Plasma Physics

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1 Introduction

Plasma is sometimes called the fourth state of matter (the first three being solid, liquid and gas.) From basic thermodyamics, we know that adding additional energy to a system allows it to progress through the states from solid to gas as the average energy per particle increases. To progress from gas to plasma, we have to ionize each of the gas particles- that is, to separate one or more electrons from each atom. To achieve this state the plasma is typically very hot. The ionization energy for hydrogen is 13.6 ev corresponding to a temperature

$$kT \sim 13.6 \text{ eV}$$

or

$$\begin{aligned} T &\sim \frac{13.6 \text{ eV}}{1.38 \times 10^{-23} \text{ J/K}} 1.6 \times 10^{-19} \text{ J/eV} \\ &= \frac{13.6 \text{ eV}}{8.625 \times 10^{-5} \text{ eV/K}} = 13.6 \text{ eV} \times 1.1594 \times 10^{4} \text{ K/eV} \qquad (1) \\ &= 1.6 \times 10^{5} \text{ K} \end{aligned}$$

From equation (1) we see that 1 eV corresponds to a temperature of about $10^4 K$. In plasma physics, temperatures are often expressed in eV. $T = 2$ eV really means $kT = 2$ eV.

In equilibrium, ionizations are balanced by recombinations. The recombination rate is

$$\alpha n_e n_i$$

with the recombination coefficient

$$\alpha = 2.5 \times 10^{-13} \text{ cm}^3\text{s}^{-1}\text{at} 10^4\text{K}$$

But if the plasma has a very low density it may be able to remain ionized even at low temperatures simply because the particles cannot find each other to recombine. Thus some plasmas, especially in astrophysical systems, are not very hot but are very low density.

2 The Saha Equation

The Saha equation expresses how the Boltzmann relation is applied to determine the ionization state. Boltzman's relation relates the relative populations of two states that differ in energy:

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} \exp\left(-\frac{\Delta E}{kT}\right)$$

where the g are the statistical weights (roughly the number of substates within the state that have the same energy), ΔE is the energy difference, and T is the temperature. We compare two states: a neutral atom on one hand, and an ion and an electron on the other.

$$\frac{n_i}{n_0} = \frac{g_i}{g_0} \exp\left(-\frac{\chi + \frac{1}{2}mv^2}{kT}\right)$$

where χ is the ionization potential. The statistical weight for the ionized state, g_i, is

$$g_i = g\,(\text{ion})\, g\,(\text{electron})$$

where

$$g\,(\text{electron}) = 2 \times \frac{\text{volume of phase space around velocity } \vec{v}}{\text{volume of phase space we can distinguish}}$$

Here the 2 is the number of spin states, and the electron may have any velocity $\vec{v}$. The ability to distinguish relates to the uncertainty principle. Then

$$g\,(\text{electron}) = 2\frac{d^3\vec{x}d^3\vec{p}}{h^3}$$

The volume of space per electron is $1/n_e$, where n_e is the number density. Thus

$$\frac{dn_i}{n_0} = \frac{g_{\text{ion}}}{g_0}\frac{2}{n_e}\frac{4\pi p^2 dp}{h^3} \exp\left(-\frac{\chi + \frac{1}{2}mv^2}{kT}\right)$$

and then, summing over all the possible velocities and assuming isotropy, we have

$$\frac{n_i n_e}{n_0} = \frac{g_{\text{ion}}}{g_0} 2\frac{4\pi m^3}{h^3} \exp\left(-\frac{\chi}{kT}\right) \int_0^\infty \exp\left(-\frac{mv^2}{2kT}\right) v^2 \ dv$$

Letting $u = mv^2/2kT$, $du = (mv/kT)\,dv$, the integral is

$$\begin{aligned} \sqrt{2}\left(\frac{kT}{m}\right)^{3/2} \int_0^\infty u^{1/2} e^{-u} du &= \sqrt{2}\left(\frac{kT}{m}\right)^{3/2} \Gamma\,(3/2) \\ &= \sqrt{2}\left(\frac{kT}{m}\right)^{3/2} \frac{1}{2}\Gamma\,(1/2) \\ &= \sqrt{\frac{\pi}{2}}\left(\frac{kT}{m}\right)^{3/2} \end{aligned} \tag{2}$$

Thus

$$\begin{aligned}\frac{n_i n_e}{n_0} &= \frac{g_{\text{ion}}}{g_0} 2 \frac{m^3}{h^3} \exp\left(-\frac{\chi}{kT}\right) \frac{4\pi^{3/2}}{\sqrt{2}} \left(\frac{kT}{m}\right)^{3/2} \\ &= \frac{g_{\text{ion}}}{g_0} 2 \frac{(2\pi mkT)^{3/2}}{h^3} \exp\left(-\frac{\chi}{kT}\right) \qquad (3)\end{aligned}$$

This is the Saha equation.

In equation (3), the statistical weight g_{ion} is actually the sum of the statistical weights for all the energy states of the ion. This is called the *partition function,* B_i It is usually dominated by the ground state.

$$n_i = \sum_{j=1}^{\infty} n_{ij} = \frac{n_{i1}}{g_{i1}} \sum_{j=1}^{\infty} g_{ij} \exp\left(-\frac{E_{ij}}{kT}\right) = \frac{n_{i1}}{g_{i1}} B_i$$

The Saha equation shows that temperature is the dominant factor affecting the ionization state, but density also plays an important role. We may rewrite it as follows:

$$\frac{n_i}{n_0}\frac{n_e}{n_0} = \frac{g_{\text{ion}}}{g_0}\frac{2}{n_0}\frac{(2\pi mkT)^{3/2}}{h^3} \exp\left(-\frac{\chi}{kT}\right)$$

This version shows that the ionization fraction also depends on density as $1/\sqrt{n_0}$. Low density systems may have high ionization at relatively low temperatures.

Once ionized, the fundamental particles in the plasma (ions and electrons) are each electrically charged. Because the electromagnetic forces are long-range forces ($\propto 1/\text{distance}^2$), this creates interesting behaviors in which the motions of plasma particles may be correlated over large distances- so-called "collective behavior". Our task this semester is to study some of these behaviors.

3 Fundamental relations:

The relations governing the behavior of plasmas are:

Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_q}{\varepsilon_0} = \frac{e\,(n_i - n_e)}{\varepsilon_0} \qquad (4)$$

where the second equality applies to singly-ionized plasmas. Much of our work will involve hydrogen plasmas, for which this seond relation is true.

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (5)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (6)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (7)$$

Newton's law with the Lorentz force law, applied to each plasma particle, with the appropriate mass m and charge q:

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{8}$$

Other forces may be included, but usually the Lorentz force is dominant.

The *distribution function* describes the velocities of the plasma particles. In equilibrium, the distribution function is a Maxwellian:

$$f\left(\vec{r}, \vec{v}, t\right) = n\left(\vec{r}, t\right)\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{1}{2}\frac{mv^2}{kT}\right) \tag{9}$$

where $n\left(\vec{r}, t\right)$ is the density of plasma particles as a function of position and time. This distribution is isotropic: there is no dependence on the direction of the particle velocities. More generally, the distribution of velocities may also depend on the direction of the velocity as well as on space and time.

This system of equations is inherently non-linear, so we frequently find it necessary to approximate in order to solve the set of equations. We'll look at several different techniques during the course.

The number density of particles is

$$n\left(\vec{r}, t\right) = \int f\left(\vec{r}, \vec{v}, t\right) d^3\vec{v}$$

The Maxwellian (9) has been normalized:

$$\int_0^\infty \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{1}{2}\frac{mv^2}{kT}\right) 4\pi v^2 dv = 1$$

The current density is

$$\vec{j} = Ze\int f_i\left(r, \vec{v}_i, t\right) \vec{v}_i\ d^3\vec{v}_i - e\int f_e\left(r, \vec{v}_e, t\right) \vec{v}_e\ d^3\vec{v}_e \tag{10}$$

If the plasma is "cold" ($T \rightarrow 0$, or every particle has the same velocity), this expression may be simplified:

$$\vec{j} = Zen_i\vec{v}_i - en_e\vec{v}_e \tag{11}$$

If the distribution function is Maxwellian, the average energy per particle is

$$< E > \ = \frac{\int \frac{1}{2}mv^2 f\left(v\right) d^3v}{\int f\left(v\right) d^3v} = \frac{3}{2}kT$$

or the usual "one half kT per degree of freedom". The corresponding rms speed is

$$\sqrt{< v^2 >} = \sqrt{\frac{3kT}{m}} = 1.7\sqrt{\frac{kT}{m}}$$

The most probable speed v_{p} is the speed at which the distribution function is maximum. To evaluate this, remember that

$$d^3v = 4\pi v^2 \; dv$$

and so we need to solve

$$\frac{d}{dv}\left[v^2 \exp\left(-\frac{1}{2}\frac{mv^2}{kT}\right)\right] = 0$$

or

$$2v \exp\left(-\frac{1}{2}\frac{mv^2}{kT}\right) + v^2\left(-\frac{mv}{kT}\right)\exp\left(-\frac{1}{2}\frac{mv^2}{kT}\right) = 0$$

$$v_{\mathrm{p}} = \sqrt{\frac{2kT}{m}} = 1.4\sqrt{\frac{kT}{m}}$$

The average speed is

$$< v > \;\; = \frac{\int v f(v)\, d^3v}{\int f(v)\, d^3v} = \frac{\int v^3 \exp\left(-\frac{1}{2}\frac{mv^2}{kT}\right)\; dv}{\int v^2 \exp\left(-\frac{1}{2}\frac{mv^2}{kT}\right)\; dv}$$

Letting $u = mv^2/2kT$, $du = (mv/kT)\, dv$, the numerator is

$$\begin{aligned} 2\left(\frac{kT}{m}\right)^2 \int_0^\infty ue^{-u}du &= 2\left(\frac{kT}{m}\right)^2 \left[-ue^{-u}\Big|_0^\infty + \int_0^\infty e^{-u}du\right] \\ &= 2\left(\frac{kT}{m}\right)^2 \end{aligned}$$

while the denominator is given by (2). Thus

$$\begin{aligned} < \;\; v > \;\; &= 2\left(\frac{kT}{m}\right)^2 \Big/ \sqrt{\frac{\pi}{2}}\left(\frac{kT}{m}\right)^{3/2} = 2\sqrt{\frac{2}{\pi}\left(\frac{kT}{m}\right)} \\ &= \;\; 1.6\sqrt{\frac{kT}{m}} \end{aligned}$$

Note that

$$v_{\mathrm{p}} < \;\; < v > \;\; < \sqrt{< v^2 >}$$

Now we're ready to explore some of the properties of plasmas.

4 Assumptions:

We almost always assume that the plasma is *quasi-neutral.* This means that when we average over a large volume of space, the system is electrically neutral. This is reasonable if the plasma was formed by ionizing a neutral gas. We also need enough particles in the system for it to be reasonable to compute the average behavior, rather than focussing on the motion of the individual particles. We'll see how to specify these constraints in terms of the plasma properties of density and temperature.