}$$

which is rather short by astronomical standards.

Additional applications include extragalactic radio sources in which the clouds of radio-emitting plasma are decelerated by the surrounding, denser, intergalactic medium. The instability may be important as a mechanism for driving turbulence, and hence acceleration of particles in the radio source. In accreting magnetic neutron stars, the instability is necessary to allow the incoming material to reach the neutron star surface.

2.4 Discontinuities with ṽ normal to the boundary

2.4.1 Shock waves

Reference: Spitzer page 218

Consider a sound wave propagating through a fluid. From equation 17 we have:

$$\vec{v}_1 = \frac{\vec{k} P_1}{\omega \rho_0}$$

and from equation 16

$$\rho_1 = \rho_0 \frac{\vec{k} \cdot \vec{v}_1}{\omega} = \frac{k^2 P_1}{\omega^2} = \frac{P_1}{c_s^2}$$

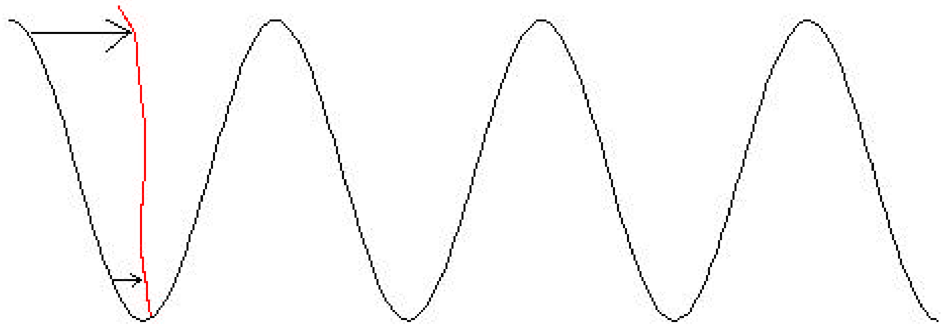
Thus all the fluid quantities oscillate in phase. However, the sound speed is

$$c_s = \sqrt{\frac{\gamma P}{\rho}}$$

and as the wave grows, the sound speed does not remain constant:

$$\begin{aligned}
c_s + \delta c_s &= \sqrt{\frac{\gamma \left(P + \delta P\right)}{\rho + \delta \rho}} = \sqrt{\frac{\gamma P \left(1 + \delta P / P\right)}{\rho \left(1 + \delta \rho / \rho\right)}} \\
&= c_s \left(1 + \frac{1}{2} \frac{\delta P}{P}\right) \left(1 - \frac{1}{2} \frac{\delta \rho}{\rho}\right) \\
&= c_s + \frac{1}{2} c_s \left(\frac{\delta P}{P} - \frac{\delta \rho}{\rho}\right) \\
&= c_s + \frac{1}{2} c_s \left(\frac{c_s^2 \delta \rho}{P} - \frac{\delta \rho}{\rho}\right) \\
&= c_s + \frac{1}{2} c_s \left(\frac{\gamma P \delta \rho}{P \rho} - \frac{\delta \rho}{\rho}\right) \\
&= c_s + \frac{1}{2} c_s \left(\gamma - 1\right) \frac{\delta \rho}{\rho}
\end{aligned}$$

Since $\gamma > 1$ always, the sound speed in the compressed part of the wave is higher than in the rarified part. The wave starts to steepen, and ultimately ”breaks”. This is a shock wave.



After a time t, the crest of the wave has advanced by a distance

$$\begin{aligned}\delta x &= t\delta c_s = c_s t\frac{\gamma-1}{2}\frac{\delta\rho}{\rho} \\ &= \lambda\frac{t}{T}\frac{\gamma-1}{2}\frac{\delta\rho}{\rho}\end{aligned}$$

The steepening is noticeable when $\delta x \sim \lambda/2$, or when

$$t \sim (\gamma-1)\,\frac{\delta\rho}{\rho}T$$

where T is the wave period. This time can become quite short as $\delta\rho/\rho$ approaches 1.

Viscosity begins to get important when the velocity shear becomes large. Viscosity acts to stabilize the steepened wave front into a zone whose thickness is of order the collisional mean free path. Usually this is small compared to other scales of interest in the problem, so we shall ignore this thickness and regard the shock as a sharp discontinuity.

The shock usually travels through the fluid in the "lab" frame. The analysis is easiest if we work in the frame moving with the shock. In this frame, the fluid properties are

upstream	shock	downstream
P_1, ρ_1	$\vert$	P_2, ρ_2
$\longrightarrow \vec{v}_1$	$\vert$	$-\!\!\longrightarrow \vec{v}_2$

We also assume a steady state, and take $\vec{g} \equiv 0$. Then the fluid equations for a plane shock are:

Continuity (equation 1):

$$\vec{\nabla}\cdot(\rho\vec{v}) = 0 = \frac{d}{dx}(\rho v) \tag{34}$$

and momentum (equation 5)

$$\begin{aligned}\rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} &= -\vec{\nabla}P \\ \rho v\frac{dv}{dx} + \frac{dP}{dx} &= 0 \end{aligned} \tag{35}$$

and the energy equation (8)

$$\frac{D}{Dt}\left(\frac{3}{2}kT\right) = \frac{\Gamma - \Lambda}{n} + \frac{kT}{n}\frac{Dn}{Dt} = -\pounds + \frac{P}{n^2}\frac{Dn}{Dt} \tag{36}$$

We can also write an equation for the specific entropy of the gas, since $dS = Q/T$ and thus

$$\frac{DS}{Dt} = v\frac{dS}{dx} = \frac{1}{T}\frac{dQ}{dt} = -\frac{1}{T}\pounds \tag{37}$$

The expression for the specific entropy S (entropy per particle)is $S = (k/m)\ln\left(T^{3/2}/\rho\right)$

The goal is to write all these equations in *conservative form*, that is in the form

$$\text{divergence}\,(something) = 0$$

Equation 34 is already in this form. We can put the momentum equation 35 into the right form by adding a term that is identically zero:

$$\begin{aligned} v\frac{d}{dx}(\rho v) + \rho v\frac{dv}{dx} + \frac{dP}{dx} &= 0 \\ \frac{d}{dx}\left(\rho v^2 + P\right) &= 0 \end{aligned} \tag{38}$$

This is Bernoulli's equation. Now we can integrate equations 34 and 38 across the shock, to get:

$$\int_2^1 \frac{d}{dx}(\rho v)\,dx = 0$$

or

$$[\rho v] = \rho_1 v_1 - \rho_2 v_2 = 0 \tag{39}$$

and similarly:

$$\left[\rho v^2 + P\right] = 0 \tag{40}$$

Digression on the energy equation Let's look at the rate of change of total energy:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \frac{3}{2}\rho\frac{kT}{m}\right) = \left(\frac{1}{2}v^2 + U\right)\frac{\partial\rho}{\partial t} + \rho\left(\vec{v}\cdot\frac{\partial\vec{v}}{\partial t} + \frac{\partial U}{\partial t}\right) \tag{41}$$

where $U = \frac{3}{2}\frac{kT}{m}$ is the specific internal energy. Then from equation 36

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{DU}{Dt} - \left(\vec{v}\cdot\vec{\nabla}\right)U = -\frac{\pounds}{m} + \frac{P}{\rho^2}\frac{D\rho}{Dt} - \left(\vec{v}\cdot\vec{\nabla}\right)U \\ &= -\frac{\pounds}{m} - \frac{P}{\rho}\vec{\nabla}\cdot\vec{v} - \left(\vec{v}\cdot\vec{\nabla}\right)U \end{aligned}$$

Now we use equations 1, 5 to simplify the RHS of equation 41:

$$\begin{aligned} & -\left(\frac{1}{2}v^2 + U\right)\vec{\nabla}(\rho\vec{v}) + \rho\left(\vec{v}\cdot\left(-\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} - \frac{1}{\rho}\vec{\nabla}P\right) - \frac{\pounds}{m} - \frac{P}{\rho}\vec{\nabla}\cdot\vec{v} - \left(\vec{v}\cdot\vec{\nabla}\right)U\right) \\ = & -\left(\frac{1}{2}v^2 + U\right)\vec{\nabla}(\rho\vec{v}) - \rho\vec{v}\cdot\left(\vec{\nabla}\left(\frac{v^2}{2}\right) + \frac{1}{\rho}\vec{\nabla}P + \vec{\nabla}U\right) - \rho\left(\frac{\pounds}{m} + \frac{P}{\rho}\vec{\nabla}\cdot\vec{v}\right) \end{aligned}$$

Now

$$\rho\vec{v}\cdot\frac{1}{\rho}\vec{\nabla}P + \rho\frac{P}{\rho}\vec{\nabla}\cdot\vec{v} = \vec{v}\cdot\vec{\nabla}P + P\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot(\vec{v}P)$$

So

$$\begin{aligned}\text{RHS} &= -\vec{\nabla}\left(\rho\vec{v}\left(\frac{1}{2}v^2+U+\frac{P}{\rho}\right)\right)-\rho\left(\frac{\pounds}{m}\right)\\ &= -\vec{\nabla}\left(\rho\vec{v}\left(\frac{1}{2}v^2+H\right)\right)-\rho\left(\frac{\pounds}{m}\right)\end{aligned}$$

where H is the enthalpy. This is the conservative form that we want. We can write the radiative loss term $\rho\left(\frac{\pounds}{m}\right)=\vec{\nabla}\cdot\vec{F}$ where $\vec{F}$ is the radiative flux vector. Thus in steady state where the total energy does not change in time $(\partial/\partial t=0)$ we can integrate across the shock to get:

$$\left[\rho v\left(\frac{1}{2}v^2+H\right)\right]=[F_x]$$

which reduces to

$$\left[\frac{1}{2}v^2+H\right]=[F_x] \tag{42}$$

since $[\rho v]=0$. Also note that

$$H=U+\frac{P}{\rho}=\frac{\gamma}{\gamma-1}\frac{P}{\rho}$$

Let

$$\tau_{\text{cool}}\approx\frac{mU}{\pounds}$$

be the cooling time behind the shock front. Then the distance material travels before cooling substantially is

$$\lambda_{\text{cool}}=v_2\tau_{\text{cool}} \tag{43}$$

Case in which cooling is unimportant in the shock itself. This case is described by

$$\lambda_{\text{cool}}\gg \text{ shock thickness and other scales of interest}$$

In this case we ignore cooling (and the jump in R_x) while analysing the shock. Define the *Mach number*

$$M\equiv\frac{v_1}{c_1}=\frac{v_1}{\sqrt{\gamma kT_1/m}}$$

Then from equation 39:

$$v_2=\frac{\rho_1}{\rho_2}v_1=\frac{\rho_1}{\rho_2}Mc_1$$

and from equation 40

$$\rho_2\left(v_2^2+\frac{P_2}{\rho_2}\right)=\rho_1\left(v_1^2+\frac{P_1}{\rho_1}\right)$$

Now eliminate the v's:

$$\begin{aligned}
\rho_2\left(\left(\frac{\rho_1}{\rho_2}Mc_1\right)^2+\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\frac{c_1^2}{\gamma}\right) &= \rho_1\left(M^2c_1^2+\frac{c_1^2}{\gamma}\right)\\
\frac{\rho_2}{\rho_1}\left(\left(\frac{\rho_1}{\rho_2}M\right)^2+\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\frac{1}{\gamma}\right) &= M^2+\frac{1}{\gamma}\\
\gamma\frac{\rho_1}{\rho_2}M^2+\frac{P_2}{P_1} &= \gamma M^2+1\\
\gamma M^2\left(1-\frac{\rho_1}{\rho_2}\right) &= \frac{P_2}{P_1}-1 \qquad (44)
\end{aligned}$$

Next we work on equation 42:

$$\begin{aligned}
\frac{1}{2}v_2^2+\frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} &= \frac{1}{2}v_1^2+\frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1}\\
\frac{1}{2}\left(\frac{\rho_1}{\rho_2}Mc_1\right)^2+\frac{\gamma}{\gamma-1}\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}\frac{c_1^2}{\gamma} &= \frac{1}{2}M^2c_1^2+\frac{\gamma}{\gamma-1}\frac{c_1^2}{\gamma}\\
\left(\frac{\rho_1}{\rho_2}M\right)^2+\frac{2}{\gamma-1}\frac{P_2}{P_1}\frac{\rho_1}{\rho_2} &= M^2c_1^2+\frac{2}{\gamma-1}\\
M^2\left(1-\left(\frac{\rho_1}{\rho_2}\right)^2\right) &= \frac{2}{\gamma-1}\left(\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}-1\right) \qquad (45)
\end{aligned}$$

Now divide equation 45 by equation 44, and write $\rho_2/\rho_1=\widetilde{\rho}$, and $P_2/P_1=\widetilde{P}$. Then

$$\begin{aligned}
\frac{M^2\left(1-\widetilde{\rho}^{-2}\right)}{\gamma M^2\left(1-\widetilde{\rho}^{-1}\right)} &= \frac{\frac{2}{\gamma-1}\left(\widetilde{P}/\widetilde{\rho}-1\right)}{\widetilde{P}-1}\\
\left(\widetilde{P}-1\right)\left(1+\widetilde{\rho}^{-1}\right) &= \frac{2\gamma}{\gamma-1}\left(\widetilde{P}/\widetilde{\rho}-1\right)
\end{aligned}$$

So

$$\widetilde{P}\left(\left(1+\widetilde{\rho}^{-1}\right)-\frac{2\gamma}{\gamma-1}\frac{1}{\widetilde{\rho}}\right)=\left(1+\widetilde{\rho}^{-1}\right)-\frac{2\gamma}{\gamma-1}$$

and so

$$\widetilde{P}=\frac{1-\frac{\gamma+1}{(\gamma-1)}\widetilde{\rho}}{\widetilde{\rho}-\frac{\gamma+1}{(\gamma-1)}} \qquad (46)$$

Now substitute back into equation 44 to get:

$$\begin{aligned}
\gamma M^2\left(1-\frac{1}{\widetilde{\rho}}\right) &= \frac{1-\frac{\gamma+1}{(\gamma-1)}\widetilde{\rho}}{\widetilde{\rho}-\frac{\gamma+1}{(\gamma-1)}}-1 \\
\gamma M^2\left(\frac{\widetilde{\rho}-1}{\widetilde{\rho}}\right) &= \frac{1-\frac{\gamma+1}{(\gamma-1)}\widetilde{\rho}-\left(\widetilde{\rho}-\frac{\gamma+1}{(\gamma-1)}\right)}{\widetilde{\rho}-\frac{\gamma+1}{(\gamma-1)}} \\
&= \frac{1+\frac{\gamma+1}{(\gamma-1)}-\left(\frac{\gamma+1}{(\gamma-1)}+1\right)\widetilde{\rho}}{\widetilde{\rho}-\frac{\gamma+1}{(\gamma-1)}} \\
&= \frac{2\frac{\gamma}{\gamma-1}\left(1-\widetilde{\rho}\right)}{\widetilde{\rho}-\frac{\gamma+1}{(\gamma-1)}}
\end{aligned}$$

and so

$$\gamma M^2\left(\frac{\gamma+1}{\gamma-1}-\widetilde{\rho}\right)=2\frac{\gamma}{\gamma-1}\widetilde{\rho}$$

and therefore

$$\begin{aligned}
\widetilde{\rho} &= \frac{M^2\left(\frac{\gamma+1}{\gamma-1}\right)}{\frac{2}{\gamma-1}+M^2} \\
&= \frac{\gamma+1}{\gamma-1}\frac{1}{\left(1+\frac{2}{M^2(\gamma-1)}\right)}
\end{aligned} \tag{47}$$

So for large values of M ($M^2 \gg 2/\left(\gamma-1\right)=3$)

$$\frac{v_1}{v_2}=\frac{\rho_2}{\rho_1}\rightarrow\frac{\gamma+1}{\gamma-1}=4$$

where the numerical value is for $\gamma=5/3$. Finally, from equation 46:

$$\begin{aligned}
\widetilde{P} &= \frac{\frac{2}{\gamma-1}+M^2-\frac{\gamma+1}{(\gamma-1)}M^2\left(\frac{\gamma+1}{\gamma-1}\right)}{M^2\left(\frac{\gamma+1}{\gamma-1}\right)-\frac{\gamma+1}{(\gamma-1)}\left(\frac{2}{\gamma-1}+M^2\right)} \\
&= \frac{2\left(\gamma-1\right)+M^2\left[\left(\gamma-1\right)^2-\left(\gamma+1\right)^2\right]}{-2\left(\gamma+1\right)} \\
&= \frac{2\left(\gamma-1\right)-4\gamma M^2}{-2\left(\gamma+1\right)} \\
&= \frac{2\gamma M^2-\left(\gamma-1\right)}{\gamma+1}
\end{aligned} \tag{48}$$

Thus while the increase in density across a shock remains bounded, the pressure increase can be large if M is large. Equations 47 and 48 are the *Rankine-Hugoniot relations* for the shock. From these relations we can also obtain an equation for the temperature jump:

$$\begin{aligned}
\frac{T_2}{T_1} &= \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{2\gamma M^2-(\gamma-1)}{\gamma+1}\left(\frac{\frac{2}{\gamma-1}+M^2}{M^2\left(\frac{\gamma+1}{\gamma-1}\right)}\right) \\
&= \frac{\left(2\gamma M^2-(\gamma-1)\right)\left(2+(\gamma-1)M^2\right)}{M^2(\gamma+1)^2} \\
&= \frac{\left(+2\gamma M^4(\gamma-1)-2\gamma+2-M^2\left(\gamma^2+1-6\gamma\right)\right)}{M^2(\gamma+1)^2} \\
&= \frac{2\left(\gamma M^4-1\right)(\gamma-1)-M^2\left(\gamma^2-6\gamma+1\right)}{M^2(\gamma+1)^2} \qquad (49)
\end{aligned}$$

which becomes for large M

$$\frac{T_2}{T_1} \to \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}M^2 = \frac{5}{16}M^2 \qquad (50)$$

The value of M By the second law of thermodynamics, the entropy must increase across the shock. Since $S \propto \ln\left(T^{3/2}/\rho\right)$, then $T^{3/2}/\rho \propto P^{3/2}/\rho^{5/2}$ must increase, and so

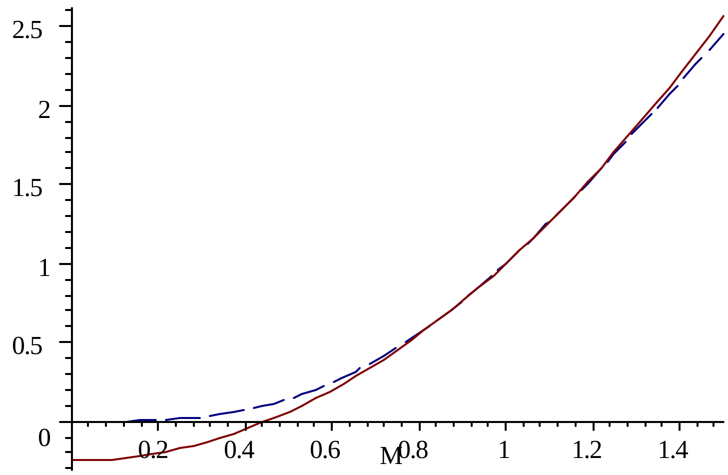
$$\frac{P_2}{P_1} > \left(\frac{\rho_2}{\rho_1}\right)^\gamma$$

Thus

$$\frac{2\gamma M^2-(\gamma-1)}{\gamma+1} > \left(\frac{M^2\left(\frac{\gamma+1}{\gamma-1}\right)}{\frac{2}{\gamma-1}+M^2}\right)^\gamma$$

Set $\gamma = 5/3$. Then we need

$$\begin{aligned}
\frac{10M^2-2}{8} &> \left(\frac{M^2\left(\frac{8}{2}\right)}{3+M^2}\right)^{5/3} \\
\frac{5M^2-1}{4} &> \left(\frac{4M^2}{M^2+3}\right)^{5/3}
\end{aligned}$$



Solid = LHS, Dashed = RHS. Both sides $= 1$ at $M = 1$.

Thus we find that we need $M > 1$ for a valid solution.

A shock wave can only occur when material approaches the shock at greater than the sound speed. In the shock, collisions between particles convert directed kinetic energy of fluid flow into random kinetic energy of thermal motion.

Cooling shocks When $\lambda_{\rm cool}$ is not very large, the structure of the shock looks like:

We can compute the jump conditions across the entire shock plus cooling region, if we pick x_2 where the temperature has returned to its pre-shock value. The previous energy equation is replaced by the ideal gas law at constant temperature T. The sound speed is

$c_s^2 = kT/m = P/\rho$ and is the same in state 1 and state 2. Then equation 40 becomes:

$$\rho_1 \left(v_1^2 + c_s^2\right) = \rho_2 \left(v_2^2 + c_s^2\right)$$

So:

$$\frac{\rho_1}{\rho_2} \left(v_1^2 + c_s^2\right) = \left(v_1^2 \left(\frac{\rho_1}{\rho_2}\right)^2 + c_s^2\right)$$

So

$$\begin{aligned} \frac{M^2+1}{\widetilde{\rho}} &= \frac{M^2}{\widetilde{\rho}^2} + 1 \\ \widetilde{\rho}^2 - \left(M^2+1\right)\widetilde{\rho} + M^2 &= 0 \end{aligned}$$

and the solution is:

$$\begin{aligned} \widetilde{\rho} &= \frac{M^2 + 1 \pm \sqrt{M^4 + 2M^2 + 1 - 4M^2}}{2} \\ &= \frac{M^2 + 1 \pm \left(M^2 - 1\right)}{2} \end{aligned}$$

With the minus sign we would get $\widetilde{\rho} = 1$, (no shock) so the correct solution is:

$$\widetilde{\rho} = M^2$$

Thus the density increase can be very large when cooling occurs.

2.4.2 Supernovae

References: Spitzer, pg 255

Review article by Chevalier, ARAA **15**, 175, 1977

Details of the initial supernova explosion are not well known, since we only see them *after* the explosion, and usually not immediately after. We expect a blast of radiation and ejection of some material at high energy. Since this material is moving at $v \gg$ sound speed of surrounding material, a shock is formed.

Phase I. Shock velocity is approximately constant. The temperature behind the shock can be found from the Rankine-Hogoniot relations, specifically equation 49 or 50:

$$\begin{aligned} T_2 &= \frac{5}{16} M^2 T_1 = \frac{5}{16} \frac{v_1^2}{\gamma k/m} \\ &= \frac{3}{16} v_1^2 \frac{m}{k} \end{aligned}$$

Thus by measuring T_2, we can determine v_1. The temperature is usually determined from x-ray measurements. *Typical* kinetic energy of a supernova is estimated to be approximately 4×10^{50} erg, and the mass of the ejecta $M_{\rm ej} \sim 0.25$ M$_\odot$ for a Type I supernova (the numbers are rather uncertain). Thus we find

$$E = \frac{1}{2} M v^2 \Rightarrow v \sim \sqrt{\frac{2E}{M}} = \sqrt{\frac{2\,(4 \times 10^{50} \text{ erg})}{0.25 \times 2 \times 10^{33} \text{ g}}} = 10^9 \text{ cm/s } = 10^4 \text{ km/s}$$

and thus

$$T_2 \sim \frac{3}{16} 10^{18} \ \ (\text{cm/s})^2 \left(\frac{1.7 \times 10^{-24} \text{ g}}{1.4 \times 10^{-16} \text{ erg/K}} \right) = 2 \times 10^9 \text{ K}$$

For a Type II, $E \sim 10^{51}$ erg, $M_{\text{ej}} \sim 5\ \text{M}_{\odot}$, so $v \sim \sqrt{\frac{2\,(10^{51}\text{ erg})}{5 \times 2 \times 10^{33}\text{ g}}} = 4.5 \times 10^3$ km/s and $T \sim 4 \times 10^8$ K.

This phase lasts until the matter swept up by the outward moving shock is of the same order as the ejected mass. At this point the shock begins to decelerate. This happens when

$$\frac{4}{3} \pi r_s^3 \rho_1 = M_{\text{ej}}$$

So for a Type I, and for surrounding ISM density of 1 cm^{-3},

$$r_s \sim \left(\frac{3 \times 0.25 \times 2 \times 10^{33} \text{ g}}{4\pi \left(1.7 \times 10^{-24} \text{ g/cm}^3\right)} \right)^{1/3} = 4 \times 10^{18} \text{ cm} \ \sim 1 \text{ pc}$$

The time to reach this radius at $v \sim 10^9$ cm/s is

$$t \sim 4 \times 10^9 \text{ s} \sim 130 \text{ y}$$

which is very short!

The Sedov phase The following phase is called the Sedov phase. We shall neglect radiation losses, which we can show *a posteriori* are small. Then the energy behind the shock wave equals the initial supernova energy, and thus the energy density is

$$u = \frac{3}{4\pi r_s^3} E$$

The mean pressure inside is

$$P = nkT = \frac{2}{3} u = \frac{E}{2\pi r_s^3}$$

The pressure right behind the shock is some numerical factor of order 1 times this mean pressure. (Numerical simulations typically find this numerical factor to be about 2.) Thus, from equation 48 for large M:

$$P_2 = \frac{5}{4} M^2 P_1 = \frac{5}{4} \frac{v_1^2}{\gamma k T_1/m} \rho_1 \frac{kT_1}{m} = \frac{3}{4} \rho_1 v_1^2$$

So

$$\frac{3}{4} \rho_1 v_1^2 = \frac{E}{2\pi r_s^3}$$

and thus

$$v_1 = \frac{dr_s}{dt} = \left(\frac{2}{3} \frac{E}{\pi \rho_1} \right)^{1/2} r_s^{-3/2} \tag{51}$$

We can integrate this equation to get:

$$\frac{2}{5} r_s^{5/2} = \left(\frac{2}{3} \frac{E}{\pi \rho_1} \right)^{1/2} t$$

or

$$r_s = \left(\frac{2}{3}\frac{E}{\pi\rho_1}\right)^{1/5}\left(\frac{5}{2}\right)^{2/5} t^{2/5} \quad (52)$$

$$= \left(8\times10^{14}\text{ cm}\right) t^{2/5} \quad (53)$$

or, expressing t in years:

$$r_s = \left(8\times10^{17}\text{cm}\right) t_{\text{yr}}^{2/5} = (0.27\text{ pc})\, t_{\text{yr}}^{2/5}$$

where we used the numbers for a type I supernova, and an ISM density of 1 cm^{-3}. Notice that r_s has a very small dependence (1/5 power) on both E and ρ_1. The temperature behind the shock decreases with time:

$$v_1 = \frac{dr_s}{dt} = \left(8\times10^{14}\text{ cm}\right)\frac{2}{5}t^{-3/5}$$

and thus

$$T_2 = \frac{3}{16}\frac{mv^2}{k} = \frac{3}{16}\frac{m}{k}\left(8\times10^{14}\text{ cm}\right)^2\frac{4}{25}t^{-6/5}$$

$$= 2.3\times10^{11} t_{\text{yr}}^{-6/5}$$

Remember this is valid for $t \gtrsim 130$ yr. Equivalently, we can find T as a function of the shock radius:

$$T_2 = 2.3\times10^{11}\left(t_{\text{yr}}^{2/5}\right)^{-3} = 2.3\times10^{11}\left(\frac{r_s}{0.27\text{ pc}}\right)^{-3}$$

$$= 4.5\times10^{9}\left(\frac{r_s}{1\text{ pc}}\right)^{-3}$$

The temperature increases toward the interior because the Mach number of the shock was higher as it passed through that material. Numerical calculations give $T \sim r^{-4.3}$, in fair agreement with this estimate. The same simple arguments suggest a constant density in the interior since $\rho_2 = 4\rho_1$ for strong shock. But that would lead to strong pressure gradients, which would change the density distribution. If the pressure in the interior were constant, then we would have to have $\rho \propto r^3$ (highest density right behind the shock). In fact the density increases more steeply than this, because the pressure is actually greatest right behind the shock.

We have observational evidence supporting this picture: both x-ray and radio maps tend to show SNR as *shells*, with the emission concentrated in a thin circular region on the sky. If the emission comes from a spherical shell of radius r and thickness $\triangle r$, then we see emission from a region of length $\ell \sim 2r\sin\theta$ where $\cos\theta \sim (r - \triangle r)/r = 1 - \triangle r/r = 1 - \theta^2/2$, so $\theta \sim \sqrt{2\triangle r/r}$ and $\ell \sim 2r\sqrt{2\triangle r/r} = 2\sqrt{2r\triangle r} \propto r^{1/2}$. Thus the emission measure

$$n^2\ell \sim r^6 r^{1/2} \sim r^{6.5}$$

a very strong dependence on r.

Isothermal phase. As expansion continues and T drops, line emission becomes more important as the ionization state decreases. The amount of energy radiated away becomes significant. For $T < 10^6$ K, colling dominates the SNR evolution.

$$4.5 \times 10^9 \left(\frac{r_s}{1 \text{ pc}}\right)^{-3} < 10^6 \text{ for } r_s > \left(4.5 \times 10^3\right)^{1/3} \text{ pc } = 17 \text{ pc}$$

In this phase we can consider the shock to be isothermal. The compression is very high, and the velocity behind the shock is very low. Thus a thin, dense, shell moves outward. The thickness of the shell is λ_{cool} (equation 43). The shock is no longer driven by internal pressure, but merely conserves momentum. Thus this phase is also called the *snowplow phase*. By now the mass inside the shell is mostly swept-up interstellar medium:

$$M \sim \frac{4}{3}\pi r_s^3 \rho_1$$

and so momentum conservation gives:

$$\frac{4}{3}\pi r_s^3 \rho_1 \frac{dr_s}{dt} = M_t v_t = \text{ constant}$$

and thus:

$$\begin{aligned} \frac{4}{3}\pi \rho_1 \frac{r_s^4}{4} &= M_t v_t t \\ r_s &= \left(\frac{3M_t v_t}{\pi \rho_1}\right)^{1/4} t^{1/4} \end{aligned} \tag{54}$$

In the final phase, the interior cools adiabatically and

$$TV^{\gamma-1} = \text{ constant}$$

so

$$T \propto \left(r^3\right)^{-2/3} = r^{-2}$$

When the heated gas interior to the shell pushes the shell outward, (a light fluid pushing on a heavier one) the shell becomes Rayleigh-Taylor unstable (section 2.3.1). In addition the shell is thermally unstable (section 2.2.3). Both effects cause the shell to break up into clumps.

Effects of supernovae

1. Heating of the ISM. As the shock wave sweeps up the ISM it also heats it.

- Shocks may overlap to form "tunnels" of hot gas that may occupy as much as 30% of the volume of the spiral arms in a galaxy (See eg Smith, Ap J **211** p 404)
- Conduction may evaporate cold clouds into the hot, "tunnel" region. (McKee and Ostriker, 1975)
- X-rays and cosmic rays from supernovae can heat and ionize gas at large distances from the supernova itself. (Salpeter, 1976, Lea and Silk 1974)
- Heating by Supernovae may lead to the formation of galactic winds that blow ISM into the intergalactic space (Mathews and Baker 1971 Ap J **170** 241)
- Supernovae lead to fast-moving clouds whose energy is ultimately dissipated as heat in the ISM.

2Relativistic particles: Particles generated in the SN itself are trapped in the remnant, because streaming particles generate plasma waves that scatter the particles. Particles thus suffer adiabatic expansion losses. Acceleration in or behind the shock wave has been discussed but remains controversial.

Radio properties: Radio observations show 3 classes of SN remnants:

1. Shells with tangential magnetic field (compressed interstellar field?)
2. Shells with radial magnetic field (stretched out stellar field?)
3. *Plerions*- filled remnants. These are usually assumed to be powered by a pulsar that ejects relativistic particles into the remnant. The proptypical plerion is the Crab Nebula.

2.4.3 Ionization fronts

Reference Spitzer page 246. Also Shu

Regions of ionized gas (HII regions) form around a source of ionizing radiation. The classical HII region is powered by a bright, young star (O or B type) that is hot enough to produce a substantial amount of its radiation in the UV. Such stars have short lifetimes (10^7years or so), so an HII region is a transient event.

Formation of an HII region Ref Stromgren Ap J **89**, 526, 1939

A photon with $h\nu > 13.6$ eV ($\lambda < 912$ Å) can travel a distance $1/n\sigma$ in neutral gas, where σ is the cross section for ionization of a neutral H atom.

$$\sigma = \left(6 \times 10^{-18} \text{ cm}^2\right)\left(\frac{\lambda}{912 \text{ Å}}\right)^3 \tag{55}$$

Therefore the mean free path is

$$\begin{aligned}\frac{1}{n\sigma} &= \frac{10^{18}}{6n}\left(\frac{912 \text{ Å}}{\lambda}\right)^3 \text{ cm} \\ &= \frac{1}{18n}\left(\frac{912 \text{ Å}}{\lambda}\right)^3 \text{ pc}\end{aligned}$$

As radiation streams out from the star it ionizes gas in a small region. The following photons can travel almost unimpeded through the "hole" created by their predecessors, and then penetrate only a small distance into the surrounding material. In this way the photons "eat" their way out into the ISM, forming an HII region. The region is fully formed when all of the flux from the star is used up reionizing those atoms that recombine within the HII region.

Heating goes along with ionization, since the photon energy is $h\nu = 13.6$ eV $+\triangle E$, and the extra energy $\triangle E$ of the liberated electron ends up as thermal kinetic energy in the HII region. Thus the pressure in the HII is greater than in the HI, and the hot gas expands outward. The expansion speed (approximately the sound speed in the hot HII) is greater than the sound speed in the surrounding cold HI and so a shock wave forms.

In a steady state we can find the radius of the HII region (the "Stromgren sphere") by noting that all the photons emitted by the star are used to reionize the atoms that recombine within the sphere.

Notation:

x	=	$\frac{n(HII)}{n}$ where $n = n(HI) + n(HII)$ is assumed uniform in the initial undisturbed medium
α	=	recombination coefficient to levels above 1: $\alpha = 2.5 \times 10^{-13}$ cm^{3}s^{-1} at 10^4 K
σ	=	mean ionization coefficient $= \int \sigma(\nu) I(\nu) d\nu / \int I(\nu) d\nu$ where $I(\nu)$ is the stellar spectrum
τ	=	optical depth $= \int_0^r n\sigma(1-x) dr$

Then the rate of change of ionization is due to net ionizations minus recombinations:

$$\frac{dx}{dt} = \frac{\Phi}{4\pi r^2} e^{-\tau} \sigma (1-x) - \alpha n x^2 \tag{56}$$

where

$$\Phi = \int_{13.6 \text{ eV}}^{\infty} \frac{\sigma(\nu)}{\sigma} \frac{L(\nu)}{h\nu} d\nu$$

is the photon flux above the ionization edge. Thus in a steady state we have:

$$\frac{1-x}{x^2} = \frac{4\pi\alpha n r^2}{\sigma \Phi e^{-\tau}} \tag{57}$$

which is a horrible equation for x. (Remember that τ also contains x.)

Now define r_S by the condition that the sphere be fully ionized, and ionizations equal recombinations in the sphere:

$$\frac{4}{3}\pi r_S^3 n^2 \alpha = \Phi \tag{58}$$

Then equation 57 becomes:

$$\frac{1-x}{x^2} = \frac{3\,(r/r_S)^2}{\sigma n r_s e^{-\tau}} \tag{59}$$

Then if $r \ll r_S$, $\tau \ll 1$ and $e^{-\tau} \approx 1$. Since

$$\tau = n\sigma \int_0^r (1-x)\,dr$$

then, using equation 59,

$$\frac{d\tau}{dr} = n\sigma\,(1-x) = \frac{3\,(r/r_S)^2}{r_s e^{-\tau}} x^2$$

and so

$$\begin{aligned} e^{-\tau} d\tau &= 3\left(\frac{r}{r_S}\right)^2 x^2 \frac{dr}{r_S} \\ &= x^2 d\left(\frac{r}{r_S}\right)^3 \end{aligned}$$

Now let $r/r_S = z$. Then

$$\frac{de^{-\tau}}{dz} = -x^2$$

So for $r \ll r_S$, $x \approx 1$ and

$$\frac{de^{-\tau}}{dz} \approx -1 \Rightarrow e^{-\tau} \approx 1 - z$$

Thus $1-x$ remains small until $e^{-\tau}$ becomes small, i.e. $z \approx 1$. In fact we need

$$\sigma n r_s e^{-\tau} \approx 1$$

or

$$1 - z \approx \frac{1}{\sigma n r_s}$$

So the thickness of the ionization front is

$$\triangle z \sim \frac{1}{\tau_S} = \frac{1}{\sigma n r_s}$$

and thus $\triangle r = r_S \triangle z = 1/n\sigma$, as predicted above.

Typical numbers for an O star are $\Phi \sim 10^{48}\ \mathrm{s}^{-1}$, which gives

$$r_S = \left(\frac{3\Phi}{4\pi n^2 \alpha}\right)^{1/3} = \left(\frac{3\times 10^{48}\ \mathrm{s}^{-1}}{4\pi n^2\,(2.5\times 10^{-13}\ \mathrm{cm}^3\mathrm{s}^{-1})}\right)^{1/3} = \frac{30\ \mathrm{pc}}{n^{2/3}\ \mathrm{cm}^2} \tag{60}$$

Dynamical evolution The HII region is not in equilibrium dynamically since the pressure inside exceeds the pressure outside. Let's investigate its evolution.

First, the size of the region is changing as the ionization front eats its way out. Let r_i be the radius of the ionization front at time t. Then

$$\begin{aligned} \frac{d}{dt}\left(\frac{4}{3}\pi r_i^3 n\right) &= \Phi - \frac{4}{3}\pi\alpha n^2 r_i^3 \\ \text{rate of increase in \# of ions} &= \text{ionization rate - recombination rate} \end{aligned}$$

Using equation 58, we have:

$$\frac{1}{\alpha n}\frac{dz}{dt} = 1 - z$$

which has the solution

$$\left(\frac{r_i}{r_s}\right)^3 = z = 1 - e^{-\alpha n t}$$

The volume grows linearly for $t \ll 1/\alpha n$ and asymptotically approaches the steady state solution. The time scale $\tau = 1/\alpha n \sim \left(10^4 \text{ y}\right)/n$ is independent of the ionization rate Φ. The timescale is rather short. Then

$$\begin{aligned} \frac{dr_i}{dt} &= r_s\frac{d}{dt}z^{1/3} = \frac{r_s}{3}z^{-2/3}\frac{dz}{dt} = \frac{\alpha n r_s}{3}\frac{1-z}{z^{2/3}} \\ &= \frac{r_s}{3\tau}\frac{e^{-t/\tau}}{\left(1-e^{-t/\tau}\right)^{2/3}} \end{aligned} \tag{61}$$

which can exceed the speed of sound in the HII region if r_s (and thus Φ) is large. The gas is essentially motionless as the front expands through it. With $T_{II} \approx 10^4$ K, $c_{II} \approx 10$ km/s. Then with $n = 10$ cm^{-3} and $r_s \approx$7 pc (equation 60),

$$v_i = \frac{dr_i}{dt} = \frac{7 \times 3 \times 10^{18} \text{ cm}}{3 \times 10^3 \times \pi \times 10^7 \text{ s}}\frac{1-z}{z^{2/3}} = 10 \times 10^5 \text{ cm/s}$$

for

$$\begin{aligned} (1-z) &= z^{2/3} 4.5 \times 10^{-3} \\ z &= 0.99547 \\ r_i &= 0.9985 r_s \end{aligned}$$

i.e. the HII region is almost formed.

Note that the pressure inside the HII region greatly exceeds the presure outside, so the gas expands outward into the neutral surroundings. We shall neglect radiation pressure here, because it is small compared with the difference in gas pressures. (You should verify this.)

Jump conditions across the ionization front Again we work in the frame moving with the front. In deriving the jump conditions across a planar shock, nothing we did depended on the fact that the discontinuity was a shock *per se*, so we get the same jump conditions here.

HII	ionization front	HI
ρ_2, v_2, T_2	$\vert$	ρ_1, v_1, T_1

$$\rho_1 v_1 = \rho_2 v_2 \tag{62}$$

and

$$\rho_1\left(v_1^2 + c_1^2\right) = \rho_2\left(v_2^2 + c_2^2\right) \tag{63}$$

Here c_1 is the isothermal sound speed $\sqrt{kT_1/m}$. With $T_1 \approx 100$ K

$$c_1 = \sqrt{\frac{(1.38 \times 10^{-16} \text{ erg/K})\,(100 \text{ K})}{1.67 \times 10^{-24} \text{ g}}} = 9. \times 10^4 \text{ cm/s } = 0.9 \text{ km/s}$$

and in the HII, $T_2 \approx 10^4$ K, so

$$c_2 = \sqrt{\frac{(1.38 \times 10^{-16}\ \text{erg/K})\,(10^4\ \text{K})}{0.5 \times 1.67 \times 10^{-24}\ \text{g}}} = 1.3 \times 10^6\ \text{cm/s}\ = 13\ \text{km/s}$$

These temperatures are enforced by the radiative terms in the energy equation.

Now use equation 62 to eliminate ρ_2 from equation 63:

$$\rho_1 \left(v_1^2 + c_1^2\right) = \rho_1 \frac{v_1}{v_2} \left(v_2^2 + c_2^2\right)$$

then solve for v_2:

$$v_2^2 - \frac{\left(v_1^2 + c_1^2\right)}{v_1} v_2 + c_2^2 = 0$$

$$v_2 = \frac{\left(v_1^2 + c_1^2\right)}{2v_1} \pm \frac{1}{2}\sqrt{\left(v_1 + \frac{c_1^2}{v_1}\right)^2 - 4c_2^2} \tag{64}$$

Real solutions exist only if the quantity inside the square root is positive:

$$v_1 + \frac{c_1^2}{v_1} \quad > \quad 2c_2 \tag{65}$$

$$v_1^2 - 2c_2 v_1 + c_1^2 \quad > \quad 0$$

With an equals sign, the quadratic $v_1^2 - 2c_2 v_1 + c_1^2 = 0$ has solutions: $v_R = c_2 + \sqrt{(c_2^2 - c_1^2)}$ and $v_D = c_2 - \sqrt{(c_2^2 - c_1^2)}$

(In the plot I have multiplied c_1 by 4 for clarity.)

Thus our problem has solutions only for $v_1 < v_D$ or $v_1 > v_R$. Since $(c_1/c_2)^2 =$

$T_1/T_2 \ll 1$,

$$\begin{aligned} v_D &= c_2 - c_2\sqrt{\left(1-\left(\frac{c_1}{c_2}\right)^2\right)} \\ &\approx c_2\left(1-\left(1-\frac{1}{2}\left(\frac{c_1}{c_2}\right)^2\right)\right) = \frac{1}{2}\left(\frac{c_1}{c_2}\right)c_1 \ll c_1 \end{aligned}$$

In fact for our numbers,

$$v_D \sim \frac{1}{2}\frac{0.9}{13}0.9 \text{ km/s } = 3\times 10^{-2} \text{ km/s} \tag{66}$$

and

$$v_R \approx 2c_2 = 26 \text{ km/s} \tag{67}$$

Since $\rho v =$ constant, small v corresponds to large ρ. Therefore D stands for "dense" while R stands for "rare".

The forbidded region arises because the sound speeds are imposed by external conditions, unlike shocks, where jump conditions determine c_2 and all values of $v_1 > c_1$ are allowed. Also note that

$$v_R v_D = c_1^2$$

so that $v_R/c_1 > 1$ while $v_D/c_1 < 1$.

All $R-$type ionization fronts have $v_1 > v_R > c_1$ and thus the front moves supersonically. Conversely, $D-$type fronts have $v_1 < v_D < c_1$, and so the front moves subsonically.

Using equation 64 to solve for v_2, we get

$$\begin{aligned} v_{2R}v_{2D} &= \frac{\left(v_1^2+c_1^2\right)}{4v_1} - \frac{1}{4}\left(\left(v_1+\frac{c_1^2}{v_1}\right)^2 - 4c_2^2\right) \\ &= c_2^2 \end{aligned} \tag{68}$$

So again one solution is subsonic in the HII region, and the other is supersonic. The solution for v_2 is:

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{1}{2}\left\{1+\left(\frac{c_1}{v_1}\right)^2 \pm \sqrt{\left(1+\left(\frac{c_1}{v_1}\right)^2\right)^2 - 4\left(\frac{c_2}{v_1}\right)^2}\right\}$$

D-type front $\quad v_1 < c_1$.

With the plus sign, we find that ρ_1/ρ_2 is large since c_1/v_1 is and the square root is > 0. This is a *strong D-type* ionization front. Then

$$\frac{v_2}{c_2} = \frac{1}{2}\frac{v_1}{c_2}\left\{1+\left(\frac{c_1}{v_1}\right)^2 + \sqrt{\left(1+\left(\frac{c_1}{v_1}\right)^2\right)^2 - 4\left(\frac{c_2}{v_1}\right)^2}\right\}$$

Since $v_1 + \frac{c_1^2}{v_1} > 2c_2$ (equation 65)

$$\frac{v_2}{c_2} > 1$$

So with the strong $D-$type front, the flow is supersonic in the HII.

With the minus sign, ρ_1/ρ_2 is small. This is a *weak D-type* ionization front. By equation 68, the flow is subsonic in the HII.

R-type $v_1 > c_1$ Equation 67 shows that in fact we have $v_1 \gg c_1$, and also $v_1 > c_2$. Thus

$$\begin{aligned} \frac{v_2}{v_1} &= \frac{\rho_1}{\rho_2} \approx \frac{1}{2}\left\{1 \pm \sqrt{1-4\left(\frac{c_2}{v_1}\right)^2}\right\} \\ &= \frac{1}{2}\left(1 \pm \left(1-2\left(\frac{c_2}{v_1}\right)^2\right)\right) \end{aligned}$$

With the plus sign, we have

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = 1 - \left(\frac{c_2}{v_1}\right)^2 \approx 1$$

This is a *weak R-type* front. We also have

$$\frac{v_2}{c_2} = \frac{v_1}{c_2} - \frac{c_2}{v_1} > 1$$

so the flow is supersonic in the HII.

With the minus sign,

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \left(\frac{c_2}{v_1}\right)^2 < 1$$

This is the *strong R-type.*

$$\frac{v_2}{c_2} = \frac{c_2}{v_1} < 1$$

so in this case the flow is subsonic in the HII.

Summarizing, we have:

Type	sign	v_1/c_1	v_2/c_2
Weak R	$+$	> 1	> 1
Strong R	$-$	> 1	< 1
Weak D	$-$	< 1	< 1
Strong D	$+$	< 1	> 1

So we trace the evolution as follows.

Initially $v_1 > c_1$. The front is $R-$type, and we have already noted that gas will not be accelerated behind the front since it passes by rapidly. So it is a weak R. But $v_1 = dr_i/dt$ decreases with time (equation 61), and so approaches v_R. But v_1 cannot decrease below $v_{R.}$ Note that if $v_1 = v_R$, then $v_2 = c_2$. Signals can begin to propagate up to the front (v_2 was previously $> c_2$) bringing news of the high pressure behind. A pressure wave catches up to and passes through the ionization front, where it steepens into a shock.

Now the structure looks like:

HII	|	dense, hot HI	|	HI
$\leftarrow$	|	$\leftarrow$	|	$\leftarrow$
v_2	|	v_1	|	v_0
	ionization front		shock wave	

The front becomes weak $D-$type. This happens when formation of the Stromgren sphere is essentially complete.

So how is it that material still flows through the front? Notice that the number of particles inside the sphere is

$$N = \frac{4}{3}\pi r_s^3 n = \frac{\Phi}{\alpha n}$$

(equation 58). Thus N increases as n decreases, i.e. as the region expands. With a lower particle density, there are fewer recombinations. As expansion continues, the pressure in the interior falls and the shock weakens. The expansion slows.

The final phase To analyze this phase, assume

1. Uniform density ρ_{II} in the HII region
2. Constant temperature T_{II} in the HII region

Then as the region expands, let r be the coordinate of a particle within the region. The total mass interior to r remains constant as the region expands, so

$$\rho_{II} r^3 = \text{const}$$

Thus

$$\frac{1}{\rho_{II}}\frac{d\rho_{II}}{dt} = -\frac{3}{r}\frac{dr}{dt} = -3\frac{v_{II}}{r}$$

Right behind the front, $r = r_{i,}$

$$\frac{1}{\rho_{II}}\frac{d\rho_{II}}{dt} = -3\frac{v_{II}}{r_i} \tag{69}$$

Now from the definition of r_s,(equation 58),

$$n_{II}^2 r_i^3 = \text{ constant}$$

So

$$\frac{2}{\rho_{II}}\frac{d\rho_{II}}{dt} = -\frac{3}{r_i}\frac{dr_i}{dt} = -3\frac{v_i}{r_i} \tag{70}$$

Then comparing equations 69 and 70, we find

$$v_{II} = \frac{1}{2}v_i$$

Thus the speed with which matter flows through the front is

$$v_2 = v_i - v_{II} = \frac{1}{2}v_i$$

Now look at the jump conditions across the whole structure (shock plus ionization front). Neglect the small difference between v_s and v_i. Then $v_0 = v_{s.}$

$\leftarrow$	$\vert$	$\vert$	$\leftarrow$
v_2	$\vert$	$\vert$	v_0
ρ_{II}	$\vert$	$\vert$	ρ_I
	ionization front	shock	

The jump conditions are:

$$\rho_I \left(v_s^2 + c_I^2\right) = \rho_{II} \left(v_2^2 + c_{II}^2\right)$$

But $v_s \gg c_I$, so

$$v_s^2 \approx c_{II}^2 \frac{\rho_{II}}{\rho_I} \left(1 + \frac{1}{4}\left(\frac{v_s}{c_{II}}\right)^2\right)$$

$$v_s^2 = c_{II}^2 \frac{\rho_{II}}{\rho_I} \frac{1}{(1 - \rho_{II}/4\rho_I)} \tag{71}$$

Now since $\rho_{II}^2 r_i^3$ =const, and initially $\rho_{II} = \rho_I$, we set $\rho_{II}/\rho_I = (r_{i0}/r_i)^{3/2}$ where r_{i0} is a constant that is close to r_s. Then since $v_s \approx v_0 \approx v_i$, and $\rho_{II} \ll \rho_I$, equation 71 gives:

$$\frac{dr_i}{dt} \approx c_{II} \left(\frac{r_{i0}}{ri}\right)^{3/4}$$

and integrating gives:

$$\frac{4}{7} r_i^{7/4} = c_{II} r_{i0}^{3/4} t + \text{ constant}$$

or

$$\frac{r_i}{r_{i0}} = \left(1 + \frac{7}{4}\frac{c_{II}t}{r_{i0}}\right)^{4/7}$$

in the final phase, where t is measured from the time at which the shock forms. The expansion stops when $P_{II} = P_I$. But $\rho_{II}/\rho_I \approx \left(1 + \frac{7}{4}\frac{c_{II}t}{r_{i0}}\right)^{-6/7} \sim 0.1$ for $t =$ an O star's lifetime, and so $P_{II}/P_I = \rho_{II}T_{II}/\rho_I T_I \approx 10$. Thus the expansion never stops. Instead, ionization ceases and recombination ends the HII region's life.

Summary

1. Dynamical evolution of the HII region. Rapid expansion of ionization front (weak R-type) to $r_i \sim 0.999 r_s.$
2. Formation of shock ahead of ionization front and slow expansion into surrounding region. Front has become weak D-type.
3. Recombination

Notice that some of the UV energy (a few percent) ends up as kinetic energy of the ISM- the star is a heat engine!

2.5 Flow under the influence of gravity: accretion and winds

2.5.1 Supersonic flow past a gravitating object.

Recall the steady state version of the momentum equation (5):

$$\tilde{\mathbf{v}} \cdot \tilde{\boldsymbol{\nabla}} \tilde{\mathbf{v}} = -\frac{\vec{\nabla} P}{\rho} + \tilde{\mathbf{g}}$$

These terms are of order:

$$\frac{v^2}{L} \sim \frac{c_s^2}{L} + \tilde{\mathbf{g}}$$

So if $v^2 \gg c_s^2$, we can ignore the pressure term to first order. Then we have:

$$\tilde{\boldsymbol{\nabla}} \left(\frac{v^2}{2} + \phi_g \right) = 0$$

or total energy (kinetic plus potential) is conserved in the flow. The fluid flows like a set of non-interacting particles. Each fluid "particle" follows an orbit around the object. Assume a straight line orbit to get a rough order of magnitude estimate. The particle is captured by the body if

$$\frac{v^2}{2} - G\frac{M}{r} < 0$$

or

$$r < \frac{2GM}{v^2} = r_A$$

the *accretion radius*. Then the accretion rate is of order $\pi r_A^2 \rho v$.

More accurately, each fluid particle orbits around the gravitating body. On the back side, the orbits intersect (particles collide) and our non-interacting particle model fails. This implies that in the *fluid* a shock forms behind the mass M.

Digression - review of orbit theory. Use polar coordinates with origin at the mass M, and assume that the orbiting particle has much smaller mass.

Conservation of angular momentum:

$$r^2 \frac{d\theta}{dt} = \ell = \text{ constant = angular momentum per unit mass} \tag{72}$$

Conservation of energy:

$$\begin{aligned} E &= \frac{1}{2} m v^2 + V(r) = \text{ constant} \\ &= \frac{1}{2} m v^2 - \frac{GMm}{r} \end{aligned} \tag{73}$$

where

$$\tilde{\mathbf{v}} = \frac{dr}{dt} \hat{\mathbf{r}} + r \frac{d\theta}{dt} \hat{\boldsymbol{\theta}}$$

and so

$$v^2 = \left(\frac{dr}{dt} \right)^2 + \left(r \frac{d\theta}{dt} \right)^2 = \left(\frac{dr}{dt} \right)^2 + \frac{\ell^2}{r^2}$$

So equation 73 becomes:

$$\frac{1}{2}\left(\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{r^2}\right) - \frac{GM}{r} = \frac{E}{m} = \varepsilon \tag{74}$$

Now make a change of variable to $u = 1/r$. Then $r = 1/u$ and so

$$-\frac{1}{u^2}\frac{du}{dt} = \frac{dr}{dt}$$

Equation 74 becomes:

$$\frac{1}{u^4}\left(\frac{du}{dt}\right)^2 + \ell^2 u^2 - 2GMu = 2\varepsilon$$

Also from equation 72,

$$\frac{d}{dt} = \frac{\ell}{r^2}\frac{d}{d\theta} = \ell u^2 \frac{d}{d\theta}$$

Thus

$$\begin{aligned} \frac{1}{u^4}\left(\ell u^2 \frac{du}{d\theta}\right)^2 + \ell^2 u^2 - 2GMu &= 2\varepsilon \\ \left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2GM}{\ell^2}u &= 2\frac{\varepsilon}{\ell^2} \end{aligned}$$

Now let $y = u - \frac{GM}{\ell^2}$. The differential equation becomes:

$$\left(\frac{dy}{d\theta}\right)^2 + y^2 = 2\frac{\varepsilon}{\ell^2} + \left(\frac{GM}{\ell^2}\right)^2 = K$$

So

$$\frac{dy}{d\theta} = \sqrt{K - y^2}$$

Integrating, we get:

$$\int \frac{dy}{\sqrt{K - y^2}} = \theta$$

Thus

$$\theta = \theta_0 + \sin^{-1}\left(\frac{y}{K^{1/2}}\right)$$

and

$$y = K^{1/2}\sin(\theta - \theta_0)$$

Returning to our original variables:

$$\frac{1}{r} = \frac{GM}{\ell^2} + \sqrt{2\frac{\varepsilon}{\ell^2} + \left(\frac{GM}{\ell^2}\right)^2}\,\sin(\theta - \theta_0)$$

With a change of constant $\theta_a = \theta_0 - \pi/2$, we get

$$r = \frac{\ell^2/GM}{1 + \epsilon\cos(\theta - \theta_a)} = \frac{D}{1 + \epsilon\cos(\theta - \theta_a)}$$

where $D = \ell^2/GM$ and

$$\epsilon = \sqrt{2\frac{E}{m\ell^2}\left(\frac{\ell^2}{GM}\right)^2 + 1} = \sqrt{2\frac{E}{m}\left(\frac{\ell}{GM}\right)^2 + 1}$$

The accretion radius Returning to our original problem, the angular momentum per unit mass for our fluid "particle" at ∞ is $\ell = pV$, where p is the impact parameter, and $E = \frac{1}{2}mV^2$. Thus

$$D = \frac{p^2V^2}{GM}$$

$$\epsilon = \sqrt{\left(\frac{pV^2}{GM}\right)^2 + 1} = \sqrt{\left(\frac{D}{p}\right)^2 + 1}$$

The particle starts out at ∞ where $\theta = \pi$, so

$$\begin{aligned} 1 + \epsilon\cos(\pi - \theta_a) &= 0 \\ 1 - \epsilon\cos\theta_a &= 0 \Rightarrow \epsilon\cos\theta_a = 1 \end{aligned}$$

The particle reaches the axis ($\theta = 0$) on the downstream side at a distance d from the mass M, where

$$d = \frac{D}{1 + \epsilon\cos\theta_a} = \frac{D}{2}$$

At this point it collides with particles orbiting in the opposite direction, and the θ component of velocity is converted to thermal energy in the resulting shock. Only the radial component remains. The "particle" is captured if

$$\frac{1}{2}mv_r^2 - \frac{GMm}{d} < 0$$

But from energy conservation:

$$\begin{aligned} \frac{1}{2}m\left(v_r^2 + v_\theta^2\right) - \frac{GMm}{d} &= \frac{1}{2}mV^2 \\ \frac{1}{2}mv_r^2 - \frac{GMm}{d} &= \frac{1}{2}m\left(V^2 - v_\theta^2\right) \end{aligned}$$

and from angular momentum conservation:

$$dv_\theta = pV$$

Thus the capture condition becomes:

$$V^2\left(1 - \frac{p^2}{d^2}\right) < 0 \Rightarrow d < p$$

or

$$\frac{p^2V^2}{2GM} < p$$

$$p < \frac{2GM}{V^2} = R_A \tag{75}$$

All fluid with impact parameter less than R_A is ultimately accreted by the mass M. The accretion rate is:

$$\frac{dM}{dt} = \pi R_A^2 \rho_\infty V = \pi \rho_\infty V \frac{4\left(GM\right)^2}{V^4} = 4\pi\rho_\infty \frac{\left(GM\right)^2}{V^3} \tag{76}$$

Notice that the accretion rate decreases as V increases.

2.5.2 Subsonic accretion

Ref: Bondi *MNRAS* **112,** 195, 1952

We study this situation by first looking at the case in which the fluid is stationary at infinity. We also use the adiabatic equation of state, $P = k\rho^\gamma$. Then the governing equations are:

Conservation of mass:

$$\frac{dM}{dt} = 4\pi\rho r^2 v = \text{ accretion rate} \tag{77}$$

and the steady state momentum equation (5) in spherical coordinates:

$$\frac{d}{dr}\left(\frac{v^2}{2}\right) = -\frac{1}{\rho}\frac{dP}{dr} + g_r$$

where

$$g_r = -\frac{d\Phi}{dr}$$

The sound speed is:

$$c_s^2 = \frac{dP}{d\rho} = \gamma k \rho^{\gamma-1} \tag{78}$$

so

$$\begin{aligned} \frac{1}{\rho}\frac{dP}{dr} &= \frac{\gamma k \rho^{\gamma-1}}{\rho}\frac{d\rho}{dr} = \frac{d}{dr}\left(\frac{\gamma}{\gamma-1}k\rho^{\gamma-1}\right) = \frac{d}{dr}\left(\frac{c_s^2}{\gamma-1}\right) \\ &= \frac{c_\infty^2}{\gamma-1}\frac{d}{dr}\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} \end{aligned}$$

where the subscript ∞ denotes values at infinity. Then integrating the momentum equation gives Bernoulli's equation:

$$\frac{v^2}{2} + \frac{c_\infty^2}{\gamma-1}\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} = -\Phi\left(r\right) + \text{ constant}$$

At infinity, $v = 0$ and $\Phi = 0$, so:

$$\frac{v^2}{2} + \frac{c_\infty^2}{\gamma-1}\left[\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} - 1\right] = -\Phi\left(r\right) = \frac{GM}{r} \tag{79}$$

Now write the mach number $\mu = v/c_s$ and use the dimensionless radius variable

$\xi = r/\left(GM/c_\infty^2\right)$. Then equation 77 becomes:

$$\begin{aligned}\frac{dM}{dt} &= 4\pi\rho r^2 v = 4\pi\frac{\rho}{\rho_\infty}\xi^2\mu\frac{c_s}{c_\infty}\rho_\infty c_\infty\left(\frac{GM}{c_\infty^2}\right)^2 \\ &= 4\pi\lambda\rho_\infty\frac{(GM)^2}{c_\infty^3}\end{aligned} \tag{80}$$

where

$$\lambda = \xi^2\frac{c_s}{c_\infty}\frac{\rho}{\rho_\infty}\mu \tag{81}$$

is the dimensionless accretion rate. Now use equation 78 to eliminate the sound speed:

$$\lambda = \xi^2\left(\frac{\rho}{\rho_\infty}\right)^{\frac{\gamma+1}{2}}\mu$$

Now we use this relation to eliminate ρ from Bernoulli's equation (79:

$$\begin{aligned}\frac{1}{2}\frac{v^2}{c_s^2} + \left(\frac{c_\infty}{c_s}\right)^2\frac{1}{\gamma-1}\left[\left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} - 1\right] &= \frac{1}{\xi}\frac{c_\infty^2}{c_s^2} \\ \frac{1}{2}\mu^2 + \frac{1}{\gamma-1}\left[1 - \left(\frac{\rho}{\rho_\infty}\right)^{-(\gamma-1)}\right] &= \frac{1}{\xi}\left(\frac{\rho}{\rho_\infty}\right)^{-(\gamma-1)} \\ \frac{\mu^2}{2} + \frac{1}{\gamma-1}\left[1 - \left(\frac{\lambda}{\xi^2\mu}\right)^{-2(\gamma-1)/(\gamma+1)}\right] &= \frac{1}{\xi}\left(\frac{\lambda}{\xi^2\mu}\right)^{-2(\gamma-1)/(\gamma+1)}\end{aligned}$$

To simplify the notation, let $\alpha = 2(\gamma-1)/(\gamma+1)$. Then

$$\frac{\mu^2}{2} + \frac{1}{\gamma-1}\left[1 - \left(\frac{\lambda}{\xi^2\mu}\right)^{-\alpha}\right] = \xi^{2\alpha-1}\mu^\alpha\lambda^{-\alpha} \tag{82}$$

Now multiply by $\mu^{-\alpha}$ and collect the terms in μ on one side:

$$\frac{\mu^{2-\alpha}}{2} + \frac{\mu^{-\alpha}}{\gamma-1} = \lambda^{-\alpha}\left(\xi^{2\alpha-1} + \frac{\xi^{2\alpha}}{\gamma-1}\right)$$

This equation for μ could be solved numerically in a specific case, but let's see what we can learn about the general case. First notice that

$$2-\alpha = 2 - 2\frac{\gamma-1}{\gamma+1} = 2\frac{2}{\gamma+1} = \frac{4}{\gamma+1}$$

and

$$2\alpha - 1 = 4\frac{\gamma-1}{\gamma+1} - \frac{\gamma+1}{\gamma+1} = \frac{3\gamma-5}{\gamma+1}$$

Since $\gamma \leq 5/3$ always, then $2\alpha - 1$ is always ≤ 0, while $2-\alpha$ is always positive. So each function

$$F(\mu) = \frac{\mu^{2-\alpha}}{2} + \frac{\mu^{-\alpha}}{\gamma-1} \tag{83}$$

and

$$G(\xi) = \xi^{2\alpha-1} + \frac{\xi^{2\alpha}}{\gamma - 1} \tag{84}$$

has one positive power and one negative power. Thus each function has a maximum or minimum. At infinity, $\mu = 0$ and $F(\mu) \to \infty$. Also $\xi \to \infty$ and $G(\xi) \to \infty$. Therefore each function has a minimum at finite ξ.

Consider $F(\mu)$ first.

$$\begin{aligned}\frac{dF}{d\mu} &= \frac{2-\alpha}{2}\mu^{1-\alpha} - \frac{\alpha}{\gamma-1}\mu^{-\alpha-1} \\ &= \frac{2}{\gamma+1}\left(\mu^{1-\alpha} - \mu^{-\alpha-1}\right)\end{aligned}$$

which is zero at $\mu = 1$. Thus F is minimum at $\mu = 1$ and

$$F_{\min} = \frac{1}{2} + \frac{1}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)}$$

Now let's look at G :

$$\frac{dG}{d\xi} = (2\alpha - 1)\,\xi^{2(\alpha-1)} + 2\alpha\frac{\xi^{2\alpha-1}}{\gamma-1}$$

which equals zero for

$$\xi = \xi_m = -\frac{(2\alpha-1)(\gamma-1)}{2\alpha} = -\left(\frac{3\gamma-5}{\gamma+1}\right)\frac{(\gamma+1)(\gamma-1)}{4(\gamma-1)} = \frac{5-3\gamma}{4} \tag{85}$$

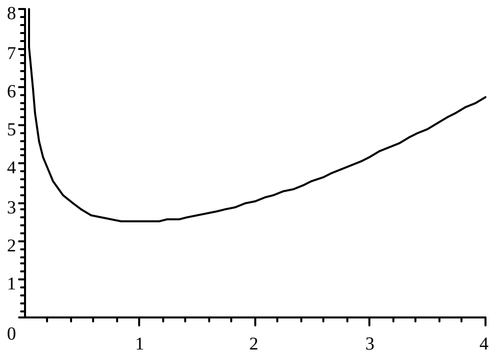
and then

$$\begin{aligned}G_{\min} &= \left(\frac{5-3\gamma}{4}\right)^{2\alpha}\left(\frac{4}{5-3\gamma} + \frac{1}{\gamma-1}\right) \\ &= \left(\frac{5-3\gamma}{4}\right)^{2\alpha}\frac{\gamma+1}{(5-3\gamma)(\gamma-1)} \\ &= \left(\frac{5-3\gamma}{4}\right)^{2\alpha-1}\frac{\gamma+1}{4(\gamma-1)}\end{aligned}$$

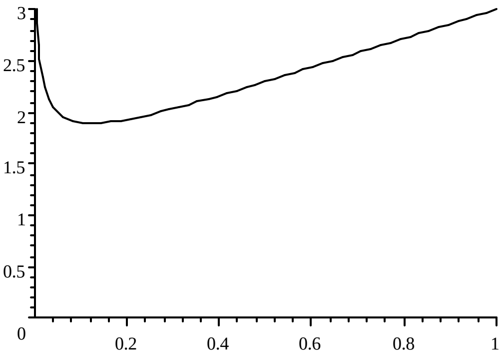
Topology of the solutions. Bernoulli's equation has been reduced to:

$$\lambda^{\alpha} F(\mu) = G(\xi) \tag{86}$$

Let's look at the functions F and G. (The plots are drawn for $\gamma = 3/2$.)



F



G

Now define $\lambda_c^\alpha = G_{\text{min}}/F_{\text{min}}$. Then

(a) If $\lambda > \lambda_c$,then $G_{\min} < \lambda^\alpha F_{\text{min}}$. There is a region of ξ for which the RHS of equation 86 is less than F_{min}, which is impossible.

(b) If $\lambda < \lambda_c$, then $\lambda^{-\alpha} G > F_{\text{min}}$ for all ξ. The solutions are always supersonic or always subsonic.

(c) If $\lambda = \lambda_c$, then $G = G_{\min}$ coincides with $F = F_{\text{min}}$. Thus the flow goes through a sonic point $(\mu = 1)$ at $\xi = \xi_m = \frac{1}{4}(5 - 3\gamma)$. A solution is possible which passes smoothly from subsonic to supersonic or vice-versa. For the accretion problem, the boundary condition is $\mu \to 0$ as $\xi \to \infty$. Case (b) which is always subsonic requires large pressure gradients, all the way out to infinity, to oppose gravity and slow the flow. This is a *settling solution.* Thus solution (c) is the usual accretion solution.

$$
\begin{aligned}
\lambda_c &= \left(\frac{G_{\min}}{F_{\min}}\right)^{1/\alpha} \\
&= \left[\left(\frac{5-3\gamma}{4}\right)^{2\alpha-1} \frac{\gamma+1}{4\,(\gamma-1)} \frac{2\,(\gamma-1)}{(\gamma+1)}\right]^{1/\alpha} \\
&= \left[\left(\frac{5-3\gamma}{4}\right)^{2\alpha-1} \frac{1}{2}\right]^{1/\alpha} \\
&= \left(\frac{5-3\gamma}{4}\right)^{2-1/\alpha} \frac{1}{2^{1/\alpha}}
\end{aligned}
$$

where

$$
2-\frac{1}{\alpha}=2-\frac{\gamma+1}{2\,(\gamma-1)}=\frac{3\gamma-5}{2\,(\gamma-1)}
$$

Thus

$$
\begin{aligned}
\lambda_c &= \left(\frac{5-3\gamma}{4}\right)^{(3\gamma-5)/2(\gamma-1)} 2^{-(\gamma+1)/2(\gamma-1)} \\
&= \left[\left(\frac{5-3\gamma}{4}\right)^{(3\gamma-5)} 2^{-(\gamma+1)}\right]^{1/2(\gamma-1)} \qquad (87)
\end{aligned}
$$

Then the accretion rate is:

$$
\frac{dM}{dt}=4\pi\rho_\infty\frac{(GM)^2}{c_\infty^3}\lambda_c \qquad (88)
$$

The value of λ_c varies from 0.87 at $\gamma = 1.2$ to 0.5 at $\gamma = 1.5$. Notice that the two values $\gamma = 5/3$ and $\gamma = 1$ cause trouble. Let's look at them.

Special case: $\gamma = 1$. We have to go back to the original derivation. In this case the sound speed is constant, and the equations are simpler:

$$
\frac{1}{\rho}\frac{dP}{dr}=c_s^2\frac{1}{\rho}\frac{d\rho}{dr}=c_s^2\frac{d}{dr}\ln\rho
$$

so that Bernoulli's equation becomes:

$$
\frac{v^2}{2}+c_s^2\ln\rho-\frac{GM}{r}=c_s^2\ln\rho_\infty
$$

and

$$
\lambda=\xi^2\frac{\rho}{\rho_\infty}\mu
$$

Thus

$$
\begin{aligned}
\frac{\mu^2}{2}+\ln\left(\frac{\lambda}{\mu\xi^2}\right) &= \frac{1}{\xi} \\
\frac{\mu^2}{2}-\ln\mu &= \frac{1}{\xi}+2\ln\xi-\ln\lambda
\end{aligned}
$$

In this case

$$F(\mu) = \frac{\mu^2}{2} - \ln \mu$$

has a minimum when

$$\mu - \frac{1}{\mu} = 0$$

or $\mu = 1$, as before. The minimum value is $F_{\min} = 1/2$.

$$G(\xi) = \frac{1}{\xi} + 2 \ln \xi$$

has a minimum for

$$\frac{-1}{\xi^2} + \frac{2}{\xi} = 0$$

or $\xi = 1/2$. The minimum value is

$$G_{\min} = 2 - 2 \ln 2$$

So λ_c is given by:

$$\begin{aligned} \ln \lambda_c &= G_{\min} - F_{\min} = -\ln 4 + 3/2 \\ \lambda_c &= \frac{1}{4} e^{3/2} \text{ for } \gamma = 1 \\ &= 1.1204 \end{aligned}$$

Special case: $\gamma = 5/3$ According to equation 85, $\xi_m = 0$ when $\gamma = 5/3$. The function has its minimum at the accreting object. This means that the flow never becomes supersonic. Then $\gamma - 1 = 2/3$, $\gamma + 1 = 8/3$, and with $5 - 3\gamma = \varepsilon$, equation 87 becomes:

$$\lambda_c = \lim_{\varepsilon \to 0} \left(\frac{\varepsilon}{4} \right)^{-3\varepsilon/4} 2^{-2} = \frac{1}{4} \lim_{\varepsilon' \to 0} \varepsilon'^{-3\varepsilon'}$$

where $\varepsilon' = \varepsilon/4$. Now let $z = \varepsilon'^{-3\varepsilon'}$. Then

$$\ln z = -3\varepsilon' \ln \varepsilon' \to 0 \text{ as } \varepsilon' \to 0$$

since the lograithm $\to \infty$ slower than any algebraic power. Then $z \to 1$ and $\lambda_c = 1/4$ for $\gamma = 5/3$.

Transitions from one branch of the solutions to another can occur. For example, when the flow goes supersonic, information about the solid object upstream cannot be transmited back through the fluid. A shock wave forms and flow makes a transition to the subsonic, settling solution. Flow cannot become supersonic again behind the shock because pressure gradients slow the flow.

2.5.3 Transonic accretion

Ref: Hunt, MNRAS,**154,** 141,1971

Let's summarize our results so far. For supersonic accretion, we have (equation 76)

$$\frac{dM}{dt} = 4\pi\rho_\infty \frac{(GM)^2}{V^3}$$

while in the subsonic case (equation 88)

$$\frac{dM}{dt} = 4\pi\rho_\infty \frac{(GM)^2}{c_\infty^3}\lambda_c; \ \frac{1}{4} < \lambda_c < 1.12$$

Thus we might guess that the transonic case must be of the form:

$$\frac{dM}{dt} = 4\pi\rho_\infty \frac{(GM)^2}{(V^2 + c_\infty^2)^{3/2}}\lambda$$

where λ is a constant of order 1. Hunt's numerical calculations confirm this guess. See his paper for details.

2.5.4 Applications

Neutron stars in binaries with stellar winds The neutron star moves through the wind as it orbits its stellar companion. The semi-major axis of the binary is not much larger than the primary star's radius, about 10^{12} cm. The orbital period is of the order of a few days, and thus the orbital speed is about

$$v_{\rm orb} \sim \frac{2\pi r}{T} = \frac{2\pi \times 10^{12}}{T_{\rm days} \times 24 \times 60 \times 60}\ \text{cm/s} = \frac{7 \times 10^7}{T_{\rm days}}\ \text{cm/s}$$

The wind speed is about 600-1000 km/s for an O-B type star. The temperature is regulated by the x-ray source itself via Compton/inverse Compton cooling/heating, so we expect $T \sim T_X \sim$ a few $\times 10^7$ K. Then the accretion radius is:

$$R_A \sim \frac{2GM_X}{v_{\rm rel}^2 (1 + \mu^{-2})} = 2.7 \times 10^{10} \frac{M_{X,\odot}}{V_8^2 (1 + \mu^{-2})}\ \text{cm}$$

where

$$V_{\rm rel}^2 = V_{\rm wind}^2 + V_{\rm orb}^2$$

$$\frac{dM_X}{dt} = \pi R_A^2 \rho_{\rm wind} V_{\rm rel}$$

is the mass accretion rate onto the x-ray source,

$$\frac{dM_*}{dt} = 4\pi R_x^2 \rho_{\rm wind} V_{\rm wind}$$

is the mass loss rate by the star, R_x is the radius of the neutron star orbit, and

$$\begin{aligned} \frac{M_X}{M_*} &= \frac{1}{4}\left(\frac{R_A}{R_x}\right)^2 \frac{V_{\rm rel}}{V_{\rm wind}} \sim \frac{\left(2.7\times 10^{-2}\right)^2}{4} \frac{M_{X,\odot}^2}{V_8^4} \frac{V_{\rm rel}}{V_{\rm wind}\left(1+\mu^{-2}\right)^2 R_{x,12}^2} \\ &\sim 1.8\times 10^{-4} \frac{M_{X,\odot}^2}{V_8^4} \frac{V_{\rm rel}}{V_{\rm wind}\left(1+\mu^{-2}\right)^2 R_{x,12}^2} \end{aligned}$$

Note the very strong dependence on the wind speed. If the stellar mass loss rate is about 3×10^{-6} $\mathrm{M}_\odot$/y, then the mass accretion rate onto the neutron star is

$$\frac{dM_X}{dt} = \frac{5.4\times 10^{-10}}{V_8^4}\ \mathrm{M}_\odot/\mathrm{y}$$

and the x-ray luminosity is then

$$\begin{aligned} L_x &\sim \frac{GM_x}{R_{\text{neutron star}}} \frac{dM_X}{dt} \sim 0.1c^2 \frac{dM_X}{dt} \\ &= \frac{3\times 10^{36}}{V_8^4}\ \text{erg/s} \end{aligned}$$

which is consistent with observed values.

2.5.5 The solar wind

Ref: Parker: *Interplanetary Dynamical Processes*

In our previous analysis, the fluid velocity appears only as v^2 (or μ^2), so v_r can have either sign - i.e. the solutions work for winds as well as for accretion. We have to modify the analysis to allow for a finite wind speed at *infinity (*or at some reference point). So we rewrite equation 79 using new variables

$$u = \frac{v}{c_a}$$

and

$$\xi = r/a$$

where $r = a$ is the reference point and c_a is the sound speed at that point. The equation becomes:

$$\begin{aligned} \frac{v^2}{2c_a^2} + \frac{1}{\gamma-1}\left(\frac{\rho}{\rho_a}\right)^{\gamma-1} - \frac{GM}{c_a^2 r} &= \frac{v_a^2}{2c_a^2} + \frac{1}{\gamma-1} - \frac{GM}{c_a^2 a} \\ \frac{u^2}{2} + \frac{1}{\gamma-1}\left(\frac{\rho}{\rho_a}\right)^{\gamma-1} - \frac{GM}{c_a^2 a}\frac{a}{r} &= \frac{u_a^2}{2} + \frac{1}{\gamma-1} - \frac{GM}{c_a^2 a} \end{aligned}$$

and from the continuity equation, we have:

$$\begin{aligned} r^2 \rho v &= a^2 \rho_a v_a \\ \frac{\rho}{\rho_a} &= \frac{1}{\xi^2}\frac{u_a}{u} \end{aligned}$$

Next write $GM/c_a^2 a = H$, and $u_1^2 \equiv u_a^2 + 2/(\gamma-1) - 2H$. Then Bernoulli's equation becomes:

$$u^2 + \frac{2}{\gamma-1}\left(\frac{u_a}{u\xi^2}\right)^{\gamma-1} - \frac{2H}{\xi} = u_1^2 \tag{89}$$

Solution at large ξ If $u \ll 1$ at large ξ, then we may ignore the term in u^2 to get:

$$\left(\frac{u_a}{u\xi^2}\right)^{\gamma-1} \approx \frac{\gamma-1}{2} u_1^2 \left(1 + \frac{2H}{\xi u_1^2}\right)$$

Taking the root:

$$\begin{aligned} \frac{u\xi^2}{u_a} &= \left(\frac{2}{(\gamma-1)\, u_1^2}\right)^{1/(\gamma-1)} \frac{1}{\left(1+\frac{2H}{\xi u_1^2}\right)^{1/(\gamma-1)}} \\ u &\approx \frac{u_a}{\xi^2}\left(\frac{2}{(\gamma-1)\, u_1^2}\right)^{1/(\gamma-1)} \left(1 - \frac{2H}{(\gamma-1)\,\xi u_1^2}\right) \end{aligned} \tag{90}$$

Whereas if u remains $\gtrsim 1$, we have

$$\begin{aligned} u^2 &= u_1^2\left(1 + \frac{2H}{\xi u_1^2} - \frac{2}{\gamma-1}\left(\frac{u_a}{u\xi^2}\right)^{\gamma-1}\frac{1}{u_1^2}\right) \\ u &\approx u_1\left(1 + \frac{H}{\xi u_1^2} - \frac{1}{\gamma-1}\left(\frac{u_a}{u\xi^2}\right)^{\gamma-1}\frac{1}{u_1^2}\right) \end{aligned} \tag{91}$$

These two solutions correspond to the upper and lower branches that we found in the accretion problem. (See Figure on page). Notice that on the upper branch, $u \to u_1$ as $\xi \to \infty$ (for $\gamma > 1$). The wind reaches a *terminal velocity.* Because energy is conserved, thermal energy in the inner regions is converted to kinetic energy in the outer regions. The flow becomes cold and supersonic at infinity. (See below.)

Solutions at small ξ.

Lower branch If $u \to 0$ as $\xi \to 0$, then equation 89 becomes:

$$\begin{aligned}
\frac{2}{\gamma-1}\left(\frac{u_a}{u\xi^2}\right)^{\gamma-1} &= \frac{2H}{\xi}+u_1^2=\frac{2H}{\xi}\left(1+\frac{u_1^2\xi}{2H}\right) \\
u &= \frac{u_a}{\xi^2}\left(\frac{\xi}{(\gamma-1)\,H}\right)^{1/(\gamma-1)}\left(1+\frac{u_1^2\xi}{2H}\right)^{-1/(\gamma-1)} \\
&\approx \xi^{1/(\gamma-1)-2}\frac{u_a}{[(\gamma-1)\,H]^{1/(\gamma-1)}}\left(1-\frac{1}{\gamma-1}\frac{u_1^2\xi}{2H}\right) \qquad (92)
\end{aligned}$$

This is self consistent (i.e. $u \to 0$) only if

$$\begin{aligned}
\frac{1}{\gamma-1}-2 &> 0 \\
\gamma-1 &< \frac{1}{2} \\
\gamma &< \frac{3}{2} \qquad (93)
\end{aligned}$$

Upper branch On the other branch, (u not small), we would have:

$$\begin{aligned}
u^2 &= \frac{2H}{\xi}+u_1^2-\frac{2}{\gamma-1}\left(\frac{u_a}{u\xi^2}\right)^{\gamma-1} \\
u &= \sqrt{2}\left(\frac{H}{\xi}\right)^{1/2}\left(1+\frac{u_1^2\xi}{2H}-\frac{\xi}{(\gamma-1)\,H}\left(\frac{u_a}{u\xi^2}\right)^{\gamma-1}\right)^{1/2} \\
&\approx \sqrt{2}\left(\frac{H}{\xi}\right)^{1/2}\left(1+\frac{u_1^2\xi}{4H}-\frac{\xi^{1-2\gamma+2}}{2\,(\gamma-1)\,H}\left(\frac{u_a}{u}\right)^{\gamma-1}\right) \\
&= \sqrt{2}\left(\frac{H}{\xi}\right)^{1/2}\left(1+\frac{u_1^2\xi}{4H}-\frac{\xi^{3-2\gamma}}{2\,(\gamma-1)\,H}\left(\frac{u_a}{u}\right)^{\gamma-1}\right)
\end{aligned}$$

We can simplify the last term by using the zeroth order term, $u \approx \sqrt{2}\left(\frac{H}{\xi}\right)^{1/2}$. Then:

$$\begin{aligned}
u &\approx \sqrt{2}\left(\frac{H}{\xi}\right)^{1/2}\left(1+\frac{u_1^2\xi}{4H}-\frac{\xi^{3-2\gamma+(\gamma-1)/2}}{2\,(\gamma-1)\,H}\left(\frac{u_a^2}{2H}\right)^{\frac{\gamma-1}{2}}\right) \\
&= \sqrt{2}\left(\frac{H}{\xi}\right)^{1/2}\left(1+\frac{u_1^2\xi}{4H}-\frac{\xi^{\frac{5}{2}-\frac{3}{2}\gamma}}{2\,(\gamma-1)\,H}\left(\frac{u_a^2}{2H}\right)^{\frac{\gamma-1}{2}}\right) \qquad (94)
\end{aligned}$$

which shows that $u \sim \xi^{-1/2}$ as $\xi \to 0$. This analysis is valid only if the power of ξ in the last term inside the parentheses is greater than zero, or $\gamma < 5/3$. Otherwise our series expansion does not converge.

The solar breeze. The case $u_1 \equiv 0$ and $\gamma = 5/3$ is a special case. For these values of the parameters, equation 89 becomes:

$$u^2 + 3\left(\frac{u_a}{u\xi^2}\right)^{2/3} - \frac{2H}{\xi} = 0$$

and the condtion $u_1 \equiv 0$ is

$$\frac{u_a^2}{2} + 1/(\gamma - 1) - H = 0 \Rightarrow H = \frac{u_a^2}{2} + \frac{3}{2}$$

Thus we have:

$$u^2 + 3\left(\frac{u_a}{u\xi^2}\right)^{2/3} - \frac{u_a^2}{\xi} - \frac{3}{\xi} = 0$$

This equation has a solution

$$u = \frac{u_a}{\xi^{1/2}} \tag{95}$$

which is called the *solar breeze*.

Boundary conditions at infinity So which solution describes the actual solar wind? We want a solution with $P \to 0$ as $\xi \to \infty$. From the continuity equation (77),

$$\rho = \frac{\rho_a u_a}{u\xi^2}$$

On the upper branch, $u \to u_1$, and so

$$\rho \to \frac{\rho_a u_a}{u_1 \xi^2} \to 0$$

and therefore $P \to 0$ as well.

On the lower branch

$$\begin{aligned} \rho &= \frac{\rho_a u_a}{\xi^2 \frac{u_a}{\xi^2}\left(\frac{2}{(\gamma-1)u_1^2}\right)^{1/(\gamma-1)}} \\ &= \rho_a \left(\frac{(\gamma-1)\,u_1^2}{2}\right)^{1/\gamma-1} \end{aligned}$$

and therefore

$$P = P_a\left(\frac{\rho}{\rho_a}\right)^{\gamma} = P_a\left(\frac{(\gamma-1)\,u_1^2}{2}\right)^{\gamma/\gamma-1}$$

is comparable to P_a.

For the solar breeze,

$$\rho = \frac{\rho_a}{\xi^{3/2}} \to 0$$

Thus the upper branch and the breeze are the only allowable solutions. But the breeze is a singular case, requiring very special values of the parameters.

Mach number Now let's look at the Mach number.

$$\mu = \frac{v}{c_s} = u\frac{c_a}{c_s} = u\left(\frac{\rho_a}{\rho}\right)^{(\gamma-1)/2}$$

Then for our various solutions we have:

1. Large ξ

- Upper branch:

$$\mu \to u_1 \left(\frac{u_1 \xi^2}{u_a}\right)^{(\gamma-1)/2} \to \infty \text{ as } \xi \to \infty$$

  Thus this solution corresponds to a cold flow.

- Lower branch:

$$\begin{aligned} \mu &\approx \frac{u_a}{\xi^2}\left(\frac{2}{(\gamma-1)\,u_1^2}\right)^{1/(\gamma-1)}\left(\frac{2}{(\gamma-1)\,u_1^2}\right)^{1/2} \\ &= \frac{u_a}{\xi^2}\left(\frac{2}{(\gamma-1)\,u_1^2}\right)^{\frac{1}{2}\frac{1+\gamma}{\gamma-1}} \to 0 \text{ as } \xi \to \infty \end{aligned}$$

  for all values of γ.

- Solar breeze:

$$\mu = \frac{u_a}{\xi^{1/2}}\xi^{3/2} = u_a \xi \to \infty \text{ as } \xi \to \infty \tag{96}$$

2Small ξ

In general:

$$\frac{\rho}{\rho_a} = \frac{u_a}{u\xi^2}$$

So

$$\mu = u\left(\frac{\rho_a}{\rho}\right)^{(\gamma-1)/2} = u\left(\frac{u\xi^2}{u_a}\right)^{(\gamma-1)/2} = u_a\left(\frac{u}{u_a}\right)^{1+(\gamma-1)/2}\xi^{\gamma-1} = u_a\left(\frac{u}{u_a}\right)^{\frac{\gamma+1}{2}}\xi^{\gamma-1} \tag{97}$$

- Lower branch:

  Here we use equation equation (92) to get:

$$\begin{aligned} \mu &= u_a\left(\frac{\xi^{(3-2\gamma)/(\gamma-1)}}{[(\gamma-1)\,H]^{1/(\gamma-1)}}\right)^{\frac{\gamma+1}{2}}\xi^{\gamma-1} \\ &= \frac{u_a}{[(\gamma-1)\,H]^{(\gamma+1)/2(\gamma-1)}}\xi^{\frac{(3-2\gamma)(\gamma+1)}{2(\gamma-1)}}\xi^{\gamma-1} \\ &= \frac{u_a}{[(\gamma-1)\,H]^{(\gamma+1)/2(\gamma-1)}}\xi^{\frac{(3-2\gamma)(\gamma+1)+2(\gamma-1)^2}{2(\gamma-1)}} \end{aligned}$$

The power of ξ is:

$$\frac{(3-2\gamma)(\gamma+1)+2(\gamma-1)^2}{2(\gamma-1)} = \frac{5-3\gamma}{2(\gamma-1)}$$

which is always positive if $\gamma < 5/3$. So $\mu \to 0$ as $\xi \to 0$ unless $\gamma = 5/3$, in which case μ approaches a constant value. Notice that there is a problem with $\gamma = 1$, but we have already noted similar problems above. This branch exists only for $\gamma < 1.5$.

- Upper branch:

Here we use equation (94) in equation (97) to get:

$$\begin{aligned}\mu &= u_a \left(\frac{\sqrt{2}}{u_a} \left(\frac{H}{\xi} \right)^{1/2} \right)^{\frac{\gamma+1}{2}} \xi^{\gamma-1} = u_a \left(\frac{\sqrt{2H}}{u_a} \right)^{\frac{\gamma+1}{2}} \xi^{\gamma-1-\frac{\gamma+1}{4}} \\ &= u_a \left(\frac{\sqrt{2H}}{u_a} \right)^{\frac{\gamma+1}{2}} \xi^{(3\gamma-5)/4} \to \infty \text{ as } \xi \to 0 \text{ for } \gamma < 5/3\end{aligned}$$

We have previously noted that this branch exists only for $\gamma < 5/3$.

- Solar breeze:

From equation 96, $\mu \to 0$ as $\xi \to 0$. For the breeze, we have the interesting situation that the lower branch in u is the upper branch in μ, and vice versa.

Thus we have the following situation:

quantity	**at infinity**		**as** $\xi \to 0$		**so**
	upper branch	lower branch	upper branch	lower branch	
speed	$\frac{u_a}{\xi^2}\left(\frac{2}{(\gamma-1)u_1^2}\right)^{\frac{1}{\gamma-1}}$	u_1	$\left(\frac{2H}{\xi}\right)^{1/2}$	$\xi^{\frac{3-2\gamma}{\gamma-1}} \frac{u_a}{[(\gamma-1)H]^{1/(\gamma-1)}}$	
pressure	$\frac{\rho_a u_a}{u_1 \xi^2}$	$P_a \left(\frac{(\gamma-1)u_1^2}{2}\right)^{\frac{\gamma}{\gg-1}}$			
Mach number	$u_1 \left(\frac{u_1 \xi^2}{u_a}\right)^{\frac{\gamma-1}{2}}$	$\frac{u_a}{\xi^2}\left(\frac{2}{(\gamma-1)u_1^2}\right)^{\frac{1+\gamma}{2(\gamma-1)}}$	$u_a \left(\frac{\sqrt{2H}}{u_a}\right)^{\frac{\gamma+1}{2}} \xi^{\frac{3\gamma-5}{4}}$	$\frac{u_a}{[(\gamma-1)H]^{\frac{\gamma+1}{2(\gamma-1)}}} \xi^{\frac{5-3\gamma}{2(\gamma-1)}}$	

Parker argues that the solution through the sonic point is always set up. His argument goes like this: Assume first that there is a high pressure at infinity, so that we are on the lower branch of the solution. Now slowly decrease P_∞. The wind will accelerate and u will increase. But P_∞ is independent of u_a, so the flow will continue to accelerate until it switches to the upper branch through the sonic point. And then we have the critical solution.

What is the significance of the constraint $\gamma < 1.5$ for the existance of the lower branch solution $u \to 0$ as $\xi \to 0$? Physically, it means that the gas must be heated as it expands. The critical solution is analagous to the expansion of a gas through a Laval nozzle. Gravity acts like the throat of the nozzle.

For the upper branch to exist we need $u_1^2 > 0$, i.e.

$$u_a^2 + \frac{2}{\gamma - 1} - \frac{2GM}{c_a^2 a} > 0$$

or

$$\frac{GM}{c_a^2 a} < \frac{u_a^2}{2} + \frac{1}{\gamma - 1} \tag{98}$$

The gravitational field must not be too strong or the gas will form a static atmosphere rather than a wind.

To find another constraint, we differentiate equation 89:

$$\begin{aligned} 2u\frac{du}{d\xi} - 4\left(\frac{u_a}{u}\right)^{\gamma-1}\xi^{-2\gamma+1} - 2\left(\frac{u_a}{\xi^2}\right)^{\gamma-1}u^{-\gamma}\frac{du}{d\xi} + \frac{2H}{\xi^2} &= 0 \\ \frac{du}{d\xi}\left(u - \left(\frac{u_a}{\xi^2}\right)^{\gamma-1}u^{-\gamma}\right) &= 2\left(\frac{u_a}{u}\right)^{\gamma-1}\xi^{-2\gamma+1} - \frac{H}{\xi^2} \end{aligned}$$

We want the velocity to be increasing outward at the reference point $\xi = 1$. At $\xi = 1$ the term in parentheses on the LHS is

$$u_a - \frac{1}{u_a}$$

which is < 0 for $u_a < 1$. Then the RHS must also be < 0, which means we need $H > 2$. This is the nozzle condition. From equation (98) with $u_a \ll 1$, we find:

$$2 < H < \frac{1}{\gamma - 1}$$

which gives the same condition (93) that we have previously found for γ.

Thermal conductivity is important in the solar wind. Heat is conducted up from the corona and keeps γ close to 1 in the inner regions, easily satisfying this constraint.

2.5.6 Radiation pressure driven winds

Now we add to our theory the interaction of the fluid with the radiation from the star (or the accreting object). For winds, this modification is important when discussing luminous stars such as O and B stars. To get the basic idea, we make some simplifying assumptions (eg Castor, Abbot and Klein, Ap.J. **195**, 160, 1975)

Radiation pressure is greater than electron scattering pressure because of absorption in spectral lines due to atomic and molecular transitions. For electron scattering only, we would have a radiation pressure force

$$F_p = \sigma_T n_e \frac{flux}{c} = \frac{\sigma_T L\rho}{4\pi r^2 cm}$$

The mass m is one half the proton mass if the fluid is ionized hydrogen. So we write the total radiation pressure force as:

$$F_p = \frac{\sigma_T L \rho}{4\pi r^2 cm} \left(1 + M\left(t\right)\right)$$

where $M\left(t\right) \propto t^{-\alpha}$ is the effect of the lines, and t is the optical depth. (CAK take $M \sim t^{-0.7}/30$). $M \to 0$ as $t \to \infty$ because all the lines become optically thick and hence less efficient at absorbing radiation.

When this extra force is included in the momentum equation, we get:

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM_*}{r^2} + \frac{\sigma_T L}{4\pi r^2 cm}\left(1 + M\right)$$

The temperature at each point in the wind is assumed to be given by the local energy balance at that point which is dominated by the radiation (heating and cooling). The timescale to reach thermal equilibrium is short compared with the flow time scale. Thus we take

$$P = \rho c_s^2$$

where c_s^2 is the local, isothermal sound speed, and is presumed known. Then:

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{c_s^2}{\rho}\frac{d\rho}{dr} + \frac{dc_s^2}{dr}$$

and as usual

$$\rho = \frac{dM/dt}{4\pi r^2 v} \tag{99}$$

Then

$$\frac{1}{\rho}\frac{d\rho}{dr} = -\frac{2}{r} - \frac{1}{v}\frac{dv}{dr}$$

and so

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{2}{r}c_s^2 - \frac{c_s^2}{v}\frac{dv}{dr} + \frac{dc_s^2}{dr}$$

Now write

$$\Gamma = \frac{\sigma_T L}{4\pi c G M_* m}$$

so that the momentum equation becomes:

$$\left(v - \frac{c_s^2}{v}\right)\frac{dv}{dr} = \frac{2}{r}c_s^2 - \frac{dc_s^2}{dr} - \frac{GM_*}{r^2}\left(1 - \Gamma - \Gamma M\right)$$

If $M = 0$, we have the previous equation with a reduced mass $M_{\text{eff}} = M_*\left(1 - \Gamma\right)$. If $\Gamma = 1$, $M_{\text{eff}} = 0$ and there is no effective gravitational force!

The optical depth depends on the wind speed gradient, since Doppler shift moves material into or out of the optically thick region near the line center.

$$t\left(\nu_0\right) = \frac{\sigma_L}{\kappa_{L,0}}\int_0^r \rho\kappa\left(\nu\right) dr$$

where σ_L is the total cross section for absorption in the line, ρ is given by equation 99 and

$$\kappa\left(\nu\right) = \kappa_{L,0}\frac{v_{th}}{\left|dv/dr\right|}$$

In this expression $\kappa_{L,0}$ is the absorption coefficient at the line center, v_{th} is the thermal velocity width of the line and dv/dr is the speed gradient in the wind. Thus $v_{th}/\left|dv/dr\right|$ is the distance before the frequency is Doppler shifted out of the line. There is one such term for each relevant line. Then:

$$\begin{aligned} t &\approx \frac{\rho\sigma_L v_{th}}{\left|dv/dr\right|} \\ &= \frac{dM/dt}{4\pi r^2 v}\frac{\sigma_L v_{th}}{\left|dv/dr\right|} \end{aligned}$$

and so

$$M = kt^{-\alpha} = k\left(\frac{dM/dt}{4\pi}\sigma_L v_{th}\right)^{-\alpha}\left(r^2 v\frac{dv}{dr}\right)^{\alpha}$$

Now define the variables

$$w = \frac{1}{2}v^2$$

$$u = -\frac{1}{r}$$

$$h\left(u\right) = -GM_*\left(1-\Gamma\right) + 2c_s^2 r - r^2\frac{dc_s^2}{dr} = \text{ a known function of } r$$

$$C = \Gamma GM_* k\left(\frac{4\pi}{\sigma_L v_{th} dM/dt}\right)^{\alpha}$$

Then Bernoulli's equation takes the form:

$$F\left(u,w,w'\right) \equiv \left(1-\frac{c_s^2}{2w}\right)w' - h\left(u\right) - C\left(w'\right)^{\alpha}$$

where $w' \equiv dw/du$. As usual we look for a solution that goes from 0 at $u = -\infty$ $(r = 0)$ to a large value at $u = 0$ $(r \to \infty)$. It turns out that there are *two* critical points. One is the usual sonic point, and one where

$$w = \frac{c_s^2}{2} - \frac{\alpha}{1-\alpha}h\left\{\text{messy function of } c_s^2 \text{ and } h\right\}$$

As $a \to 0$, this point approaches the sonic point. For $h < 0$ (which is always true) the Mach number is greater than 1 at this 2nd critical point.

The wind density must be below a maximum value. If the density is too high, the wind gets optically thick too soon and there is not enough pressure at high altitude to drive the wind.

Accretion flows are similarly affected by radiation pressure. With $\Gamma = 1$, the accretion is stopped. This is the *Eddington limit.*

$$\begin{aligned} L_{\text{Edd}} &= \frac{4\pi cGM_* m}{\sigma_T} \\ &= \frac{4\pi\left(3\times10^{10}\text{ cm/s}\right)\left(6.67\times10^{-8}\text{ cm}^3/\text{g}\cdot\text{s}^2\right)\frac{1}{2}\left(1.67\times10^{-24}\text{g}\right)}{6.6\times10^{-25}\text{ cm}^2}\left(2\times10^{33}\text{ g}\right)\frac{M_*}{M_\odot} \\ &= 6.4\times10^{37}\frac{M_*}{M_\odot}\text{ erg/s} \end{aligned}$$

Steady state spherical accretion is not possible above the Eddington limit.

3 Accretion disks

Reference: Pringle, Annual Reviews of Astronomy and Astrophysics, **19,** 137, 1981

3.1 Introduction

When the accreting material has high angular momentum, we get an accretion disk. Material *must* orbit the gravitating object, and cannot move in arbitrarily close without losing angular momentum. If the angular momentum per unit mass is ℓ, then the tangential velocity at distance r from the central object is

$$v_\phi = \frac{\ell}{r}$$

and the force per unit mass required to keep the fluid in circular motion is:

$$F_C = \frac{v^2}{r} = \frac{\ell^2}{r^3}$$

The inward gravitational force equals this centripetal force at a distance r_ℓ where

$$\frac{GM}{r_\ell^2} = \frac{\ell^2}{r_\ell^3} \Rightarrow r_\ell = \frac{\ell^2}{GM} \tag{100}$$

Gas cannot move in any further since all the inward force is needed to maintain the orbit and hence there is no dr/dt.

In the absence of forces preventing it, each parcel of gas moves inwards to the r_ℓ corresponding to its ℓ, and its speed there is $v_\phi = \frac{\ell}{r} = \sqrt{\frac{GM}{r_\ell}}$. Thus the fluid parcel is in a Keplerian orbit. Since $v = v(r)$, we have velocity shear and viscosity and instabilities act to smooth out the shear. Angular momentum is transported outward and material is transported inward through the disk.

Disks are common phenomena.

1. **Cataclysmic variables and dwarf novae**. The disk contributes most of the light from the system. Observational evidence for the existence of disks includes Doppler broadening of spectral lines and observed *hot spots* where the accretion stream from the companion meets the disk.

2. **X-ray binaries**. Disks are mostly conjectured rather than directly observed. But in certain sources such as HER X-1 the warped, processing disk occasionally occults the x-ray emitter.

3. **Gas disks observed in galactic nucle**i (in 21-cm radio emission and molecular lines. Also HST images.) May be disks around central black holes.

4. **Disks around young stars** may be proto-planetary systems. These are observed primarily in the infra-red.

5. **Symmetry** of extra-galactic radio sources may be explained if there is a disk around the central *engine* that acts as a collimator for the jets.

6. Disks are conjectured to be present in many stages of binary star evolution- e.g x-ray

binaries. SS433 may have one.

3.2 Basic theory

As usual we shall simplify the discussion by assuming the system to be in a steady state. We also assume that the mass M of the central object remains constant and that density in the disk is high enough for cooling to occur. Thus the flow is supersonic.

3.2.1 The momentum equation

a) Radial component

$$v_r \frac{\partial v_r}{\partial r} - \frac{v_\phi^2}{r} + \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{GM}{r^2} = 0 \tag{101}$$

As usual we work with the mach number $\mu = v_\phi / c_s$, where c_s is the isothermal sound speed $\sqrt{P/\rho}$. Then equation 101 becomes:

$$\begin{aligned} \frac{\partial}{\partial r}\left(\frac{v_r^2}{2}\right) - \frac{v_\phi^2}{r} + \frac{GM}{r^2} + \frac{\partial}{\partial r}\left(c_s^2 \ln \rho\right) &= 0 \\ \frac{\partial}{\partial r}\left(\frac{v_r^2}{2}\right) - \frac{v_\phi^2}{r} + \frac{GM}{r^2} + \frac{\partial}{\partial r}\left(\frac{v_\phi^2}{\mu^2} \ln \rho\right) &= 0 \end{aligned}$$

If $\mu \gg 1$, then the last term may be ignored with respect to the second. Then if we also have $v_r \ll 1$, we find

$$v_\phi = \sqrt{\frac{GM}{r}}$$

as we concluded in the introduction.

b) z-direction (perpendicular to the disk) We assume that $v_z = 0$ so that the momentum equation becomes the equation of hydrostatic equilibrium:

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{GM}{r^2}\frac{z}{r}$$

where $z/r = \cos\theta$ for $z \ll r$. Then we can estimate the disk thickness:

$$\begin{aligned} c_s^2 \frac{\partial}{\partial z} \ln \rho &= -\frac{GMz}{r^3} \\ \ln \rho &= C - \frac{GM}{r^3}\frac{z^2}{2c_s^2} = C - \frac{v_\phi^2}{2c_s^2}\left(\frac{z}{r}\right)^2 \end{aligned}$$

(C is an integration constant), and thus

$$\rho = \rho_0 \exp\left(-\frac{v_\phi^2}{2c_s^2}\left(\frac{z}{r}\right)^2\right) = \rho_0 \exp\left(-\frac{1}{2}\left(\frac{z}{H}\right)^2\right) \tag{102}$$

where the scale height of the disk is

$$H = \frac{r}{\mu}$$

and is much less than r, as required for self-consistency. Note that the disc "flares" - H increases as r increases. As material flows inward through the disc, its $z-$component of velocity is of order:

$$v_z \sim \frac{H}{T} = \frac{r}{\mu}\frac{v_\phi}{r} = \frac{v_\phi}{\mu} \ll v_\phi$$

consistent with our previous approximation $v_z \approx 0$.

3.2.2 Mass flow inward

Mass flows inward through the disk at a rate

$$\frac{dM}{dt} = 2\pi r \Sigma \left(-v_r\right) \tag{103}$$

where $\Sigma\left(r\right)$ is the surface mass density in the disk:

$$\Sigma\left(r\right) = \int_{-\infty}^{+\infty} \rho\left(r, z\right) dz \sim 2H\rho\left(r\right)$$

To make further progress, we need v_r. Recall that it is viscous drag that allows material to fall inward, so we expect v_r to depend on viscosity. Since we found v_ϕ from the radial momentum equation, we look at the ϕ component to get v_r. Write $v_\phi = r \quad$ where $\quad = \sqrt{GM/r^3}$ is the keplerian angular speed of the disk at radius r. Then the angular momentum contained in an annular ring of thickness dr is $dL = dMr^2 \quad = 2\pi r \Sigma r^2 \quad dr$. This angular momentum is carried outward by viscous torques. The net change of angular momentum in the annulus is:

$$\begin{aligned} \text{flux in - flux out} &= 2\pi r \Sigma r^2 \quad v_r\Big|_{r+dr} - 2\pi r \Sigma r^2 \quad v_r\Big|_r \\ &= \frac{d}{dr}\left(2\pi \Sigma r^3 \quad v_r\right) dr \end{aligned}$$

Now the viscous force is given in terms of the kinematic viscosity ν :

$$F\left(r\right) = 2\pi r \nu \Sigma \left(\text{rate of velocity shear}\right) = 2\pi r \nu \Sigma r \frac{d}{dr}$$

and thus the viscous torque is

$$G\left(r\right) = rF\left(r\right) = 2\pi r^3 \nu \Sigma \frac{d}{dr}$$

The net torque on an annulus of thickness dr is

$$dG = G\left(r + dr\right) - G\left(r\right) = \frac{d}{dr}\left(2\pi r^3 \nu \Sigma \frac{d}{dr}\right) dr$$

Then setting torque equal to rate of change of angular momentum, we have:

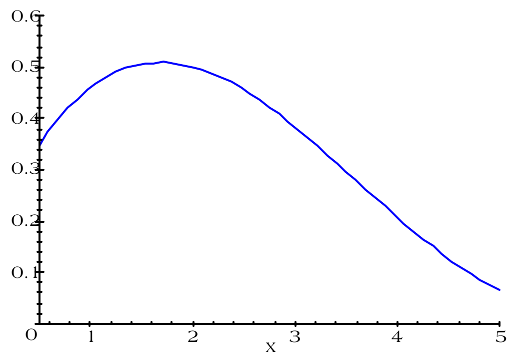
$$\begin{aligned} \frac{d}{dr}\left(2\pi \Sigma r^3 \quad v_r\right) &= \frac{d}{dr}\left(2\pi r^3 \nu \Sigma \frac{d}{dr}\right) \\ v_r &= \frac{\nu}{} \frac{d}{dr} + \frac{k}{r^3 \Sigma} \end{aligned} \tag{104}$$

where k is an integration constant that must be determined by the boundary conditions.

For a disk around a star with radius R_* and angular speed $_*$, there must be a boundary layer where changes rapidly from the Keplerian value $\sqrt{GM/R_*^3}$ to $_*$. Since we must have

$$\frac{GM}{R_*^2} > {}_*^2 R_*$$

for the star to hold together, then $_* < {}_{\mathrm{Kep}}(R_*)$. The angular speed as a function of radius looks like:



Thus $d\ \ /dr$ must be zero at some r_0 not much larger than R_*, and $(r_0) \approx \sqrt{GM/R_*^3}$. Then from equation 104,

$$v_r(R_*) \approx \frac{k}{R_*^3 \Sigma \sqrt{GM/R_*^3}} = \frac{k}{\Sigma(R_*)\sqrt{(GMR_*^3)}}$$

and hence

$$\begin{aligned} v_r &= \frac{\nu}{}\frac{d}{dr} + \frac{v_r(R_*)\Sigma(R_*)\sqrt{(GMR_*^3)}}{r^3\Sigma} \\ &= \frac{\nu}{}\frac{d}{dr} - \left(\frac{r}{R_*}\right)^{-3/2}\frac{dM/dt}{2\pi\Sigma R_*} \end{aligned} \qquad (105)$$

where we used equation 103 evaluated at R_* to simplify the last term. Then using this equation again, we find:

$$\begin{aligned} \frac{dM}{dt} &= -2\pi r\Sigma\left(\frac{\nu}{}\frac{d}{dr} - \left(\frac{r}{R_*}\right)^{-3/2}\frac{dM/dt}{2\pi\Sigma R_*}\right) \\ \frac{dM}{dt}\left(1-\left(\frac{r}{R_*}\right)^{-1/2}\right) &= -2\pi r\Sigma\frac{\nu}{}\frac{d}{dr} = 3\pi r\Sigma\frac{\nu}{r} = 3\pi\Sigma\nu \end{aligned} \qquad (106)$$

So $\Sigma\nu$ must vary with r. Note also that $dM/dt \to 0$ as $\nu \to 0$, as expected.

From equation 105, we find that for $r \gg R_*$,

$$v_r \approx -\frac{3}{2}\frac{\nu}{r}$$

which is negative (i.e. the flow is inward) and depends directly on the viscosity, as expected.

The viscosity also dissipates energy. The power expended by the vsicous forces on our annulus of thickness dr is:

$$\begin{aligned} dP &= \tilde{\mathbf{F}} \cdot \tilde{\mathbf{v}} = 2\pi r^2 \nu \Sigma \frac{d}{dr} \left((r+dr) \quad (r+dr) - r \quad (r) \right) \\ &= 2\pi r^3 \nu \Sigma \left(\frac{d}{dr} \right)^2 dr \end{aligned}$$

and thus the power per unit area is

$$\frac{dP}{dA} = \frac{2\pi r^3 \nu \Sigma \left(\frac{d}{dr} \right)^2 dr}{2\pi r\, dr} = \nu \Sigma \left(r \frac{d}{dr} \right)^2$$

This energy is dissipated as heat. using the Keplerian expression for , and using equation 106 for $\nu\Sigma$, we get:

$$\begin{aligned} D(r) &= \frac{1}{3\pi} \frac{dM}{dt} \left(1 - \left(\frac{r}{R_*} \right)^{-1/2} \right) \left(-\frac{3}{2} \right)^2 \frac{GM}{r^3} \\ &= \frac{3}{4\pi} \frac{dM}{dt} \frac{GM}{r^3} \left(1 - \left(\frac{r}{R_*} \right)^{-1/2} \right) \end{aligned} \tag{107}$$

In this expression, the viscosity ν has disappeared! The total luminosity of the disk is then:This is one half of the total accretion energy.

1. **Solid Star**: the other half of the energy must be released in a boundary layer at the stellar surface.
2. **Neutron star with magnetosphere:** Replace R_* with R_M. Energy can be carried through the magnetosphere and released at the star itself.
3. **Black hole:** R_* must be the radius of the last stable orbit. The other half of the energy disappears down the hole!

Now let's look at the energy radiated between radii R_1 and R_2, both of which are $\gg R_*$.

$$L(R_1 - R_2) = \frac{3}{2} \frac{dM}{dt} GM \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

This is 1.5 times the energy released in the region between R_1 and R_2. Thus energy must be transported out from the inner regions of the disk.

3.2.3 Local disk structure

We already found the density profile (equation 102). The density falls off rapidly above the central plane of the disk. For supersonic flow, as we have assumed, $H \ll r$, i.e. the disk is *thin*. The temperature gradient perpendicular to the disk is much greater than the radial gradient, thus radiation escapes primarily perpendicular to the disk. In equilibrium, the disk reaches a temperature $T(r)$ such that the radiated energy $2\sigma T^4$ equals the heat input $D(r)$.

(σ is the Stefan-Boltzmann constant.) Thus:

$$2\sigma T^4 = \frac{3}{4\pi}\frac{dM}{dt}\frac{GM}{r^3}\left(1-\left(\frac{r}{R_*}\right)^{-1/2}\right)$$

and thus:

$$T = \left(\frac{3}{8\sigma\pi}\frac{dM}{dt}\frac{GM}{r^3}\left(1-\sqrt{\frac{R_*}{r}}\right)\right)^{1/4}$$

For $r \gg R_*$,

$$T = T_1\left(\frac{r}{R_*}\right)^{-3/4}$$

where

$$T_1 = \left(\frac{3}{8\sigma\pi}\frac{dM}{dt}\frac{GM}{R_*^3}\right)^{1/4}$$

Thus the temperature is greatest in the inner regions of the disk.

The total radiated spectrum is:

$$F_v = \int_{R_*}^{R_{\max}} B_\nu\left(T\left(r\right)\right)2\pi r dr$$

At high frequencies, we get an exponential drop off. At low frequencies, $\nu \ll kT_{\text{out}}/h$, radiation comes from the outer regions of the disk, and the spectrum $\propto \nu^2$. To investigate the intermediate region, let $x = h\nu/kT$. Then

$$dx = -\frac{h\nu}{kT^2}\frac{dT}{dr}dr = -\frac{h\nu}{kT^2}\left(-\frac{3}{4r}T\right)dr = \frac{3x}{4r}dr$$

$$B_\nu\left(T\right) = \frac{2h\nu^3}{e^x-1}$$

and

$$r = R_*\left(\frac{T_1}{T}\right)^{4/3} = R_*\left(\frac{kT_1}{h\nu}x\right)^{4/3}$$

So

$$\begin{aligned} F_v &= 4\pi h\nu^3\int_{x_*}^{x_{\max}}\frac{1}{e^x-1}r\frac{4r}{3x}dx \\ &= \frac{16}{3}\pi h\nu^3 R_*^2\left(\frac{kT_1}{h\nu}\right)^{8/3}\int_{x_*}^{x_{\max}}\frac{1}{e^x-1}x^{5/3}dx \\ &\propto \nu^{1/3}\int_{x_*}^{x_{\max}}\frac{x^{5/3}}{e^x-1}dx \end{aligned}$$

So the spectrum looks like:

3.2.4 Viscosity

The usual molecular viscosity proves inadequate to explain disk structure. Therefore, viscosity is usually assumed to be due to magnetic fields or turbulence.

Turbulence The flow is strongly sheared and supersonic. Thus the Reynolds number $R = Lv/\nu$ is high. This leads us to believe that turbulence will develop in the disk, but this has not been proven.

Magnetic viscosity The flow in the disk winds up the magnetic field lines, and reconnection will occur. The field may be chaotic. The magnetic field energy is constrained to be

$$\frac{B^2}{8\pi} \lessapprox \frac{1}{2}\rho c_s^2$$

since reconnection will generate heat in the disk.

For both of these processes, we can parametrize our ignorance with a parameter called the $\alpha-$parameter as follows

1. Turbulent viscosity:

   $$\nu \approx \ell v$$

   where ℓ is the scale length and v is the speed of the largest turbulent eddies. Obviously $\ell < H,$ the disk scale height, and probably $v < c_s$, so

   $$\nu \lessapprox Hc_s = \alpha Hc_s$$

   where the parameter $\alpha \leqslant 1.$ But $H/r \approx c_s/v_\phi$ (equation102), so

   $$\nu \approx \alpha\frac{c_s^2 r}{v_\phi} \tag{108}$$

2. Magnetic viscosity: The magnetic stress is of order $\frac{1}{8\pi}\left(B_r B_\phi\right) \sim B^2/8\pi \sim \rho c_s^2\left(v_A^2/c_s^2\right)$ where v_A is the Alfven speed. Now we derived the viscous stress =

force/length as $\nu \Sigma r \quad '$, so

$$\left| \nu \left(\int \rho dz \right) r \quad ' \right| = \int \rho c_s^2 \left(v_A^2 / c_s^2 \right) dz$$

so, evaluating $\rho \; ' \, '$ for a keplerian disk,

$$\nu \approx c_s^2 \frac{v_A^2}{c_s^2 \, (3/2) \quad} = \alpha c_s^2 \frac{r}{v_\phi}$$

where in this case $\alpha \approx \frac{2}{3} \left(v_A^2 / c_s^2 \right) \lessapprox 1$ again.

So the usual method is to use the so-called $\alpha-$viscosity given by equation 108. Then we have a radial velocity component

$$v_r \approx \nu \frac{' \, '}{\quad} = \alpha c_s^2 \frac{' \, '}{\quad 2} = \alpha c_s \frac{c_s}{r} = \alpha \frac{c_s}{\mu} \ll 1 \tag{109}$$

Note that the inward velocity is directly proportional to the value of α. Then the Reynolds number is

$$R \sim \frac{Lv}{\nu} = \frac{r v_\phi^2}{\alpha c_s^2 r} = \frac{\mu^2}{\alpha} \gg 1$$

which is self-consistent.

3.2.5 Radiation from the disk:

If the disk is optically thick, then the spectrum does not depend on ν (or α), as we have seen. But if energy is dissipated in optically thin regions, we do not have complete thermalization of the radiation and hence we can have a higher effective temperature T_{eff}. If the surface of the disk gets hot, there can be a *disk corona* above and below the central plane. For high luminosity systems we can even have a disk wind. Radiation from the inner regions can be reabsorbed farther out and cause heating and evaporation.

3.2.6 Timescales

1. Rotational timescale is

$$t_\phi \sim \frac{r}{v_\phi} \sim \frac{1}{\quad}$$

2. z-structure.

$$t_z \sim \frac{H}{c_s} \sim \frac{H}{r} \frac{r}{v_\phi} \frac{v_\phi}{c_s} \sim \frac{H}{r} t_\phi \mu \sim t_\phi$$

3. surface density is governed by dM/dt and ν :

$$t_\nu \sim \frac{r}{v_r} \sim \frac{r}{\alpha c_s^2} \quad r = \frac{\left(\quad r \right)^2}{\alpha c_s^2} \frac{1}{\quad} = \frac{1}{\alpha} \mu^2 t_\phi \gg t_\phi$$

4. Thermal timescale

$$t_{th} \sim \frac{\Sigma c_s^2}{D(r)} = \frac{\Sigma c_s^2}{\nu \Sigma \quad ^2} = \frac{c_s^2 v_\phi}{\alpha c_s^2 r \quad ^2} = \frac{t_\phi}{\alpha} \gtrapprox t_\phi$$

3.2.7 Instabilities

- **Thermal**. Since $t_{th} \ll t_\nu$, we may assume $\Sigma =$ constant. In equilibrium, heating = cooling. But if the heating rate increases with temperature faster than the cooling rate, a small perturbation will grow. Since the scale height $H \propto c_s$, increased T $\Rightarrow$increased $H \Rightarrow$ decreased ρ and therefore decreased cooling on the timescale $t_z \sim t_\phi$. Thus the disk should be thermally unstable.

- **Viscous**. Since $t_\nu \gg t_{th}$, thermal equilibrium always holds, heating balances cooling and the temperature $T(r)$ is fixed. Then $\nu = \nu(\Sigma, r)$. Define $\lambda = \nu\Sigma$, and consider a perturbation $\Sigma \to \Sigma + \delta\Sigma$, $\lambda \to \lambda + \delta\lambda$, where

$$\delta\lambda = \frac{\partial\lambda}{\partial\Sigma}\delta\Sigma$$

The continuity equation gives us:

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma v_r) = 0 \tag{110}$$

while the $\phi-$component of the momentum equation gives:

$$\frac{\partial}{\partial t}\left(\Sigma r^2 \quad\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Sigma v_r r^2 \quad\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu\Sigma r^3 \quad '\right)$$

where the LHS equals the rate of change of angular momentum and the LHS equals the net viscous torque. Expanding the second term, we get:

$$\frac{\partial}{\partial t}\left(\Sigma r^2 \quad\right) + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma v_r) r^2 \quad + \frac{r\Sigma v_r}{r}\frac{\partial}{\partial r}\left(r^2 \quad\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu\Sigma r^3 \quad '\right)$$

Now does not depend on time, and we can use equation 110 to simplify the second term:

$$\begin{aligned} r^2 \quad \frac{\partial}{\partial t}\Sigma - \frac{\partial\Sigma}{\partial t} r^2 \quad + \frac{r\Sigma v_r}{r}\frac{\partial}{\partial r}\left(r^2 \quad\right) &= \frac{1}{r}\frac{\partial}{\partial r}\left(\nu\Sigma r^3 \quad '\right) \\ r\Sigma v_r \frac{\partial}{\partial r}\left(r^2 \quad\right) &= \frac{\partial}{\partial r}\left(\nu\Sigma r^3 \quad '\right) \\ r\Sigma v_r &= \frac{1}{(r^2 \quad)'}\frac{\partial}{\partial r}\left(\nu\Sigma r^3 \quad '\right) \end{aligned}$$

Now we substitute back into equation 110:

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{(r^2 \quad)'}\frac{\partial}{\partial r}\left(\nu\Sigma r^3 \quad '\right)\right) = 0$$

Now for a keplerian disk, $\quad = \sqrt{GM/r^3}$, so $r^2 \quad = \sqrt{GMr}$ and $\left(r^2 \quad\right)' = \frac{1}{2}\sqrt{GM/r}$. Also, $\quad ' = -\frac{3}{2} \quad /r$. Thus:

$$\begin{aligned} \frac{\partial\Sigma}{\partial t} &= -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{2}{\sqrt{GM/r}}\frac{\partial}{\partial r}\left(-\frac{3}{2}\nu\Sigma r^2\sqrt{\frac{GM}{r^3}}\right)\right) \\ &= \frac{3}{r}\frac{\partial}{\partial r}\left(\sqrt{r}\frac{\partial}{\partial r}\left(\nu\Sigma\sqrt{r}\right)\right) \end{aligned}$$

Then

$$\frac{\partial}{\partial t}\delta\lambda = \frac{\partial\lambda}{\partial\Sigma}\frac{\partial\delta\Sigma}{\partial t} = \frac{\partial\lambda}{\partial\Sigma}\frac{3}{r}\frac{\partial}{\partial r}\left(\sqrt{r}\frac{\partial}{\partial r}\left(\delta\lambda\sqrt{r}\right)\right) \tag{111}$$

Let's look at

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\sqrt{r}\frac{\partial}{\partial r}\left(\delta\lambda\sqrt{r}\right)\right) &= \frac{1}{2}r^{-1/2}\frac{\partial}{\partial r}\left(\delta\lambda\sqrt{r}\right) + \sqrt{r}\frac{\partial^2}{\partial r^2}\left(\delta\lambda\sqrt{r}\right) \\
&= \frac{1}{2}r^{-1/2}\left(\sqrt{r}\frac{\partial}{\partial r}(\delta\lambda) + \frac{1}{2\sqrt{r}}\delta\lambda\right) + \sqrt{r}\frac{\partial}{\partial r}\left(\sqrt{r}\frac{\partial}{\partial r}(\delta\lambda) + \frac{1}{2\sqrt{r}}\delta\lambda\right) \\
&= \frac{1}{2}\frac{\partial}{\partial r}(\delta\lambda) + \frac{1}{4r}\delta\lambda + \sqrt{r}\left(\frac{1}{2\sqrt{r}}\frac{\partial}{\partial r}(\delta\lambda) + \sqrt{r}\frac{\partial^2}{\partial r^2}(\delta\lambda) - \frac{1}{4r^{3/2}}\delta\lambda + \frac{1}{2\sqrt{r}}\frac{\partial}{\partial r}(\delta\lambda)\right) \\
&= \frac{1}{2}\frac{\partial}{\partial r}(\delta\lambda) + \frac{1}{4r}\delta\lambda + \frac{1}{2}\frac{\partial}{\partial r}(\delta\lambda) + r\frac{\partial^2}{\partial r^2}(\delta\lambda) - \frac{1}{4r}\delta\lambda + \frac{1}{2}\frac{\partial}{\partial r}(\delta\lambda) \\
&= \frac{3}{2}\frac{\partial}{\partial r}(\delta\lambda) + r\frac{\partial^2}{\partial r^2}(\delta\lambda) = \frac{1}{2}\frac{\partial}{\partial r}(\delta\lambda) + \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(\delta\lambda)\right) \\
&= \frac{1}{2}\frac{\partial}{\partial r}(\delta\lambda) + r\nabla^2\delta\lambda
\end{aligned}$$

Thus equation 111 is of the form:

$$\frac{\partial}{\partial t}\delta\lambda = \frac{\partial\lambda}{\partial\Sigma}3\nabla^2\delta\lambda + \frac{\partial\lambda}{\partial\Sigma}\frac{3}{2r}\frac{\partial}{\partial r}(\delta\lambda)$$

which is a diffusion equation for $\delta\lambda$.

The standard form of the diffusion equation is:

$$\frac{\partial F}{\partial t} = D\nabla^2 F$$

To solve it, separate variables: $F = X(x)\,T(t)$, to get

$$\frac{1}{T}\frac{dT}{dt} = \frac{D}{X}\frac{d^2X}{dt^2} = \text{constant } = k$$

The solution is:

$$T = T_0 e^{kt}$$

and

$$X = X_0 e^{\sqrt{k/D}x}$$

For stability, we need $k < 0$ (perturbation decays in time) in which case $X = X_0 \exp\left(\pm i\sqrt{\frac{|k|}{D}}x\right)$ which is an oscillating disturbance in space if $D > 0$. For the disk, spatial growth is as bad as temporal growth, since the disk is disrupted either way. So we need $D > 0$.

Digression: solving the equation for $\delta\lambda$ Our equation is a little more complicated because of the extra term, but the same general ideas apply.

$$\frac{\partial}{\partial t}\delta\lambda = \frac{\partial\lambda}{\partial\Sigma}\frac{3}{r}\frac{\partial}{\partial r}\left(\sqrt{r}\frac{\partial}{\partial r}\left(\delta\lambda\sqrt{r}\right)\right)$$

Let $\delta\lambda = R(r)\,T(t)$. Then

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{R}\frac{\partial\lambda}{\partial\Sigma}\frac{3}{r}\frac{\partial}{\partial r}\left(\sqrt{r}\frac{\partial}{\partial r}\left(R\sqrt{r}\right)\right) = \text{constant } = k$$

Then $T = T_0 e^{kt}$ and

$$\frac{\partial}{\partial r}\left(\sqrt{r}\frac{\partial}{\partial r}\left(R\sqrt{r}\right)\right) = \frac{kRr}{3D}$$

where $D = \frac{\partial\lambda}{\partial\Sigma}$. Expand the LHS:

$$r\frac{\partial^2 R}{\partial r^2} + \frac{3}{2}\frac{\partial R}{\partial r} - \frac{krR}{3D} = 0$$

Look for a solution of the form $R = g(r)\exp\left(\sqrt{k/3D}r\right)$. Then:

$$\frac{\partial R}{\partial r} = \frac{dg}{dr}\exp\left(\sqrt{k/3D}r\right) + \sqrt{\frac{k}{3D}}g\exp\left(\sqrt{k/3D}r\right)$$

and

$$\frac{\partial^2 R}{\partial r^2} = \left(\frac{d^2 g}{dr^2} + 2\sqrt{\frac{k}{3D}}\frac{dg}{dr} + \frac{k}{3D}g\right)\exp\left(\sqrt{k/3D}r\right)$$

Stuffing into the differential equation gives:

$$\begin{aligned} r\left(\frac{d^2 g}{dr^2} + 2\sqrt{\frac{k}{3D}}\frac{dg}{dr} + \frac{k}{3D}g\right) + \frac{3}{2}\left(\frac{dg}{dr} + \sqrt{\frac{k}{3D}}g\right) - \frac{kr}{3D}g &= 0 \\ r\frac{d^2 g}{dr^2} + \left(r2\sqrt{\frac{k}{3D}} + \frac{3}{2}\right)\frac{dg}{dr} + \frac{3}{2}\sqrt{\frac{k}{3D}}g &= 0 \end{aligned}$$

which is ugly. Look for a series solution: $g = \sum_n a_n r^{n+p}$

$$\begin{aligned} r\sum a_n(n+p)(n+p-1)r^{n+p-2} + \left(2r\sqrt{\frac{k}{3D}} + \frac{3}{2}\right)\sum a_n(n+p)r^{n+p-1} + \frac{3}{2}\sqrt{\frac{k}{3D}}\sum a_n r^{n+p} &= 0 \\ \sum a_n\left((n+p)(n+p-1)r^{n+p-1} + (n+p)\left(2r^{n+p}\sqrt{\frac{k}{3D}} + \frac{3}{2}r^{n+p-1}\right) + \frac{3}{2}\sqrt{\frac{k}{3D}}r^{n+p}\right) &= 0 \end{aligned}$$

Looking at the lowest power of r, which is r^{p-1}, we get:

$$a_0 p(p-1) + \frac{3p}{2}a_0 = 0$$

which has roots $p = 0$ pr $p = 1 - \frac{3}{2} = -\frac{1}{2}$. Then the recursion relation is found by looking at the power r^{p+m} :

$$a_{m+1}(p+m+1)(p+m) + a_m(m+p)2\sqrt{\frac{k}{3D}} + a_{m+1}\frac{3}{2}(p+m+1) + \frac{3}{2}\sqrt{\frac{k}{3D}}a_m = 0$$

$$
\begin{aligned}
a_{m+1} &= -a_m \sqrt{\frac{k}{3D}} \frac{3/2 + 2\,(m+p)}{(p+m+1)\,(p+m+3/2)} \\
&= -\frac{a_m}{(p+m+1)} \frac{3/2 + 2\,(m+p)}{(p+m+3/2)} \sqrt{\frac{k}{3D}}
\end{aligned}
$$

So for $p = 0$ we get:

$$
\begin{aligned}
a_m &= -\frac{a_{m-1}}{m} \frac{3/2 + 2\,(m-1)}{m+1/2} \sqrt{\frac{k}{3D}} \\
&= (-1)^m \frac{a_0}{m!} \frac{2m-1/2}{m+1/2} \frac{2m-5/2}{m-1/2} \cdots \frac{3/2}{3/2} \left(\frac{k}{3D}\right)^{m/2}
\end{aligned}
$$

giving the solution

$$
R = R_0 \exp\left(\sqrt{k/3D}r\right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{2m-1/2}{m+1/2} \frac{2m-5/2}{m-1/2} \cdots \frac{3/2}{3/2} \left(\frac{kr^2}{3D}\right)^{m/2}
$$

while for $p = -1/2$

$$
\begin{aligned}
a_m &= -\frac{a_{m-1}}{m-1/2} \frac{2m-1}{m} \sqrt{\frac{k}{3D}} \\
&= (-1)^m \frac{a_0}{m!} \frac{2m-1}{m-1/2} \frac{2m-3}{m-3/2} \cdots \frac{1}{1/2} \left(\frac{k}{3D}\right)^{m/2}
\end{aligned}
$$

giving

$$
R = \frac{R_0}{\sqrt{r}} \exp\left(\sqrt{k/3D}r\right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{2m-1}{m-1/2} \frac{2m-3}{m-3/2} \cdots \frac{1}{1/2} \left(\frac{kr^2}{3D}\right)^{m/2}
$$

Both series converge, so our conclusion about stability is not changed. We need $D > 0$ for stability.

3.2.8 The two-temperature accretion disk

Reference: Shapiro, Lightman and Eardley. Ap.J. **204,** 187, 1976

Consider an optically thin disk, where cooling is primarily by Bremsstrahlung. Then

$$
\Lambda = 5 \times 10^{20} \rho^2 T^{1/2}\ \mathrm{erg \cdot s^{-1} \cdot cm^{-3}}
$$

and therefore the heat removed from each unit area of the disk is

$$
Q^- = 2\Lambda H = 10^{21} \rho^2 H T^{1/2} \mathrm{erg \cdot s^{-1} \cdot cm^{-2}}
$$

Equate this cooling rate to heating by viscous dissipation (equation 107):

$$
\frac{3}{4\pi} \frac{dM}{dt} \frac{GM}{r^3} \left(1 - \left(\frac{r}{R_*}\right)^{-1/2}\right) = 10^{21} \rho^2 H T^{1/2} \mathrm{erg \cdot s^{-1} \cdot cm^{-2}} \qquad (112)
$$

We previously found

$$
\begin{aligned}
H &= \frac{r}{\mu} = \frac{c_s}{v_\phi} r = \frac{c_s}{} \\
&= \left(\frac{T_i + T_e}{10^9 \text{ K}}\right)^{1/2} \left(4.0 \times 10^8 \text{ cm/s}\right) \sqrt{\frac{r}{GM}} r
\end{aligned}
$$

Now let's express the radius in terms of the Schwarzschild radius of a mass M :

$$r_s = \frac{2GM}{c^2}$$

$$r_* = 2r/r_s = \frac{c^2 r}{GM}$$

Then:

$$
\begin{aligned}
H &= \left(\frac{T_i + T_e}{10^9 \text{ K}}\right)^{1/2} \left(4.0 \times 10^8 \text{ cm/s}\right) \sqrt{\frac{r_*}{c^2}} r_* \frac{GM}{c^2} \\
&= T_9^{1/2} \left(4 \times 10^8\right) r_*^{3/2} \frac{6.7 \times 10^{-11} \left(2 \times 10^{33}\right)}{\left(3 \times 10^{10}\right)^3} \frac{M}{M_\odot} \\
&= \left(2 \times 10^3 \text{ cm}\right) T_9^{1/2} r_*^{3/2} \frac{M}{M_\odot}
\end{aligned}
$$

From equations 103 and 109, we find Σ :

$$\Sigma = \frac{dM}{dt} \frac{}{2\pi\alpha c_s^2}$$

and then

$$
\begin{aligned}
\rho &= \frac{\Sigma}{2H} = \frac{dM}{dt} \frac{}{4\pi\alpha c_s^2 H} = \frac{dM}{dt} \frac{2}{4\pi\alpha c_s^3} \\
&= \left(10^{17} \text{ g/s}\right) M_{17} \frac{GM}{r^3 4\pi\alpha \left(4.0 \times 10^8 \text{ cm/s}\right)^3 T_9^{3/2}} \\
&= \frac{\left(10^{17} \text{ g/s}\right)}{4\pi \left(4.0 \times 10^8 \text{ cm/s}\right)^3} \frac{M_{17}}{\alpha T_9^{3/2}} \frac{GM}{r_*^3} \left(\frac{c^2}{GM}\right)^3 \\
&= \frac{\left(10^{17} \text{ g/s}\right)}{4\pi \left(4.0 \times 10^8 \text{ cm/s}\right)^3} \frac{M_{17}}{\alpha T_9^{3/2}} \frac{1}{r_*^3} \frac{c^6}{(GM)^2} \\
&= \frac{\left(10^{17} \text{ g/s}\right)}{4\pi \left(4.0 \times 10^8 \text{ cm/s}\right)^3} \frac{M_{17}}{\alpha T_9^{3/2}} \frac{1}{r_*^3} \frac{\left(3 \times 10^{10} \text{ cm/s}\right)^6}{\left(6.7 \times 10^{-8} \text{ cm}^3/gs^2\right)^2} \frac{\left(M_\odot/M\right)^2}{\left(2 \times 10^{33} \text{ g}\right)^2} \\
&= \left(5.0 \text{ g/cm}^3\right) \left(\frac{M_\odot}{M}\right)^2 \frac{M_{17}}{\alpha T_9^{3/2}} \frac{1}{r_*^3}
\end{aligned}
$$

Then we can solve equation 112 for the temperature:

$$\begin{aligned}
T^{1/2} &= \frac{1}{10^{21}\rho^2 H \mathrm{erg}\cdot\mathrm{s}^{-1}\cdot\mathrm{cm}^{-2}}\frac{3}{4\pi}\frac{dM}{dt}\frac{GM}{r^3}\left(1-\left(\frac{r}{R_*}\right)^{-1/2}\right) \\
&\simeq \frac{1}{10^{21}\left(5.0\left(\frac{M_\odot}{M}\right)^2\frac{M_{17}}{\alpha T_9^{3/2}}\frac{1}{r_*^3}\right)^2\left(2\times 10^3\ T_9^{1/2}r_*^{3/2}\frac{M}{M_\odot}\right)}\frac{3}{4\pi}10^{17}M_{17}\frac{GM}{r_*^3}\left(\frac{c^2}{GM}\right)^3 \\
10^{4.5}T_9^{1/2} &= \alpha^2 T_9^{5/2} r_*^{9/2}\frac{M_\odot}{M}\frac{1}{M_{17}r_*^3}19.385
\end{aligned}$$

and thus

$$\begin{aligned}
T_9^2 &= 1.6\times 10^3\frac{M}{M_\odot}\frac{M_{17}}{\alpha^2 r_*^{3/2}} \\
T_9 &= 40\left(\frac{M}{M_\odot}\right)^{1/2}\frac{M_{17}^{1/2}}{\alpha r_*^{3/4}}
\end{aligned}$$

This disk is very hot!