Effect of scattering on the emitted spectrum

S.M.Lea

Here we are going to consider what happens when a system is effected by scattering as well as emission and absorption. For simplicity we consider a simple model with only one emission mechanism. Many astrophysical systems may be understood using this simple model. Much of the early work was done to understand the spectra of compact x-ray sources and supernovae.

1 Coherent scattering

Consider a slab of ionized hydrogen. The emission and absorption is due to brensstrahlung (free-free), and we also have Thomson scattering. Then:

$$\alpha_\nu = \left(4\times 10^8 \text{ cm}^{-1}\right)\frac{n^2}{\nu^3 T^{1/2}}\left(1-e^{-h\nu/kT}\right)g$$

where g is the Gaunt factor, and

$$\alpha_s = n\sigma_T$$

1.1 large cloud, τ_s large

1.1.1 low frequency:

At a low enough frequency, the absorption optical depth will be much larger than the scattering optical depth, since α_ν increases as ν decreases. In this regime we can ignore the effects of scattering, since $\tau_{eff} = \sqrt{\tau_\nu\left(\tau_\nu+\tau_s\right)} \simeq \tau_\nu$. Thus the emitted radiation is a black body spectrum:

$$I_\nu = B_\nu\left(T\right) = \frac{2h\nu^3/c^2}{e^{h\nu/kT}-1} \simeq \frac{2h\nu^3/c^2}{h\nu/kT} = 2\frac{\nu^2}{c^2}kT$$

which is the Rayleigh-Jeans law.

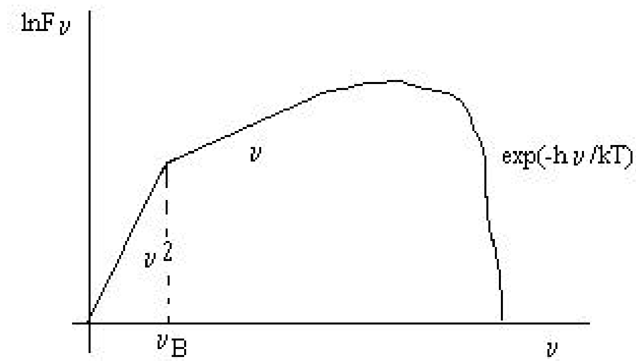
1.1.2 high frequency

At a high enough frequency, the absorption optical depth becomes small enough that $\tau_\nu \ll \tau_s$. Then the "skin depth" of the source is approximately $\left[\alpha_\nu\left(\alpha_\nu+\alpha_s\right)\right]^{-1/2}$ and the emitted spectrum is:

$$I_\nu = j_\nu \left[\alpha_\nu \left(\alpha_\nu + \alpha_s\right)\right]^{-1/2} = \frac{j_\nu}{\alpha_\nu}\sqrt{\frac{\alpha_\nu}{\alpha_\nu + \alpha_s}} \simeq B_\nu\left(T\right)\sqrt{\frac{\alpha_\nu}{\alpha_s}}$$

a "diluted" black body.

If we still have $h\nu \ll kT$ in this regime, then $B_\nu \sim \nu^2$, $\alpha_\nu \sim 1/\nu^2$ and $I_\nu \sim \nu$. Eventually, at higher ν, $B_\nu \sim e^{-h\nu/kT}$ and $I_\nu \sim e^{-h\nu/kT}$.



The transition from the low frequency ($I_\nu \sim \nu^2$) to high frequency ($I_\nu \sim \nu$) regimes occurs where

$$\tau_\nu \simeq \tau_s$$

or at a frequency ν_B where

$$\left(4 \times 10^8 \text{ cm}^5\text{s}^{-3}\text{K}^{1/2}\right)\frac{n^2}{\nu^3 T^{1/2}}\left(\frac{h\nu}{kT}\right) g \approx n\sigma_T$$

and so

$$\begin{aligned}\nu_B &= \sqrt{\frac{\left(4 \times 10^8 \text{ cm}^5\text{s}^{-3}\text{K}^{1/2}\right) n}{T^{3/2}\sigma_T}\frac{h}{k} g} = \sqrt{\frac{\left(4 \times 10^8 \text{ cm}^5\text{s}^{-3}\text{K}^{1/2}\right) n}{T^{3/2}\left(6.6 \times 10^{-25}\text{ cm}^2\right)}\frac{6.6 \times 10^{-27}\text{ erg}\cdot\text{s}}{1.4 \times 10^{-16}\text{ erg/K}} g} \\ &= \sqrt{2.9 \times 10^{22}\text{ cm}^3\text{s}^{-2}\text{K}^{3/2}\frac{n}{T^{3/2}} g} = \frac{1.7 \times 10^{11}\text{ cm}^{3/2}\text{K}^{3/4}}{T^{3/4}} n^{1/2} g^{1/2}\end{aligned}$$

The exact solution that is also valid at the "knee" is

$$I_\nu = B_\nu \left(1 - \varepsilon\right)^{1/2} H$$

where $\varepsilon = \alpha_s/\left(\alpha_s + \alpha_\nu\right)$ and H is a weak function of ε, varying between 1 and 3.

1.2 Thin slab, $\tau_\nu < 1$ at high frequencies, but $\tau_s > 1$.

The slab becomes translucent (not transparent) at high ν. Photons escape without absorption for $\tau_{eff} < 1$,or

$$\sqrt{\tau_\nu\left(\tau_\nu + \tau_s\right)} < 1$$

Since $\tau_\nu \ll \tau_s$,this becomes:

$$\tau_\nu \tau_s < 1$$

The transition occurs at $\tau_\nu \sim \tau_s^{-1}$ at frequency ν_t :

$$\left(4 \times 10^8 \text{ cm}^5\text{s}^{-3}\text{K}^{1/2}\right) \frac{n^2}{\nu^3 T^{1/2}} \left(\frac{h\nu}{kT}\right) g\ell \approx \frac{1}{n\sigma_T \ell}$$

Thus:

$$\begin{aligned} \nu_t &\approx \sqrt{\left(4 \times 10^8 \text{ cm}^5\text{s}^{-3}\text{K}^{1/2}\right) \frac{n^3 \sigma_T g \ell^2 h}{kT^{3/2}}} \\ &= \sqrt{\left(4 \times 10^8 \text{ cm}^5\text{s}^{-3}\text{K}^{1/2}\right) (6.6 \times 10^{-25} \text{ cm}^2) \frac{6.6 \times 10^{-27} \text{ erg} \cdot \text{s}}{1.4 \times 10^{-16} \text{ erg/K}} \frac{n^3 g \ell^2}{T^{3/2}}} \\ &= \sqrt{1.2 \times 10^{-26} \frac{\text{cm}^7}{\text{s}^2} \text{K}^{3/2} \frac{n^3 g \ell^2}{T^{3/2}}} = 1.1 \times 10^{-13} \text{ Hz} \cdot \text{cm}^{7/2}\text{K}^{3/4} \sqrt{\left(\frac{n^3 g}{T^{3/2}}\right)} \ell \end{aligned}$$

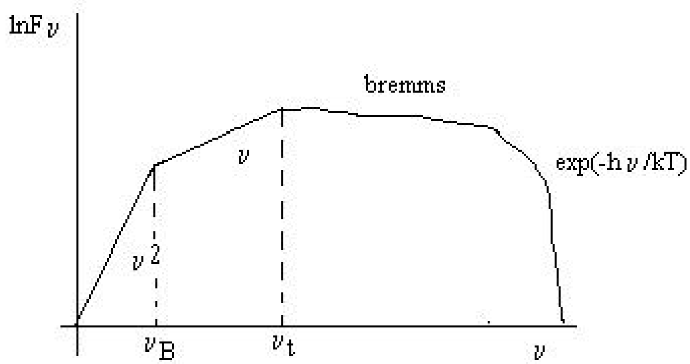
Note that ν_B is defined by $\tau_\nu \sim \tau_s$ while ν_t is defined by $\tau_\nu \sim \tau_s^{-1}$, and since $\tau_\nu \sim \nu^{-2}$, then

$$\frac{\nu_t^{-2}}{\nu_B^{-2}} \sim \frac{1}{\tau_s^2}$$

or

$$\nu_t / \nu_B \sim \tau_s$$

Above ν_t,the source is optically thin and the spectrum is a bremsstrahlung spectrum.



2 Incoherent scattering

The frequency of the scattered photon may differ from that of the incoming photon if

1. the incoming photon has a high frequency ($h\nu \gtrsim mc^2$) (Compton scattering) or
2. the electron is relativistic, moving with speed $v = \beta c$ and large $\gamma = 1/\sqrt{1 - v^2/c^2}$.

In calculating the frequency of the scattered photon in the second case, we first move into the rest frame of the incoming electron. In this frame, the incoming photon is doppler shifted to a frequency $\sim \gamma\nu_0$, but the frequency shift is probably small (Thomson scattering). Then we shift back to the lab frame, Doppler shifting the frequency by a factor γ again. (For an exact derivation of this result, see Rybicki and Lightman p 195-199) The net power radiated by the scattered photons is approximately

$$P \sim u_{ph} c \sigma_T \gamma^2$$

where $u_{ph} = hv_0 n_{ph}$ is the energy density in the photon field. More exactly (R&L eqn 7.16a)

$$P = \frac{4}{3} u_{ph} c \sigma_T \gamma^2 \beta^2 \tag{1}$$

This process is called Inverse Compton Scattering.

In this section we'll consider how the frequency shifts in each scattering can effect the emergent spectrum from a source.

2.1 Single scattering

2.1.1 Approximate treatment

(For a more exact analysis, see R&L §7.3)

We consider a cloud with a particle distribution $n(\gamma)$ particles per unit volume per unit energy and an incident photon spectrum $f(\varepsilon)$ photons per unit volume per unit energy. We may write an approximate emission coefficient due to scattered photons as:

$$j(\varepsilon') = \frac{1}{4\pi} \int \delta\left(\varepsilon' - \gamma^2 \varepsilon\right) n(\gamma) f(\varepsilon) P(\gamma, \varepsilon)\, d\varepsilon d\gamma$$

where $P(\gamma, \varepsilon)$ is the power radiated in a single scattering,

$$P(\gamma, \varepsilon) = \frac{4}{3} c \sigma_T \gamma^2 \beta^2 \varepsilon \tag{2}$$

Now we want to do the integral over the electron energy spectrum, so we rewrite the $\delta-$function as:

$$\delta\left(\varepsilon' - \gamma^2 \varepsilon\right) = \frac{1}{2\gamma\varepsilon} \delta\left(\gamma - \sqrt{\frac{\varepsilon'}{\varepsilon}}\right) = \frac{1}{2\sqrt{\varepsilon'\varepsilon}} \delta\left(\gamma - \sqrt{\frac{\varepsilon'}{\varepsilon}}\right)$$

and then after integration we get

$$\begin{aligned} j(\varepsilon') &= \frac{1}{4\pi} \int \frac{1}{2\sqrt{\varepsilon'\varepsilon}} n\left(\sqrt{\frac{\varepsilon'}{\varepsilon}}\right) f(\varepsilon) \frac{4}{3} c \sigma_T \frac{\varepsilon'}{\varepsilon} \beta^2 \varepsilon d\varepsilon \\ &= \frac{c\sigma_T}{6\pi} \int \sqrt{\frac{\varepsilon'}{\varepsilon}} n\left(\sqrt{\frac{\varepsilon'}{\varepsilon}}\right) f(\varepsilon) \beta^2 d\varepsilon \end{aligned}$$

Now let $n(\gamma)$ be a power law

$$n(\gamma) = K\gamma^{-p}$$

so

$$\begin{aligned} j(\varepsilon') &= \frac{c\sigma_T}{6\pi}\int \sqrt{\frac{\varepsilon'}{\varepsilon}}K\left(\sqrt{\frac{\varepsilon'}{\varepsilon}}\right)^{-p} f(\varepsilon)\beta^2 d\varepsilon \\ &= K\frac{c\sigma_T}{6\pi}\int \left(\sqrt{\frac{\varepsilon'}{\varepsilon}}\right)^{-p+1} f(\varepsilon)\beta^2 d\varepsilon \\ &= K\frac{c\sigma_T}{6\pi}\varepsilon'^{-(p-1)/2}\int \varepsilon^{(p-1)/2} f(\varepsilon)\, d\varepsilon \end{aligned}$$

where in the last step I took $\beta \approx 1$. The limits of the integral are the maximum and minimum values of the incident spectrum. Thus the emission coefficient leads to a power law spectrum with index $-\alpha = -(p-1)/2$.

A more exact analysis gives almost the same result. There is a multiplicative factor of:

$$\frac{9}{16}A(p)$$

where $A(p)\, is$ a slowly varying function of p.

p	$A(p)$
0	1.96
1	1.33
2	1.4

and thus the difference between my result and the exact result is a factor of between 0.9 and 1.6.

The integral $\int \varepsilon^{(p-1)/2} f(\varepsilon)\, d\varepsilon$ can be done and gives an analytical closed form solution when $f(\varepsilon)$ is a black body spectrum, as discussed in R&L (p208). The result is important when the scattered photons are the microwave BB radiation (cosmic background radiation).

In some sources the photons are also produced within the same source, e.g. by synchrotron radiation. In this case we have $f(\varepsilon) \sim \varepsilon^{-\alpha-1}$ (more about this later) leading to

$$j(\varepsilon') \propto \varepsilon'^{-(p-1)/2}\int \varepsilon^{\alpha}\varepsilon^{-\alpha-1}d\varepsilon = \varepsilon'^{-(p-1)/2}\ln(\varepsilon_{\max}/\varepsilon_{\min})$$

which diverges unless $\varepsilon_{\max}$ is finite. This is always the case due to rapid radiation losses at the high energy end of the electron spectrum.

2.2 Multiple scatterings

2.2.1 Energy change per scattering

The small energy change that occurs in Compton scattering can become important when multiple scatterings occur. we have:

$$\frac{\triangle\varepsilon}{\varepsilon} \simeq -\frac{\varepsilon}{mc^2}$$

in the electron rest frame. If we have a thermal distribution of electrons, some of them will have less energy than ε and some will have more. The less energetic electrons gain energy from the photon, but the more energetic electrons lose energy to the photon. Averaging over the entire distribution of electrons we expect to find:

$$\frac{\triangle\varepsilon}{\varepsilon} \simeq \alpha\frac{kT}{mc^2} - \frac{\varepsilon}{mc^2} \tag{3}$$

in the lab frame. If the system is isolated, it should eventually come to equilibrium at some temperature. If the only interaction between photons and matter is scattering (no absorption and emission) then the equilibrium photon distribution is NOT a black body spectrum, but rather a "relativistic Maxwellian"

$$N(\varepsilon) \propto \varepsilon^2 e^{-\varepsilon/kT}$$

For this distribution

$$<\varepsilon>= \frac{\int \varepsilon N(\varepsilon)\, d\varepsilon}{\int N(\varepsilon)\, d\varepsilon} = \frac{\int \varepsilon^3 e^{-\varepsilon/kT} d\varepsilon}{\int \varepsilon^2 e^{-\varepsilon/kT} d\varepsilon}$$

Letting $y = \varepsilon/kT$, we get

$$<\varepsilon>= \frac{(kT)^4}{(kT)^3}\frac{\int y^3 e^{-y} dy}{\int y^2 e^{-y} dy}$$

Now integrate by parts:

$$\int_0^\infty y^3 e^{-y} dy = -y^3 e^{-y}\Big|_0^\infty + \int_0^\infty 3y^2 e^{-y} dy = \int_0^\infty 3y^2 e^{-y} dy$$

and thus

$$<\varepsilon>= 3kT$$

Also:

$$<\varepsilon^2>= \frac{\int \varepsilon^2 N(\varepsilon)\, d\varepsilon}{\int N(\varepsilon)\, d\varepsilon} = \frac{\int \varepsilon^4 e^{-\varepsilon/kT} d\varepsilon}{\int \varepsilon^2 e^{-\varepsilon/kT} d\varepsilon} = 4 \times 3\,(kT)^2$$

Now once equilibrium is established, there is no further net energy transfer between electrons and photons, so averaging equation 3, we have:

$$<\triangle\varepsilon>\simeq \alpha\frac{kT}{mc^2} <\varepsilon> - \frac{<\varepsilon^2>}{mc^2} = 0$$

and thus

$$\alpha kT\,(3kT) - 3 \times 4\,(kT)^2 = 0 \Rightarrow \alpha = 4$$

So we may rewrite equation 3 as

$$\triangle\varepsilon \simeq \frac{\varepsilon}{mc^2}\,(4kT - \varepsilon) \tag{4}$$

Now we may reinterpret equation 1 for thermal electrons with $<\beta^2>= 3kT/mc^2$:

$$\begin{aligned} < \quad P >&= \frac{4}{3} u_{ph} c\sigma_T \frac{3kT}{mc^2} \\ &= 4\frac{kT}{mc^2} <\varepsilon> n_{ph} c\sigma_T \\ &= (\text{average energy shift per scattering})\,(\text{rate of scattering}) \end{aligned}$$

As an aside, we note here that this scattering process also acts as a heating or cooling process for the electrons. The heating rate Γ may be written as:

$$\Gamma = F(\varepsilon)\, n_e \sigma_T \frac{\varepsilon}{mc^2}(\varepsilon - 4kT)$$

where $F(\varepsilon)$ is the photon flux to which the electrons are exposed. (If Γ is negative the result is a net cooling of the plasma.)

2.2.2 Emergent spectrum

The net change in photon energy due to multiple scatterings is determined using the Compton "y" parameter:

$$\begin{aligned} y &= \frac{4kT}{mc^2}\max\left(\tau, \tau^2\right) \qquad (5) \\ &= \text{(fractional energy change per scattering) (mean \# of scatterings)} \end{aligned}$$

where $\varepsilon \ll 4kT$ has been assumed in the interpretation in words, and τ is the scattering optical depth.

If absorption is important, we need the number of scatterings within one mean free path of the surface; that is, the "τ" that we need in y is

$$\begin{aligned} \tau_y &= \alpha_s \text{ (mfp between emission and absorption)} \\ &= \alpha_s \left[\alpha_{abs}\left(\alpha_{abs} + \alpha_s\right)\right]^{-1/2} \\ &= \frac{\tau_s}{\tau_{eff}} \qquad (\tau_{eff} \gtrsim 1) \qquad (6) \end{aligned}$$

Emergent spectrum for small τ Let A be the amplification factor per scattering. (For example, $A = \gamma^2$ for relativistic electrons). Consider an input beam of photons at energy $hv = \varepsilon_i$. After k scatterings, the incidents photon is scattered to energy

$$\varepsilon_k \sim \varepsilon_i A^k \qquad (7)$$

For $\tau_s < 1$, the probability of k scatterings is τ^k, and so

$$I(\varepsilon_k) \simeq I_0(\varepsilon_i)\, \tau^k$$

and from equation 7,

$$k = \frac{\ln(\varepsilon_k/\varepsilon_i)}{\ln A}$$

To proceed, note that

$$\ln\left(a^{\ln b/\ln c}\right) = \frac{\ln b}{\ln c}\ln a = \frac{\ln a}{\ln c}\ln b = \ln\left(b^{\ln a/\ln c}\right)$$

so

$$\tau^{\ln(\varepsilon_k/\varepsilon_i)/\ln A} = (\varepsilon_k/\varepsilon_i)^{\ln\tau/\ln A}$$

Since $\tau < 1$ in this example, it is useful to define $\alpha \equiv -\ln\tau/\ln A$, so that $\alpha > 0$. Then the emitted spectrum has the form:

$$I(\varepsilon_k) \simeq I_0(\varepsilon_i)\left(\frac{\varepsilon_k}{\varepsilon_i}\right)^{-\alpha}$$

i.e. the spectrum is a power law of index α. Note that this derivation makes sense only if

$A > 1$ (e.g. scattering off relativistic electrons.) The result is important since so many astronomical objects have power law spectra over a fairly large range of frequencies. One of the other common models- synchrotron radiation- can run into trouble because of the "Compton catastrophe". (Photons are Compton scattered up to higher energy, drastically reducing the lifetime of the relativistic electrons, and thus increasing the energy requirements for the source.)

The Kompaneets equation (Reference: Kompaneets, **JETP 4,** 730, 1957)

In the event that the amplification factor A is not large, the previous analyis fails. We must develop a more exact formalism. We derive the Boltzmann equation, which describes the change in the distribution function $n(\omega)$ of the photons in phase space due to scattering off electrons.

Let $n(\omega)$ be the number of photons/cm^3/momentum space volume at frequency ω.

Let $f_e(\vec{p})$ be the number of electrons/cm^3/momentum space volume $d^3\vec{p}$ at momentum $\vec{p}$.

$n(\omega)$ is changed because photons are scattered to a new frequency ω'.

$$\#\text{ scattered out/time } = \int\int f_e(\vec{p})\, n(\omega)\, c\,(1 + n(\omega'))\, \frac{d\sigma}{d}d \quad d^3\vec{p}$$

where $n(\omega)\,c$ is the photon flux and the factor $n(\omega')$ is due to induced scattering. (Induced scattering is similar to stimulated emission, and occurs because photons are bosons.) The values of $\vec{p}$, ω and ω' are related by the physics of the scattering event. Similarly:

$$\#\text{ scattered in/time } = \int\int f_e(\vec{p}_1)\, n(\omega')\, c\,(1 + n(\omega))\, \frac{d\sigma}{d}d \quad d^3\vec{p}_1$$

The rate of change of $n(\omega)$ is due to these two processes:

$$\frac{\partial n(\omega)}{dt} = c\int \frac{d\sigma}{d}d \quad \left[\int f_e(\vec{p}_1)\, n(\omega')\, c\,(1 + n(\omega))\, d^3\vec{p}_1 - \int f_e(\vec{p})\, n(\omega)\, c\,(1 + n(\omega'))\, d^3\vec{p}\right] \tag{8}$$

To simplify this equation, we make 2 approximations:

1. Electrons are non-relativistic

$$f_e(\vec{p}) = \frac{n_e}{(2\pi mkT)^{3/2}} \exp\left(-p^2/2mkT\right)$$

and

2. Energy transfer per scattering is small.

$$\triangle \equiv \frac{\hbar(\omega' - \omega)}{kT} \ll 1 \tag{9}$$

Then we expand $n(\omega')$ in a Taylor series:

$$\begin{aligned} n(\omega') &= n(\omega) + (\omega' - \omega)\frac{\partial n}{\partial \omega} + \frac{1}{2}(\omega' - \omega)^2 \frac{\partial^2 n}{\partial \omega^2} + \cdots \\ &= n(\omega) + \triangle\frac{\partial n}{\partial x} + \frac{1}{2}\triangle^2\frac{\partial^2 n}{\partial x^2} + \cdots \end{aligned}$$

where $x = \hbar\omega/kT$.

Now we need to relate $\vec{p}_1$ to $\vec{p}$ and $\triangle$. By conservation of energy:

$$\frac{p_1^2}{2m} + \hbar\omega' = \frac{p^2}{2m} + \hbar\omega$$

so

$$\frac{p_1^2}{2m} = \frac{p^2}{2m} + \hbar\omega - \hbar\omega' = \frac{p^2}{2m} - kT\triangle \tag{10}$$

Thus

$$f_e\left(\vec{p}_1\right) \propto \exp\left(-p_1^2/2mkT\right) = \exp\left(-p^2/2mkT\right)e^{\triangle}$$

and expanding $e^{\triangle}$ to second order, we get:

$$f_e\left(\vec{p}_1\right) = f_e\left(\vec{p}\right)\left(1 + \triangle + \triangle^2/2\right)$$

Then:

$$\begin{aligned} f_e\left(\vec{p}_1\right) n\left(\omega'\right)\left(1 + n\left(\omega\right)\right) &= f_e\left(\vec{p}\right)\left(1 + \triangle + \triangle^2/2\right)\left(n + \triangle n' + \triangle^2 n''/2\right)\left(1 + n\right) \\ &= f_e\left(\vec{p}\right)\left(n + \triangle\left(n' + n\right) + \frac{1}{2}\triangle^2\left(n'' + 2n' + n\right)\right)\left(1 + n\right) \end{aligned}$$

and

$$f_e\left(\vec{p}\right) n\left(\omega\right)\left(1 + n\left(\omega'\right)\right) = f_e\left(\vec{p}\right) n\left(1 + n + \triangle n' + \frac{1}{2}\triangle^2 n''\right)$$

In our primary equation 8, we need

$$\begin{aligned} \frac{\partial n\left(\omega\right)}{dt} &= c\int \frac{d\sigma}{d}d\quad\left(\int f_e\left(\vec{p}_1\right) n\left(\omega'\right) c\left(1 + n\left(\omega\right)\right) d^3\vec{p}_1 - \int f_e\left(\vec{p}\right) n\left(\omega\right) c\left(1 + n\left(\omega'\right)\right) d^3\vec{p}\right) \\ &= c\int \frac{d\sigma}{d}d\quad\int d^3\vec{p}\left[\begin{array}{c} f_e\left(\vec{p}\right)\left(n + \triangle\left(n' + n\right) + \frac{1}{2}\triangle^2\left(n'' + 2n' + n\right)\right)\left(1 + n\right) \\ -f_e\left(\vec{p}\right) n\left(1 + n + \triangle n' + \frac{1}{2}\triangle^2 n''\right) \end{array}\right] \\ &= c\int \frac{d\sigma}{d}d\quad\int d^3\vec{p} f_e\left(\vec{p}\right)\left(\left(\triangle\left(n' + n + n^2\right) + \frac{1}{2}\triangle^2\left(n'' + \left(1 + n\right)\left(n + 2n'\right)\right)\right)\right) \end{aligned} \tag{11}$$

To proceed further we need $\triangle$ in terms of $\vec{p}$. So we use conservation of momentum. In the non-relativistic case:

$$\frac{\hbar\omega}{c}\hat{k} + \vec{p} = \frac{\hbar\omega'}{c}\hat{k}' + \vec{p}_1$$

where $\hat{k}$, $\hat{k}'$ are unit vectors in the direction of photon propagation (unit wave vectors). Then:

$$\vec{p}_1 = \frac{\hbar\omega}{c}\hat{k} + \vec{p} - \frac{\hbar\omega'}{c}\hat{k}' = \vec{p} + \frac{\hbar}{c}\left(\omega\hat{k} - \omega'\hat{k}'\right)$$

and

$$p_1^2 = p^2 + \frac{\hbar^2}{c^2}\left(\omega^2 + \omega'^2 - 2\omega\omega'\vec{k}\cdot\hat{k}'\right) + 2\frac{\hbar}{c}\left(\omega\hat{k} - \omega'\hat{k}'\right)\cdot\vec{p} \tag{12}$$

We use relation 12 to eliminate p_1 from equation 10:

$$\triangle = \frac{p^2 - p_1^2}{2mkT} = -\frac{\frac{\hbar^2}{c^2}\left(\omega^2 + \omega'^2 - 2\omega\omega'\vec{k}\cdot\hat{k}'\right) + 2\frac{\hbar}{c}\left(\omega\hat{k} - \omega'\hat{k}'\right)\cdot\vec{p}}{2mkT}$$

and then using equation 9

$$\hbar\omega' = \hbar\omega + \triangle kT$$

so to first order in $\triangle$:

$$\triangle = -\frac{2\frac{\hbar^2}{c^2}\omega^2 + 2\triangle kT\frac{\hbar\omega}{c^2} - 2\frac{\hbar}{c^2}\omega\left(\hbar\omega + \triangle kT\right)\vec{k}\cdot\hat{k}' + \frac{2}{c}\left(\hbar\omega\hat{k} - \left(\hbar\omega + \triangle kT\right)\hat{k}'\right)\cdot\vec{p}}{2mkT}$$

$$= -\frac{2\frac{\hbar^2}{c^2}\omega^2\left(1 - \vec{k}\cdot\hat{k}'\right) + 2\triangle kT\frac{\hbar\omega}{c^2} - 2\frac{\hbar}{c^2}\omega\triangle kT\vec{k}\cdot\hat{k}' + \frac{2}{c}\left(\hbar\omega\left(\hat{k} - \hat{k}'\right) - \triangle kT\hat{k}'\right)\cdot\vec{p}}{2mkT}$$

$$= -\frac{\hbar\omega}{mc^2}\frac{\hbar\omega}{kT}\left(1 - \vec{k}\cdot\hat{k}'\right) - \triangle\left(\frac{\hbar\omega}{mc^2}\left(1 - \vec{k}\cdot\hat{k}'\right) - \hat{k}'\cdot\frac{\vec{p}}{mc}\right) - \frac{\hbar\omega}{kT}\left(\hat{k} - \hat{k}'\right)\cdot\frac{\vec{p}}{mc}$$

Collecting terms:

$$\triangle\left(1 + \frac{\hbar\omega}{mc^2}\left(1 - \vec{k}\cdot\hat{k}'\right) - \hat{k}'\cdot\frac{\vec{p}}{mc}\right) = -\frac{\hbar\omega}{mc^2}\frac{\hbar\omega}{kT}\left(1 - \vec{k}\cdot\hat{k}'\right) - \frac{\hbar\omega}{kT}\left(\hat{k} - \hat{k}'\right)\cdot\frac{\vec{p}}{mc}$$

For photons with $\hbar\omega \ll mc^2 = 511$ keV, the bracket on the LHS is approximately $= 1$, and we may ignore the first term on the RHS, to get

$$\triangle = \frac{\hbar\omega}{kT}\left(\hat{k}' - \hat{k}\right)\cdot\frac{\vec{p}}{mc} = x\left(\hat{k}' - \hat{k}\right)\cdot\frac{\vec{p}}{mc}$$

Now we return to equation 11. The differntial scattering cross section is:

$$\frac{d\sigma}{d} = \frac{3\sigma_T}{16\pi}\left(1 + \cos^2\theta\right)$$

where θ is the angle between $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$. Evaluating the $\triangle^2$ term first, we have:

$$I_1 = \int\frac{d\sigma}{d}d \quad \int d^3\vec{p}f_e\left(\vec{p}\right)\triangle^2 = \int\frac{d\sigma}{d}d \quad \int d^3\vec{p}f_e\left(\vec{p}\right)x^2\frac{p^2}{m^2c^2}\left(\hat{\mathbf{k}}' - \hat{\mathbf{k}}\right)^2\cos^2\xi$$

where ξ is the angle between $\vec{p}$ and $\hat{k}' - \hat{k}$. We do the integral over $\vec{p}$ first, taking ξ as the polar angle in a spherical coordinate system, so that

$$d^3\vec{p} = p^2dp\sin\xi d\xi d\phi' = p^2dpd\mu_\xi d\phi'$$

where μ_ξ = cosξ. Then

$$I_1 = \frac{3\sigma_T}{16\pi}\frac{n_ex^2}{\left(2\pi mkT\right)^{3/2}}\int\left(1 + \cos^2\theta\right)\left(\hat{\mathbf{k}}' - \hat{\mathbf{k}}\right)^2\sin\theta d\theta d\phi\int\frac{p^2}{m^2c^2}\exp\left(-p^2/2mkT\right)p^2dp\mu_\xi^2d\mu_\xi d\phi'$$

$$= \frac{3\sigma_T}{16\pi}\frac{n_ex^2}{\left(2\pi mkT\right)^{3/2}}\int\left(1 + \cos^2\theta\right)\left(\hat{\mathbf{k}}' - \hat{\mathbf{k}}\right)^2\sin\theta d\theta d\phi\int\frac{p^4}{m^2c^2}\exp\left(-p^2/2mkT\right)dp\,\frac{1}{3}\mu_\xi^3\Big|_{-1}^{+1}2\pi$$

$$= \frac{3\sigma_T}{16\pi}\frac{n_ex^2}{\left(2\pi mkT\right)^{3/2}}\int\left(1 + \cos^2\theta\right)\left(\hat{\mathbf{k}}' - \hat{\mathbf{k}}\right)^2\sin\theta d\theta d\phi\frac{1}{m^2c^2}\int_0^\infty p^4\exp\left(-p^2/2mkT\right)dp\frac{4\pi}{3}$$

Change variables to $u = p^2/2mkT$ Then $du = pdp/mkT$, and

$$\begin{aligned}\int_0^\infty p^4 \exp\left(-p^2/2mkT\right) dp &= \int_0^\infty (2mkT)^{3/2} u^{3/2} \exp(-u)\, mkTdu \\ &= \frac{1}{2}(2mkT)^{5/2} \int_0^\infty u^{3/2} \exp(-u)\, du \\ &= \frac{1}{2}(2mkT)^{5/2}\, \Gamma(5/2)\end{aligned}$$

Now

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

So:

$$\begin{aligned}I_1 &= \frac{3\sigma_T}{16\pi}\frac{n_e x^2}{(2\pi mkT)^{3/2}} \int \left(1+\cos^2\theta\right)\left(\hat{\mathbf{k}}'-\hat{\mathbf{k}}\right)^2 \sin\theta d\theta d\phi \frac{1}{m^2c^2}\frac{3}{8}\sqrt{\pi}\,(2mkT)^{5/2}\,\frac{4\pi}{3} \\ &= \frac{3}{16\pi}\frac{n_e\sigma_T}{mc^2} kTx^2 \int \left(1+\cos^2\theta\right)\left(\hat{\mathbf{k}}'-\hat{\mathbf{k}}\right)^2 \sin\theta d\theta d\phi\end{aligned}$$

where

$$\left(\hat{\mathbf{k}}'-\hat{\mathbf{k}}\right)^2 = 2 - 2\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}' = 2\,(1-\cos\theta)$$

Making the usual substitution $\mu = \cos\theta$,the integral is:

$$\begin{aligned}2\int \left(1+\mu^2\right)(1-\mu)\, d\mu d\phi &= 4\pi \int_{-1}^{+1} \left(1-\mu+\mu^2-\mu^3\right) d\mu \\ &= 4\pi\left(\mu - \frac{\mu^2}{2} + \frac{\mu^3}{3} - \frac{\mu^4}{4}\right)\Bigg|_{-1}^{+1} \\ &= 4\pi\left(\frac{8}{3}\right)\end{aligned}$$

and then

$$I_1 = 4\pi\left(\frac{8}{3}\right)\frac{3}{16\pi}\frac{n_e\sigma_T}{mc^2}kTx^2 = 2n_e\sigma_T\frac{kT}{mc^2}x^2 \qquad (13)$$

Next we turn to the term in Δ. In principle, we could proceed the same way, but the term in $\left(\hat{\mathbf{k}}'-\hat{\mathbf{k}}\right)\cdot\tilde{\mathbf{p}}$ is much harder to work with than its square was. So instead we use the method from Kompaneets' original paper (also your text, p 215).

First note that since we are considering scattering only, the total number of photons is conserved. The total number of quanta is found by integrating the phase space density over momentum space.

$$N = \int n(\omega)\, 4\pi\left(\frac{\hbar\omega}{c}\right)^2 \frac{\hbar}{c} d\omega \propto \int nx^2 dx$$

So the conservation of photon number is expressed as

$$\frac{d}{dt}\int nx^2 dx = 0$$

or, equivalently,

$$x^2\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}\left(x^2 j\right)=0 \tag{14}$$

where j is the flux of photons in momentum space. (This is like the conservation law for charge or mass. Interpret x^2dx as the volume element dV in spherical coordinates. The density can change only by flux of particles into or out of the volume.) Now we need j. Note that in equilibrium j must vanish identically. But we know what n looks like in equilibrium. Kompaneets just assumed n_{eq} to be a Planck funtion, but that's not quite right. When the photon number is conserved, the equilibrium distribution is a Bose-Einstein distribution

$$n_{\text{eq}}=\frac{1}{e^{\alpha+x}-1} \tag{15}$$

where α depends on the occupation number N as $\alpha\propto 1/N$. So when the occupation number is small $\alpha\gg 1$ and

$$n_{\text{eq}}\propto e^{-x}$$

the Wien law.

Next we write the as-yet-undetermined integral as $n_e\sigma_T\frac{kT}{mc^2}I$, so the Boltzmann equation becomes:

$$\frac{1}{c}\frac{\partial n}{dt}=n_e\sigma_T\frac{kT}{mc^2}\left(x^2n''+2x^2n'\left(1+n\right)+n'I+n\left(1+n\right)\left(x^2+I\right)\right) \tag{16}$$

which must look like equation 14:

$$\frac{\partial n}{\partial t}=-j'-2\frac{j}{x} \tag{17}$$

and thus j has the form:

$$j=g\left(x\right)\left(n'+h\left(n,x\right)\right)$$

so that j' has a term in n'' nut no higher derivatives. We can determine the function h by looking at the equilibrium form (equation 15). For this function:

$$n_{\text{eq}}'=\frac{-e^{\alpha+x}}{\left(e^{\alpha+x}-1\right)^2}=-\left(\frac{1}{n_{\text{eq}}}+1\right)n_{\text{eq}}^2=-n_{\text{eq}}\left(1+n_{\text{eq}}\right)$$

If j is to be zero in equilibrium, then $h\left(n,x\right)=n\left(1+n\right)$. Now we compare equations 16 and 17 in detail.

$$j'=g'\left(n'+n\left(1+n\right)\right)+g\left(n''+n'\left(1+n\right)+nn'\right)=g'\left(n'+n\left(1+n\right)\right)+g\left(n''+n'+2nn'\right)$$

and so

$$\begin{aligned}-j'-\frac{2j}{x} &= -g'\left(n'+n\left(1+n\right)\right)-g\left(n''+n'+2nn'\right)-\frac{2g}{x}\left(n'+n\left(1+n\right)\right)\\ &= -\left[gn''+n'\left(g'+g\left(1+2n+\frac{2}{x}\right)+n\left(1+n\right)\left(g'+2\frac{g}{x}\right)\right)\right]\end{aligned}$$

Comapring the n'' terms in the two equations:

$$g=-n_e\sigma_T c\frac{kT}{mc^2}x^2$$

and comparing the n' term:

$$g' + g\left(1 + 2n - \frac{2}{x}\right) = -n_e\sigma_T c\frac{kT}{mc^2}\left(2x^2\left(1+n\right) + I\right) \tag{18}$$

and the $n\left(1+n\right)$ term:

$$\begin{aligned} n_e\sigma_T c\frac{kT}{mc^2}\left(x^2 + I\right) &= -\left(g' + 2\frac{g}{x}\right) \\ &= n_e\sigma_T c\frac{kT}{mc^2}\left(2x + 2x\right) \end{aligned}$$

From this last relation, we find

$$I = \left(4 - x\right)x$$

You should convince yourself that this also satisfies the n' term condition (18).

Notice that I (the term in Δ) gives us the secular change in frequency, which is positive for $4 > x$ $(4kT > \hbar\omega)$ and negative for $4 < x$ $(4kT < \hbar\omega)$, in agreement with equation 4. The term in Δ^2 may be thought of as the random walk change in energy. The final result is the Kompaneets equation:

$$\frac{\partial n}{\partial t} = n_e\sigma_T c\frac{kT}{mc^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\left(n' + n + n^2\right)\right) \tag{19}$$

It is valid under the conditions that

1. The electrons are non-relativistic $(kT/mc^2 \ll 1)$ and
2. we are in the Thomson limit $\hbar\omega/mc^2 \ll 1$.

The Kompaneets equation can always be solved numerically, but a few important cases can be treated analytically.

Evolution of total energy If the occupation number is small, $n \ll 1$, then the Kompaneets equation (19) can be simplified:

$$\frac{\partial n}{\partial t} = n_e\sigma_T c\frac{kT}{mc^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\left(n' + n\right)\right)$$

The total energy radiated is:

$$E = \int nx x^2 dx 4\pi\frac{\left(kT\right)^4}{c^3}$$

where x^2dx represents the volume element in momentum space. Then the rate of change of the energy is:

$$\frac{\partial E}{\partial t} = \int\frac{\partial n}{\partial t}x^3dx4\pi\frac{\left(kT\right)^4}{c^3}$$

At low frequencies, $x \ll 1$, we also have $n' \sim n/x \gg n$. This is true throughout most of the spectrum, and so

$$\begin{aligned}
\frac{\partial E}{\partial t} &= \int n_e \sigma_T c \frac{kT}{mc^2} \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^4 n' \right) x^3 dx 4\pi \frac{(kT)^4}{c^3} \\
&= n_e \sigma_T 4\pi \frac{(kT)^5}{mc^4} \int \frac{\partial}{\partial x} \left(x^4 n' \right) x dx \\
&= n_e \sigma_T 4\pi \frac{(kT)^5}{mc^4} \left(x^5 n' \big|_0^\infty - \int_0^\infty x^4 n' dx \right) \\
&= n_e \sigma_T 4\pi \frac{(kT)^5}{mc^4} \left(0 - x^4 n \big|_0^\infty + \int_0^\infty 4x^3 n dx \right) \\
&= n_e \sigma_T c \frac{kT}{mc^2} 4E \\
&= \frac{kT}{mc^2} \frac{4E}{t_c}
\end{aligned}$$

where $t_c = 1/n_e \sigma_T c$ is the mean time between scatterings. Thus the energy increases exponentially:

$$E(t) = E_0 \exp \left(4 \frac{kT}{mc^2} \frac{t}{t_c} \right)$$

The result is true provided that the input frequencies satisfy $\hbar\omega/kT \ll 1$. The quantity inside the exponential is:

(mean fractional energy change per scattering)×(mean number of scatterings)

Frequency shifts in bremsstrahlung plus scattering. We have previously defined two significant frequencies: ν_B and ν_t, such that:

$\nu < \nu_B$	optically thick	$I_\nu \propto B_\nu \propto \nu^2$ in R-J limit
$\nu = \nu_B$	$\tau_B = \tau_s$	
$\nu_B < \nu < \nu_t$		$I_\nu \propto \nu$
$\nu = \nu_t$	$\tau_B = 1/\tau_s$	
$\nu > \nu_t$	optically thin, translucent	$I_\nu \propto \exp(-h\nu kT)$

Now we introduce a third frequency ν_{coh} which defines the region in which incoherence is important. The definition is

$$y(\nu_{\text{coh}}) \equiv 1 \tag{20}$$

For $\nu > \nu_{\text{coh}}$ there are enough scatterings per mfp for a substantial energy shift to occur. From equation 20,

$$\frac{4kT}{mc^2} \tau_y^2 = 1 \tag{21}$$

where (equation 6)

$$\begin{aligned}
\tau_y &= \frac{\alpha_s}{\sqrt{\alpha_\nu (\alpha_\nu + \alpha_s)}} = \frac{\alpha_s}{\alpha_\nu \sqrt{1 + \alpha_s/\alpha_\nu}} \\
&\simeq \frac{\alpha_s}{\alpha_\nu} \ll 1 \text{ for } \frac{\alpha_s}{\alpha_\nu} \ll 1 \\
&\simeq \sqrt{\frac{\alpha_s}{\alpha_\nu}} \gg 1 \text{ for } \frac{\alpha_s}{\alpha_\nu} \gg 1
\end{aligned}$$

Notice that if $kT < mc^2$, we only need to consider the case $\tau_y > 1$, which means $\alpha_s/\alpha_\nu > 1$.

If $x_{\text{coh}} \gg 1$, then since most photons are produced at $x \lesssim 1 \ll x_{\text{coh}}$, the impact of incoherence on the spectrum is minimal and we need not consider it. Thus the interesting case has $x_{\text{coh}} < 1$.

Recall

$$\alpha_s = 6.6 \times 10^{-25} n$$

and

$$\begin{aligned} \alpha_\nu &= 4 \times 10^8 \frac{n^2}{v^3 T^{1/2}} \left(1 - e^{-h\nu/kT}\right) \text{ in cgs units} \\ &= 4 \times 10^{-23} \frac{n^2}{x^3 T^{1/2}} \left(1 - e^{-x}\right) \end{aligned}$$

For $x < 1$ we may approximate the exponential:

$$\alpha_\nu = 4 \times 10^{-23} \frac{n^2}{x^3 T^{1/2}} (x) = 4 \times 10^{-23} \frac{n^2}{x^2 T^{1/2}}$$

Thus

$$\frac{\alpha_s}{\alpha_\nu} = \frac{6.6 \times 10^{-25} n}{4 \times 10^{-23} n^2} x^2 T^{1/2} = \frac{0.0165}{n} x^2 T^{1/2}$$

For a fixed n and T, we may write this as:

$$\frac{\alpha_s}{\alpha_\nu} = \left(\frac{x}{x_B}\right)^2$$

since α_s/α_B = 1 at $x = x_B$. Then equation 21 becomes:

$$\begin{aligned} \frac{4kT}{mc^2} \frac{\alpha_s}{\alpha_\nu} &= 1 \\ \frac{4kT}{mc^2} \left(\frac{x_{\text{coh}}}{x_B}\right)^2 &= 1 \end{aligned}$$

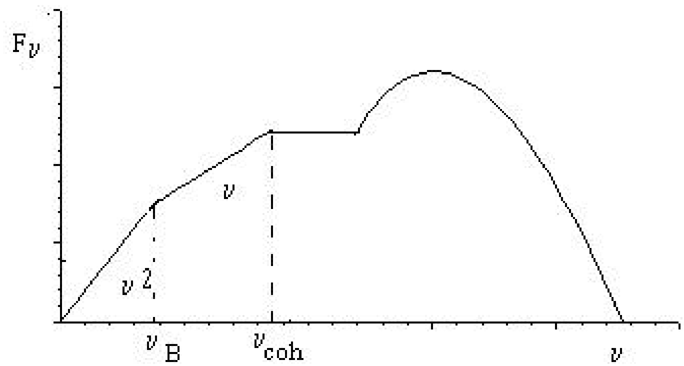
or

$$x_{\text{coh}} = x_B \frac{mc^2}{4kT} \gg x_B$$

Now as frequency increases above x_{coh}, y increases. For $y \gg 1$ there will be enough scatterings for the spectrum to "saturate", i.e. we get a Wien spectrum, $n(x) \propto e^{-x}$. Then

$$I_\nu = \frac{2h\nu^3}{c^2} n(x) = \frac{2h\nu^3}{c^2} e^{-\alpha} e^{-h\nu/kT}$$

and the spectrum looks like:



The height of the peak is determined by the factor $e^{-\alpha}$ (= 25 in this plot) which we have yet to find.

Total flux: The photons are all generated by Bremssrahlung, but the energy of each is increased by scattering. For large y, almost all the photons end up with an energy near $3kT$, so

$$\frac{dE}{dtdV} \sim 3kT \int \frac{4\pi j_\nu}{h\nu} d\nu$$

where j_ν is the emission coefficient for bremsstrahlung, $j_\nu \propto \left(n^2/\sqrt{T}\right) \exp\left(-h\nu/kT\right)$. Integrating over ν, we find

$$\varepsilon_{ff} = \int 4\pi j_\nu d\nu = \varepsilon_o n^2 k \sqrt{T}/h$$

and so

$$\frac{dE}{dtdV} \sim 3\varepsilon_{ff} \int \frac{g\left(\nu\right) \exp\left(-h\nu/kT\right)}{\nu} d\nu = A\varepsilon_{ff}$$

where the amplification factor A is

$$A = 3\int_{x_{\text{coh}}}^{\infty} g\left(x\right) \frac{e^{-x}}{x} dx = \frac{3}{4} \ln^2 \left(\frac{2.25}{x_{\text{coh}}}\right)$$

as discovered by Kompaneets in 1957.

For a thin cloud with $x_t \ll 1$,

$$I \sim \frac{dE}{dtdV} \frac{R}{4\pi}$$

while for $x_t \gg 1$

$$I \sim \frac{dE}{dtdV} \frac{R_*}{4\pi}$$

where R_* is defined by

$$\tau_{\text{eff}}\left(R_*, x = 3\right) = 1$$

Since all photons end up around $x = 3$, that is the appropriate frequency at which to compute the optical depth.

Unsaturated Comptonization If $y \gtrsim 1$ but $x_{\text{coh}} \approx 1$, we do not get a Wien spectrum.

We'll look for a steady state solution in the case that photons are produced by a source within the cloud and ultimately escape after scattering. Thus we set $\partial/\partial t \equiv 0$ in equation 19, but add source and sink terms.

$$0 = n_e \sigma_T c \frac{kT}{mc^2} \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^4 \left(n' + n + n^2 \right) \right) + Q\left(x\right) - S\left(x\right)$$

In addition, if $n \ll 1$, we may neglect the terms in n^2. The sink term $S\left(x\right)$ has the form

$$\frac{n}{t_{\text{escape}}} = \frac{n}{t_{\text{compt}} \times \text{(number of scatterings)}} = \frac{n}{t_{\text{compt}} \times \max\left(\tau, \tau^2\right)}$$

where $t_{\text{compt}} = 1/n_e \sigma_T c$ is the mean time between scatterings. Then from the definition of y (equation 5) we get:

$$S\left(x\right) = n \frac{n_e \sigma_T c}{\max\left(\tau, \tau^2\right)} = \frac{4kT}{mc^2} \frac{n}{y} n_e \sigma_T c$$

and the equation becomes:

$$\frac{1}{x^2} \frac{\partial}{\partial x} \left(x^4 \left(n' + n \right) \right) + q\left(x\right) - \frac{4n}{y} = 0$$

where $q = Q t_{\text{compt}}$.

If the source is soft, $q\left(x\right) \to 0$ for $x > x_s$.. Then for $x_s \ll x \ll 1$ we find

$$\frac{1}{x^2} \frac{\partial}{\partial x} \left(x^4 \left(n' + n \right) \right) - \frac{4n}{y} \simeq \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^4 n' \right) - \frac{4n}{y} = 0$$

and the equation simplifies to

$$x^2 n'' + 4xn' - \frac{4n}{y} = 0$$

This equation has a power law solution, $n \propto x^p$ where

$$p\left(p-1\right) + 4p - \frac{4}{y} = 0$$

with

$$p = \frac{-3 \pm \sqrt{9 + 16/y}}{2}$$

There is a particularly nice result for $y = 1$

$$p = \frac{-3 \pm 5}{2} = -4, 1$$

The positive power is not physically possible, because it implies infinite energy in the spectrum. Thus the solution we need is

$$n \propto x^{-4}$$

or

$$F_\nu \propto x^3 n \propto x^{-1}$$

This is a power law that fits the observed spectrum of a number of sources.