Green's function for the wave equation

Non-relativistic case

1 The wave equations

In the Lorentz Gauge, the wave equations for the potentials in Lorentz Gauge and Gaussian units are:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j} \tag{1}$$

and

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -4\pi\rho \tag{2}$$

The Gauge condition is:

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0 \tag{3}$$

2 The Green's function

For both potentials we have a wave equation of the form

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -4\pi\,(\text{source})$$

where Φ can be either the scalar potential or a cartesian component of $\vec{A}$. The corresponding Green's function problem is:

$$\nabla^2 G\left(\vec{x}, t; \vec{x}', t'\right) - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta\left(\vec{x} - \vec{x}'\right)\delta\left(t - t'\right)$$

where the source is now a unit event located at $\vec{x} = \vec{x}'$ and happening at $t = t'$. To solve this equation we first Fourier transform in time:

$$G\left(\vec{x}, t; \vec{x}', t'\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G\left(\vec{x}, \omega; \vec{x}', t'\right) e^{-i\omega t} d\omega$$

and the equation becomes

$$\begin{aligned}\left(\nabla^2 + \frac{\omega^2}{c^2}\right) G\left(\vec{x}, \omega; \vec{x}', t'\right) &= -4\pi\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta\left(\vec{x} - \vec{x}'\right)\delta\left(t - t'\right) e^{i\omega t} dt \\ &= -2\sqrt{2\pi}\delta\left(\vec{x} - \vec{x}'\right) e^{i\omega t'}\end{aligned}$$

So let $G(\vec{x},\omega;\vec{x}',t') = g(\vec{x},\vec{x}')\,e^{i\omega t'}/\sqrt{2\pi}$ and then g satisfies the equation

$$\left(\nabla^2 + k^2\right) g = -4\pi\delta\left(\vec{x}-\vec{x}'\right)$$

Now in free space without boundaries, g must be a function only of $R = |\vec{x}-\vec{x}'|$ and must posess spherical symmetry about the source point. Thus in spherical coordinates, we can write:

$$\frac{1}{R}\frac{d^2}{dR^2}(Rg) + k^2 g = -4\pi\delta\left(\vec{R}\right) \tag{4}$$

For $\vec{R} \neq 0$, the right hand side is zero. Then the function Rg satisfies the exponential equation, and the solution is:

$$\begin{aligned} Rg &= Ae^{ikR} + Be^{-ikR} \\ g &= \frac{1}{R}\left(Ae^{ikR} + Be^{-ikR}\right) \end{aligned} \tag{5}$$

Near the origin, where the delta-function contributes, the second term on the LHS is negligible compared with the first, and the eequation becomes:

$$\nabla^2 g = -4\pi\delta\left(\vec{x}-\vec{x}'\right)$$

and we know that this has solution

$$g = \frac{1}{R}$$

This is consistent with equation 5 provided that

$$A + B = 1$$

(You should convince yourself that this solution is correct by differentiating and stuffing back into equation 4.)

Thus we have the solution

$$G(\vec{x},\omega;\vec{x}',t') = \frac{1}{\sqrt{2\pi}R}\left(Ae^{ikR} + Be^{-ikR}\right)e^{i\omega t'}$$

Now we do the inverse transform:

$$\begin{aligned} G(\vec{x},t;\vec{x}',t') &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}R}\left(Ae^{ikR} + Be^{-ikR}\right)e^{i\omega t'}e^{-i\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A\exp\left(i\omega\left(R/c + t' - t\right)\right) + B\exp\left(i\omega\left(-R/c + t' - t\right)\right)d\omega \\ &= A\delta\left(t' - (t - R/c)\right) + B\delta\left(t' - (t + R/c)\right) \end{aligned} \tag{6}$$

The second term is usually rejected because it predicts a response to an event occurring in the future. However, Feynman and Wheeler have proposed a theory in which both terms are kept. They show that this theory can be consistent with observed causality provided that the universe is perfectly absorbing in the infinite future. The time $t - R/c$ that appears in the first term is called the *retarded time* $t_{\text{ret.}}$

The symmetry of this Green's function is:

$$G(\vec{x},t;\vec{x}',t') = G(\vec{x}',-t';\vec{x},-t)$$

(See Morse and Feshbach Ch 7 pg 834-835). Causality requires:

$$G(\vec{x},-\infty;\vec{x}',t') = 0$$

and

$$G\left(\vec{x},t;\vec{x}',t'\right)=0 \text{ for } t<t'$$

3 The potentials

Now that we have the Green's function, we can solve our original equations.

$$\begin{aligned}\Phi\left(\vec{x},t\right) &= \int \frac{\rho\left(\vec{x}',t'\right)}{R}\delta\left(t'-t_{\text{ret}}\right)dt'd^3x' \qquad (7)\\ &= \int \frac{\rho\left(\vec{x}',t_{\text{ret}}\right)}{R}d^3x' \qquad (8)\end{aligned}$$

in Lorentz Gauge, and similarly:

$$\vec{A}\left(\vec{x},t\right)=\frac{1}{c}\int\frac{\vec{j}\left(\vec{x}',t_{\text{ret}}\right)}{R}d^3x' \qquad (9)$$

Notice that these equations have the same form as the static potentials (equations 1.17 and 5.32 in Jackson).

4 Radiation from a moving point charge (non-relativistic case)

4.1 The Lienard-Wiechert potentials

Our source is a point charge moving with velocity $\vec{v}\left(t\right)$. Then the charge and current densities are

$$\rho\left(\vec{x},t\right)=q\delta\left(\vec{x}-\vec{r}\left(t\right)\right)$$

and

$$\vec{j}\left(\vec{x},t\right)=q\vec{v}\delta\left(\vec{x}-\vec{r}\left(t\right)\right)$$

Then from equation 7, we have:

$$\Phi\left(\vec{x},t\right)=\int\frac{q\delta\left(\vec{x}'-\vec{r}\left(t'\right)\right)}{R}\delta\left(t'-t_{\text{ret}}\right)dt'd^3x'$$

We do the integral over the spatial coordinates first.

$$\Phi\left(\vec{x},t\right)=\int q\frac{\delta\left(t'+R\left(t'\right)/c-t\right)}{R\left(t'\right)}dt'$$

where $R\left(t'\right)=\left|\vec{x}-\vec{r}\left(t'\right)\right|$. Now to do the t' integral, we must reexpress the delta-function. Recall:

$$\delta\left(f\left(x\right)\right)=\sum\frac{1}{\left|f'\left(x_i\right)\right|}\delta\left(x-x_i\right)$$

where $f\left(x_i\right)=0$. In this case:

$$f\left(t'\right)=t'+\frac{R\left(t'\right)}{c}-t$$

and, since $\vec{v}=d\vec{r}/dt$,

$$\begin{aligned} f'\left(t'\right) &= 1+\frac{1}{c}\frac{dR}{dt'}=1-\frac{1}{c}\frac{\left(\vec{x}-\vec{r}\left(t'\right)\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}\cdot\frac{d}{dt'}\left(\vec{x}-\vec{r}\left(t'\right)\right) \\ &= 1-\frac{\vec{v}\cdot\left(\vec{x}-\vec{r}\left(t'\right)\right)}{c\left|\vec{x}-\vec{r}\left(t'\right)\right|}=1-\frac{\vec{v}\cdot\vec{R}}{cR} \end{aligned}$$

The function f is zero when $t'=t_{\text{ret}}=t-R/c$. Thus, evaluating the integral, we get:

$$\Phi\left(\vec{x},t\right)=\left.\frac{q}{R\left(1-\frac{\vec{v}\cdot\vec{R}}{cR}\right)}\right|_{t_{\text{ret}}} \tag{10}$$

And similarly

$$\vec{A}\left(\vec{x},t\right)=\left.\frac{q\vec{v}}{R\left(1-\frac{\vec{v}\cdot\vec{R}}{cR}\right)}\right|_{t_{\text{ret}}} \tag{11}$$

These are the Lienhard-Wiechert potentials. It is convenient to use the shorthand

$$r_v=R\left(1-\frac{\vec{v}\cdot\vec{R}}{cR}\right)=R-\frac{\vec{v}\cdot\vec{R}}{c} \tag{12}$$

4.2 Calculating the fields

In Lorentz Gauge, the fields are found using

$$\vec{E}=-\vec{\nabla}\Phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

and

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

But our expressions for the potentials are in terms of $\vec{x}$ and t_{ret}, not $\vec{x}$ and t, so we have to be very careful in taking the partial derivatives. We can put the origin at the instantaneous position of the charge to simplify things. Then $R=r$. Our potential may be written:

$$\Phi\left(\vec{x},t\right)\equiv\Psi\left(\vec{x},t_{\text{ret}}\right)$$

Then

$$d\Phi=\left.\vec{\nabla}\Phi\right|_{\text{const t}}\cdot d\vec{x}+\frac{\partial\Phi}{\partial t}dt\equiv\left.\vec{\nabla}\Psi\right|_{\text{const t}_{ret}}\cdot d\vec{x}+\frac{\partial\Psi}{\partial t_{\text{ret}}}dt_{\text{ret}}$$

But $dt_{\text{ret}}=dt-dr/c$, so

$$\left.\vec{\nabla}\Phi\right|_{\text{const t}}\cdot d\vec{x}+\frac{\partial\Phi}{\partial t}dt\equiv\left.\vec{\nabla}\Psi\right|_{\text{const t}_{ret}}\cdot d\vec{x}-\frac{\partial\Psi}{\partial t_{\text{ret}}}\frac{dr}{c}+\frac{\partial\Psi}{\partial t_{\text{ret}}}dt$$

Thus the $r-$component of $\vec{\nabla}\Phi$ must be modified:

$$\left.\frac{\partial\Phi}{\partial r}\right|_{\text{const t}} = \left.\frac{\partial\Psi}{\partial r}\right|_{\text{const t}_{ret}} - \frac{1}{c}\frac{\partial\Psi}{\partial t_{\text{ret}}} \tag{13}$$

Now we can calculate the fields:

$$\vec{\nabla}\Phi = \vec{\nabla}\frac{q}{r_v} = -\frac{q}{r_v^2}\vec{\nabla}r_v$$

and

$$\vec{\nabla}r_v = \frac{\partial}{\partial r}\left(r - \frac{\vec{r}\cdot\vec{v}}{c}\right)\hat{r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta}\left(r - \frac{\vec{r}\cdot\vec{v}}{c}\right) + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r - \frac{\vec{r}\cdot\vec{v}}{c}\right)$$

We can choose our axes with polar axis along the instantaneous direction of $\vec{v}$. Then $\vec{r}\cdot\vec{v} = rv\cos\theta$, and

$$\vec{\nabla}r_v = \left(1 - \frac{v}{c}\cos\theta\right)\hat{r} + \frac{\hat{\theta}}{r}\left(r\frac{v}{c}\sin\theta\right)$$

In the non-relativistic limit, $v/c \ll 1$, to zeroth order in v/c, this is

$$\vec{\nabla}r_v = \hat{r}$$

We are also going to need

$$\frac{\partial r_v}{\partial t} = -\frac{\vec{r}\cdot\vec{a}}{c}$$

Then we have

$$\begin{aligned}
\vec{E} &= -\left.\tilde{\boldsymbol{\nabla}}\Phi\right|_{\text{const t}_{ret}} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\hat{r} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \\
& \frac{q}{r_v^2}\hat{\mathbf{r}} - \frac{q}{r_v^2}\left(-\frac{\vec{r}\cdot\vec{a}}{c}\right)\hat{\mathbf{r}} - \frac{q}{c}\frac{\partial}{\partial t}\frac{\vec{v}}{r_v} \\
&= \frac{q}{r_v^2}\hat{\mathbf{r}}\left(1 + \frac{\vec{r}\cdot\vec{a}}{c}\right) - \frac{q}{c}\frac{\vec{a}}{r_v} + \frac{q\vec{v}}{cr_v^2}\left(r - \frac{\vec{r}\cdot\vec{a}}{c}\right)
\end{aligned}$$

and again taking the non-relativistic limit, this becomes:

$$\vec{E} = \frac{q}{r^2}\hat{r}\left(1 + \frac{\vec{r}\cdot\vec{a}}{c}\right) - \frac{q}{c}\frac{\vec{a}}{r}$$

The first term is the usual Coulomb field. The other two terms depend on $\vec{a}$: these are the radiation field.

$$\begin{aligned}
\vec{E}_{\text{rad}} &= \frac{q}{rc}\left[(\hat{r}\cdot\vec{a})\,\hat{r} - \vec{a}\right] \\
&= \frac{q}{rc}\left(\hat{r}\times(\hat{r}\times\vec{a})\right)
\end{aligned} \tag{14}$$

4.3 Radiated power

The Poynting vector is

$$\begin{aligned}
\vec{S} &= \frac{c}{4\pi}\vec{E}_{\text{rad}} \times \vec{B}_{\text{rad}} \\
&= \frac{c}{4\pi}\vec{E}_{\text{rad}} \times \left(\hat{n} \times \vec{E}_{\text{rad}}\right) \\
&= \frac{c}{4\pi}E_{\text{rad}}^2 \hat{n} \\
&= \frac{c}{4\pi}\left(\frac{q}{cR}\hat{n}\times\left[\hat{n} \times \frac{d\vec{\beta}}{dt}\right]\right)^2 \hat{n} \\
&= \frac{q^2}{4\pi R^2 c}\left|\hat{n}\times\left[\hat{n} \times \frac{d\vec{\beta}}{dt}\right]\right|^2 \hat{n}
\end{aligned}$$

Thus the power radiated per unit solid angle is:

$$\begin{aligned}
\frac{dP}{d} &= R^2 \vec{S}\cdot\hat{n} \\
&= \frac{q^2}{4\pi c}\left|\hat{n}\times\left(\hat{n} \times \frac{d\vec{\beta}}{dt}\right)\right|^2 \\
& \frac{q^2}{4\pi c^3} a^2 \sin^2\theta
\end{aligned} \qquad (15)$$

Thus is the Larmor formula, where $a = dv/dt$ is the acceleration and θ is the angle between $\vec{a}$ and $\hat{n}$. The total power radiated in the non-relativistic case is:

$$\begin{aligned}
P &= \int \frac{dP}{d} d \quad = \int_{-1}^{+1} d\mu \int_0^{2\pi} d\phi \frac{q^2}{4\pi c^3} a^2 \left(1-\mu^2\right) \\
&= \frac{q^2}{2c^3}a^2\left(\mu - \frac{\mu^3}{3}\right)\Big|_{-1}^{+1} = \frac{2q^2}{3c^3}a^2
\end{aligned} \qquad (16)$$