Vlasov theory with magnetic effects

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When we include magnetic fields in the Vlasov equation several interesting new effects arise. These include cyclotron damping and waves at harmonics of ω_c (the Bernstein modes).

The Vlasov equation, now with magnetic field included. is:

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\vec{\nabla}f+\frac{q}{m}\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\frac{\partial f}{\partial\vec{v}}=0$$

Initially $\vec{B}_0\neq 0$ but $\vec{E}_0=0$. When we perturb, there are perturbations to both B and E.

$$\frac{\partial f_1}{\partial t}+\vec{v}\cdot\vec{\nabla}f_1+\frac{q}{m}\left(\vec{E}_1+\vec{v}\times\vec{B}_1\right)\cdot\frac{\partial f_0}{\partial\vec{v}}+\frac{q}{m}\vec{v}\times\vec{B}_0\cdot\frac{\partial f_1}{\partial\vec{v}}=0$$

Or, equivalently:

$$\frac{df_1}{dt}=-\frac{q}{m}\left(\vec{E}_1+\vec{v}\times\vec{B}_1\right)\cdot\frac{\partial f_0}{\partial\vec{v}} \tag{1}$$

where df_1/dt is the total time derivative of f_1 taken along the unperturbed orbit in the magnetic field $\vec{B}_0=B_0\hat{\mathbf{z}}$. We can find a formal solution for f_1 by integrating along the orbit.

$$\int_{t_0}^{t}\frac{df_1}{dt'}dt'=f_1(t)-f_1(t_0)=\int_{t_0}^{t}-\frac{q}{m}\left(\vec{E}_1+\vec{v}\times\vec{B}_1\right)\cdot\frac{\partial f_0}{\partial\vec{v}}dt' \tag{2}$$

The unperturbed trajectory is described by:

$$v_z(t')=v_z(t)=v_{\parallel} \tag{3}$$

$$v_y(t')=v_{\perp}\cos\left[\omega_c(t'-t)+\phi\right] \tag{4}$$

and

$$v_x(t')=v_{\perp}\sin\left[\omega_c(t'-t)+\phi\right] \tag{5}$$

(This works for both signs of charge if we keep the sign of the charge in ω_c.) We can integrate the z-component directly to get

$$z(t')-z(t)=v_{\parallel}(t'-t) \tag{6}$$

while for the x and y components we get:

$$y(t')-y(t)=\frac{v_{\perp}}{\omega_c}\left\{\sin\left[\omega_c(t'-t)+\phi\right]-\sin\phi\right\}=r_L\left\{\sin\left[(\omega_c(t'-t)+\phi)-\sin\phi\right]\right\} \tag{7}$$

and similarly for x.

We won't have time to explore all the possible waves, so first let's look at electrostatic (longitudinal) waves so that $\vec{B}_1 \equiv 0$ and $\vec{k} \| \vec{E}$. We also consider propagation across the magnetic field, in the $y-$direction, so that

$$\vec{E} = E_a \exp\left[iky\left(t'\right) - i\omega t'\right]\ \hat{y}$$

Then the solution for f_1 (2) becomes:

$$\begin{aligned} f_1(t) - f_1(t_0) &= \int_{t_0}^{t} -\frac{q}{m}\vec{E}_1 \cdot \frac{\partial f_0}{\partial \vec{v}} dt' \\ &= -\frac{q}{m}\int_{t_0}^{t} \vec{E}_a e^{iky(t') - i\omega t'} \cdot \frac{\partial f_0}{\partial \vec{v}} dt' \end{aligned}$$

Using expression (7) for $y\left(t'\right)$, the exponential is:

$$\exp\left(ik\left[r_L\left\{\sin\left(\omega_c\left(t'-t\right)+\phi\right) - \sin\phi\right\} + y(t)\right] - i\omega t'\right)$$

Now let $\tau = t' - t$. We have:

$$\begin{aligned} & e^{-ikr_L\sin\phi} e^{iky(t)} \exp\left[ikr_L\left\{\sin\left(\omega_c\tau + \phi\right)\right\} - i\omega\left(\tau + t\right)\right] \\ = \ & e^{-ikr_L\sin\phi} e^{iky(t) - i\omega t} \exp\left[ikr_L\left\{\sin\left(\omega_c\tau+\phi\right)\right\} - i\omega\tau\right] \end{aligned}$$

If the wave is damped, we can take $t_0 \to \infty$, with $f_1(t) \to 0$ as $t \to \infty$ to get:

$$f_1(t) = \int_{\infty}^{t} -\frac{q}{m}\vec{E}_1 \cdot \frac{\partial f_0}{\partial \vec{v}} dt' \tag{8}$$

while for a growing wave, we let $t_0 \to -\infty$, with $f_1(t) \to 0$ as $t \to -\infty$ to get:

$$f_1(t) = \int_{-\infty}^{t} -\frac{q}{m}\vec{E}_1 \cdot \frac{\partial f_0}{\partial \vec{v}} dt' \tag{9}$$

We'll consider the damped wave case (8), but we can get a growing wave with only minor changes to the analysis. Changing variables to τ, the integral becomes

$$f_1(t) = -\frac{q}{m}E_a e^{iky(t) - i\omega t}\int_{\infty}^{0} e^{-ikr_L\sin\phi}\exp\left[ikr_L\left\{\sin\left(\omega_c\tau+\phi\right)\right\} - i\omega\tau\right]\frac{\partial f_0}{\partial v_y} d\tau \tag{10}$$

We can simplify this expression by making use of the generating function for Bessel functions (see e.g. Lea 8.93 or Jackson problem 3.16c)

$$\exp\left(\frac{kr}{2}\left(u - \frac{1}{u}\right)\right) = \sum_{n=-\infty}^{\infty} u^n J_n(kr)$$

Here we let $u = e^{i(\omega_c\tau + \phi)}$. Then

$$u - \frac{1}{u} = e^{i\omega_c(\omega_c\tau+\phi)} - e^{-i\omega_c(\omega_c\tau+\phi)} = 2i\sin\left(\omega_c\tau+\phi\right)$$

Thus

$$\exp\left[ikr_L\sin\left(\omega_c\tau+\phi\right)\right] = \exp\left[\frac{ikr_L}{2i}\left(u - \frac{1}{u}\right)\right] = \sum_{n=-\infty}^{+\infty}\left(e^{i(\omega_c\tau+\phi)}\right)^n J_n\left(kr_L\right)$$

and so (10) becomes

$$f_1(t) = -\frac{q}{m}E_a e^{iky(t)-i\omega t}\int_{\infty}^{0}\sum_{n=-\infty}^{+\infty}\left(e^{i(\omega_c\tau+\phi)}\right)^n J_n\left(kr_L\right)\sum_{m=-\infty}^{+\infty}\left(e^{-i\phi}\right)^m J_m\left(kr_L\right)e^{-i\omega\tau}\frac{\partial f_0}{\partial v_y}d\tau \tag{11}$$

Now we expect f_0 to be a function of $v_\perp$ and $v_\|$, so using cylindrical coordinates in the velocity space, we have

$$\begin{aligned}\vec{\nabla}_v f_0 &= \hat{\mathbf{x}}\frac{\partial f_0}{\partial v_x} + \hat{\mathbf{y}}\frac{\partial f_0}{\partial v_y} + \hat{\mathbf{z}}\frac{\partial f_0}{\partial v_z} = \hat{\perp}\frac{\partial f_0}{\partial v_\perp} + \frac{\hat{\boldsymbol{\theta}}}{v_\perp}\frac{\partial f_0}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial f_0}{\partial v_z} \\ &= (\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta)\frac{\partial f_0}{\partial v_\perp} + 0 + \hat{\mathbf{z}}\frac{\partial f_0}{\partial v_z}\end{aligned}$$

from the cylindrical symmetry, where θ is the angle that $\vec{v}$ makes with the x-axis. Thus, (cf eqn 4):

$$\frac{\partial f_0}{\partial v_y} = \frac{v_y}{v_\perp}\frac{\partial f_0}{\partial v_\perp} = \cos(\omega_c\tau + \phi)\frac{\partial f_0}{\partial v_\perp}$$

Putting this expression into eqn (11) we have:

$$\begin{aligned} f_1(t) &= -\frac{q}{m}E_a e^{iky(t)-i\omega t} \times \\ &\quad \int_\infty^0 \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} e^{in(\omega_c\tau+\phi)} J_n(kr_L)\, e^{-im\phi} J_m(kr_L)\, e^{-i\omega\tau}\cos(\omega_c\tau+\phi)\frac{\partial f_0}{\partial v_\perp} d\tau \\ &= -\frac{q}{2m}E_a e^{iky(t)-i\omega t} \times \\ &\quad \int_\infty^0 \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\left[e^{i(n+1)(\omega_c\tau+\phi)} + e^{i(n-1)(\omega_c\tau+\phi)}\right] e^{-im\phi} J_n(kr_L)\, J_m(kr_L)\, e^{-i\omega\tau}\frac{\partial f_0}{\partial v_\perp} d\tau\end{aligned}$$

We can now do the integration over τ easily to get:

$$f_1(t) = -\frac{q}{2m}E_0 e^{iky(t)-i\omega t}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} J_n(kr_L)\, J_m(kr_L)\frac{\partial f_0}{\partial v_\perp}F_{nm} \qquad (12)$$

where

$$F_{nm} = \left.\frac{e^{i(n+1-m)\phi}e^{[i(n+1)\omega_c - i\omega]\tau}}{i[(n+1)\omega_c - \omega]} + \frac{e^{i(n-1-m)\phi}e^{[i(n-1)\omega_c - i\omega]\tau}}{i[(n-1)\omega_c - \omega]}\right|_\infty^0$$

Since we are considering the damped case ($\omega = \omega_r - i\gamma$ with $\gamma > 0$), then $-i\omega\tau = -i\omega_r\tau - \gamma\tau$ and the integrated term vanishes as $\tau \to \infty$. (It is straightforward to repeat the derivation for the growing wave case $\gamma < 0$ starting from equation (9) and show that the end result is the same.) Then:

$$F_{nm} = e^{-im\phi}\left\{\frac{e^{i(n+1)\phi}}{i[(n+1)\omega_c - \omega]} + \frac{e^{i(n-1)\phi}}{i[(n-1)\omega_c - \omega]}\right\}$$

Then we have:

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} J_n(kr_L) F_{nm} &= e^{-im\phi} \sum_{n=-\infty}^{+\infty} J_n(kr_L) \left\{ \frac{e^{i(n+1)\phi}}{i\left[(n+1)\omega_c - \omega\right]} + \frac{e^{i(n-1)\phi}}{i\left[(n-1)\omega_c - \omega\right]} \right\} \\ &= e^{-im\phi} \sum_{\nu=-\infty}^{+\infty} \frac{e^{i\nu\phi}}{i\left[\nu\omega_c - \omega\right]} \left[J_{\nu-1}(kr_L) + J_{\nu+1}(kr_L)\right] \\ &= e^{-im\phi} \sum_{\nu=-\infty}^{+\infty} \frac{e^{i\nu\phi}}{i\left[\nu\omega_c - \omega\right]} \frac{2\nu}{kr_L} J_\nu(kr_L)\end{aligned}$$

where we have made use of a recursion relation for the Bessel functions (eg equation 3.87 in Jackson or Lea 8.89). Finally then (12) is:

$$f_1(t) = -\frac{q}{m} E_a e^{iky(t) - i\omega t} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{e^{i(n-m)\phi}}{i\left[n\omega_c - \omega\right]} \frac{n\omega_c}{kv_\perp} J_n(kr_L) J_m(kr_L) \frac{\partial f_0}{\partial v_\perp} \quad (13)$$

Next we use Poisson's equation:

$$ikE_1 = \frac{n_1 q}{\varepsilon_0} = \frac{q}{\varepsilon_0} \int f_1 dv_\parallel v_\perp dv_\perp d\phi$$

The integration over the azimuthal angle ϕ is easy:

$$\int_0^{2\pi} e^{i(n-m)\phi} d\phi = 2\pi \delta_{nm}$$

and thus the double sum collapses, leaving:

$$\begin{aligned}ik &= -\frac{q^2}{m\varepsilon_0} \int \sum_{n=-\infty}^{+\infty} \frac{2\pi}{i(n\omega_c - \omega)} \frac{n\omega_c}{kv_\perp} \left[J_n\left(\frac{kv_\perp}{\omega_c}\right)\right]^2 \frac{\partial f_0}{\partial v_\perp} dv_\parallel v_\perp dv_\perp \\ k^2 &= 2\pi \frac{q^2}{m\varepsilon_0} \sum_{n=-\infty}^{+\infty} \frac{n\omega_c}{(n\omega_c - \omega)} \int \left[J_n\left(\frac{kv_\perp}{\omega_c}\right)\right]^2 \frac{\partial f_0}{\partial v_\perp} dv_\parallel dv_\perp\end{aligned}$$

Here we wrote the Larmor radius in terms of $v_\perp$ explicitly, since we are integrating over $v_\perp$. Now we can factor the plasma density from the distribution function to get:

$$k^2 = 2\pi\omega_p^2 \sum_{n=-\infty}^{+\infty} \frac{n\omega_c}{(n\omega_c - \omega)} \int \left[J_n\left(\frac{kv_\perp}{\omega_c}\right)\right]^2 \frac{\partial \hat{f}_0}{\partial v_\perp} dv_\parallel dv_\perp$$

For a Maxwellian, the integral over $v_\parallel$ is straightforward, leaving

$$\begin{aligned}\hat{f}_{0,\perp} &= \frac{\beta}{2\pi} e^{-\beta v_\perp^2/2} \\ \frac{\partial \hat{f}_{0,\perp}}{\partial v_\perp} &= -\beta v_\perp \hat{f}_{0,\perp}\end{aligned}$$

where $\beta = m/kT$, and then:

$$k^2 = -\omega_p^2 \sum_{n=-\infty}^{+\infty} \frac{n\omega_c}{(n\omega_c - \omega)} \int_0^\infty \left[J_n\left(\frac{kv_\perp}{\omega_c}\right)\right]^2 \beta^2 v_\perp e^{-\beta v_\perp^2/2} dv_\perp$$

where the $n = 0$ term gives zero. To do the integral we can use the result (G&R 6.633#2) valid for $n > -1$

$$\int_0^\infty J_n^2(x)\, e^{-x^2/2\sigma^2} x dx = \sigma^2 e^{-\sigma^2} I_n\left(\sigma^2\right)$$

where I_n is a modified Bessel function. Since $J_{-n} = (-1)^n J_n$, $(J_{-n})^2 = J_n^2$, and $I_{-n} = I_n$, the same result holds for the negative n terms. . Here we have $x = kr_L = kv_\perp/\omega_c$, and $x/\sigma = \sqrt{\beta} v_\perp$, so

$$\sigma = \frac{x}{\sqrt{\beta} v_\perp} = \frac{kv_\perp/\omega_c}{\sqrt{\beta} v_\perp} = \frac{k}{\sqrt{\beta}\omega_c}$$

Thus:

$$\begin{aligned} k^2 &= -\omega_p^2 \beta \sum_{n=-\infty}^{+\infty} \frac{n\omega_c}{[n\omega_c - \omega]} \frac{\beta\omega_c^2}{k^2} \int_0^\infty J_n^2(x)\, e^{-x^2/2\sigma^2} x\, dx \\ &= -\frac{1}{\lambda_D^2} \sum_{n=-\infty}^{+\infty} \frac{n\omega_c}{[n\omega_c - \omega]} \exp\left(-\frac{k^2}{\beta\omega_c^2}\right) I_n\left(\frac{k^2}{\beta\omega_c^2}\right) \end{aligned}$$

where we used the fact that $\lambda_D^2 = 1/\omega_p^2\beta$. Notice that we can write σ as:

$$\sigma = \frac{k}{\sqrt{\beta}\omega_c} = \frac{kv_{\text{th}}}{\omega_c} = kr_{\text{th}}$$

where r_{th} is the Larmor radius of a particle moving at the thermal speed $\sqrt{kT/m}$.Thus the dispersion relation is:

$$\begin{aligned} k^2 &= \frac{1}{\lambda_D^2} \sum_{n=-\infty}^{\infty} \frac{n\omega_c}{[\omega - n\omega_c]} \exp\left(-k^2 r_{\text{th}}^2\right) I_n\left(k^2 r_{\text{th}}^2\right) \\ k^2\lambda_D^2 &= 2\sum_{n=1}^{\infty} \frac{(n\omega_c)^2}{\omega^2 - n^2\omega_c^2} \exp\left(-k^2 r_{\text{th}}^2\right) I_n\left(k^2 r_{\text{th}}^2\right) \qquad (14) \end{aligned}$$

Now if σ is large (wavelength small compared with the thermal gyro-radius) we may use the large argument expansion of the Bessel function

$$I_n\left(\sigma^2\right) \approx \frac{e^{\sigma^2}}{\sqrt{2\pi}\sigma}$$

and then the dispersion relation simplifies:

$$\begin{aligned} k^2\lambda_D^2 &= 2\sum_{n=1}^{\infty} \frac{(n\omega_c)^2}{\omega^2 - n^2\omega_c^2} \frac{1}{\sqrt{2\pi} k r_{\text{th}}} \\ \sqrt{\frac{\pi}{2}} k^3 \lambda_D^2 r_{\text{th}} &= \sum_{n=1}^{\infty} \frac{(n\omega_c)^2}{\omega^2 - n^2\omega_c^2} \qquad (15) \end{aligned}$$

so as $k \to \infty$, $\omega \to n\omega_c$, the harmonics of the cyclotron frequency. For k large but not infinite and ω near $n\omega_c$, one term dominates and we have:

$$\omega^2 - (n\omega_c)^2 = \sqrt{\frac{2}{\pi}} \frac{(n\omega_c)^2}{k^3 \lambda_D^2 r_{\rm th}}$$

The right hand side is positive, so the limit is approached from above.

Now as $k \to 0$, we should use the small argument approximation to the Bessel function: $I_n \approx \frac{1}{n!}\left(\frac{k^2 r_{\rm th}^2}{2}\right)^n$ (Lea eqn 8.101). The exponential in eqn (14) is approximately 1 in this case. Then (14) becomes:

$$k^2 \lambda_D^2 = 2 \sum_{n=1}^{\infty} \frac{(n\omega_c)^2}{\omega^2 - n^2 \omega_c^2} \frac{1}{n!} \left(\frac{k^2 r_{\rm th}^2}{2}\right)^n \left(1 - k^2 r_{\rm th}^2 + \cdots\right) \tag{16}$$

The sum is dominated by the $n = 1$ term, which gives:

$$\begin{aligned} \frac{\lambda_D^2}{r_{\rm th}^2} &= \frac{\beta\omega_c^2}{\beta\omega_p^2} = \frac{\omega_c^2}{\omega^2 - \omega_c^2} \\ \frac{1}{\omega_p^2} &= \frac{1}{\omega^2 - \omega_c^2} \\ \omega^2 - \omega_c^2 &= \omega_p^2 \\ \omega^2 &= \omega_p^2 + \omega_c^2 = \omega_{\rm UH}^2 \end{aligned}$$

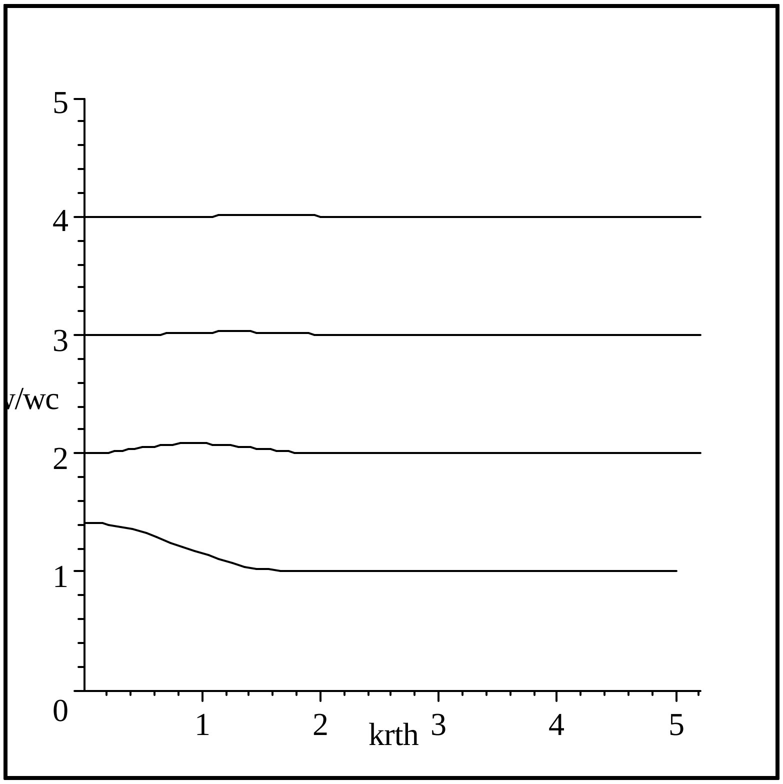
the upper hybrid frequency. However, if the frequency is very close to one of the resonances at $\omega = n\omega_c$ (other than $n = 1$), then we cannot ignore the resonant term. So again we get $\omega \approx n\omega_c$.

In more detail, for $n \neq 1$ but $\omega \approx n\omega_c$, we have, including the $n = 1$ term,

$$\begin{aligned} k^2 \lambda_D^2 &= \frac{\omega_c^2}{\omega^2 - \omega_c^2} + \frac{2(n\omega_c)^2}{\left[\omega^2 - (n\omega_c)^2\right]} \frac{1}{n!} \left(\frac{k^2 r_{\rm th}^2}{2}\right)^n \\ \omega^2 - (n\omega_c)^2 &= \frac{2(n\omega_c)^2}{k^2 \lambda_D^2} \frac{1}{n!} \left(\frac{k^2 r_{\rm th}^2}{2}\right)^n + \frac{\omega_c^2 \left[\omega^2 - (n\omega_c)^2\right]}{(\omega^2 - \omega_c^2)\, k^2 \lambda_D^2} \\ &= \frac{2(n\omega_c)^2}{k^2 \lambda_D^2} \frac{1}{n!} \left(\frac{k^2 r_{\rm th}^2}{2}\right)^n + \frac{\omega_c^2 \left[\omega^2 - \omega_c^2 - (n-1)\,\omega_c^2\right]}{(\omega^2 - \omega_c^2)\, k^2 \lambda_D^2} \\ &= \frac{\omega_c^2}{k^2 \lambda_D^2} \left[1 + \frac{2n}{(n-1)!} \left(\frac{k^2 r_{\rm th}^2}{2}\right)^n - \frac{(n-1)\,\omega_c^2}{\omega^2 - \omega_c^2}\right] \end{aligned} \tag{17}$$

Generally if $\omega \sim n\omega_c$, the last term in [] is small ($\sim (n+1)^{-1}$), the right hand side is positive, and so the limit is approached from above.

With $\omega_p = \omega_c$, the plot of ω versus k looks like this:



These are the Bernstein modes.

Note: to get the plot we used eqn(14) with the Bessel function series evaluated to 10 terms.

First write it in terms of dimensionless variables. $y_n = \left(\omega^2 - n^2\omega_c^2\right)/n^2\omega_c^2$ and $k\lambda_D = x$, so that

$$k^2 r_{\text{th}}^2 = k^2\lambda_D^2 \frac{r_{\text{th}}^2}{\lambda_D^2} = x^2 \frac{v_{\text{th}}^2}{\omega_c^2}\frac{\omega_p^2}{v_{\text{th}}^2} = x^2\alpha^2$$

where $\alpha = \omega_p/\omega_c$. Then for $\omega \simeq n\omega_c$, (14) becomes:

$$1 = 2\sum_{n=1}^{\infty} \frac{1}{y_n} \frac{\exp\left(-x^2\alpha^2\right) I_n\left(x^2\alpha^2\right)}{x^2} \simeq \frac{2}{y_n} \frac{\exp\left(-x^2\alpha^2\right) I_n\left(x^2\alpha^2\right)}{x^2}$$

The small x limit is slightly different if $\omega_{\text{UH}} > 2\omega_c$. To investigate this, let $\omega_{\text{UH}}/\omega_c = \gamma = \sqrt{1+\alpha^2}$

Let's look at (16) keeping one more term in the exponential.

$$\begin{aligned}
k^2\lambda_D^2 &= 2\sum_{n=1}^{\infty}\frac{(n\omega_c)^2}{\omega^2-n^2\omega_c^2}\frac{1}{n!}\left(\frac{k^2r_{\text{th}}^2}{2}\right)^n\left(1-k^2r_{\text{th}}^2+\cdots\right)\\
1 &= \frac{\alpha^2\omega_c^2}{\omega^2-\omega_c^2}\left[1+\sum_{n=2}^{\infty}\frac{n^2\left(\omega^2-\omega_c^2\right)}{\omega^2-n^2\omega_c^2}\frac{1}{n!}\left(\frac{k^2r_{\text{th}}^2}{2}\right)^{n-1}\right]\left(1-k^2r_{\text{th}}^2+\cdots\right)\\
\omega^2-\omega_c^2-\alpha^2\omega_c^2 &= \alpha^2\omega_c^2\left[\sum_{m=1}^{\infty}\frac{\left(-k^2r_{\text{th}}^2\right)^m}{m!}+\sum_{n=2}^{\infty}\frac{n\left(\omega^2-\omega_c^2\right)}{\omega^2-n^2\omega_c^2}\frac{1}{(n-1)!}\left(\frac{k^2r_{\text{th}}^2}{2}\right)^{n-1}\right]\\
\omega^2-\omega_{\text{UH}}^2 &= \alpha^2\omega_c^2\left[\sum_{m=1}^{\infty}\frac{\left(-k^2r_{\text{th}}^2\right)^m}{m!}+\sum_{n=2}^{\infty}\frac{n\left(\omega^2-\omega_c^2\right)}{\omega^2-n^2\omega_c^2}\frac{1}{(n-1)!}\left(\frac{k^2r_{\text{th}}^2}{2}\right)^{n-1}\right]
\end{aligned}$$

where we will keep the same order of terms in the exponential as in the Bessel function.

$$\omega^2-\gamma^2\omega_c^2=\alpha^2\omega_c^2\left[\sum_{m=1}^{\infty}\frac{\left(-k^2r_{\text{th}}^2\right)^m}{m!}+\sum_{n=2}^{\infty}\frac{n\left(\omega^2-\omega_c^2\right)}{\omega^2-n^2\omega_c^2}\frac{1}{(n-1)!}\left(\frac{k^2r_{\text{th}}^2}{2}\right)^{n-1}\right]$$

If $2<\gamma<3$, then we keep the $n=2$ and $n=3$ terms, and truncate the first sum at $m=2$.

$$\begin{aligned}
\omega^2-\gamma^2\omega_c^2 &= \alpha^2\omega_c^2k^2r_{\text{th}}^2\left[-1+\frac{k^2r_{\text{th}}^2}{2}+\frac{\left(\omega^2-\omega_c^2\right)}{\omega^2-4\omega_c^2}+\frac{3\left(\omega^2-\omega_c^2\right)}{\omega^2-9\omega_c^2}\frac{1}{2!}\frac{k^2r_{\text{th}}^2}{4}\right]\\
&= \alpha^2\omega_c^2k^2r_{\text{th}}^2\left[\frac{3\omega_c^2}{\omega^2-4\omega_c^2}+\frac{k^2r_{\text{th}}^2}{2}+\frac{3\left(\omega^2-\omega_c^2\right)}{\omega^2-9\omega_c^2}\frac{1}{2!}\frac{k^2r_{\text{th}}^2}{4}\right]
\end{aligned}$$

Since the first term is positive, we still approach the limit from above.

We also have, including the $n=1$ and 2 terms,

$$\begin{aligned}
\omega^2-\omega_c^2 &= \alpha^2\omega_c^2\left[1-k^2r_{\text{th}}^2+\frac{\left(\omega^2-\omega_c^2\right)}{\omega^2-4\omega_c^2}k^2r_{\text{th}}^2\right]\\
&\simeq \alpha^2\omega_c^2\left\{1+\frac{3\omega_c^2}{\omega^2-4\omega_c^2}k^2r_{\text{th}}^2+\cdots\right\}
\end{aligned}$$

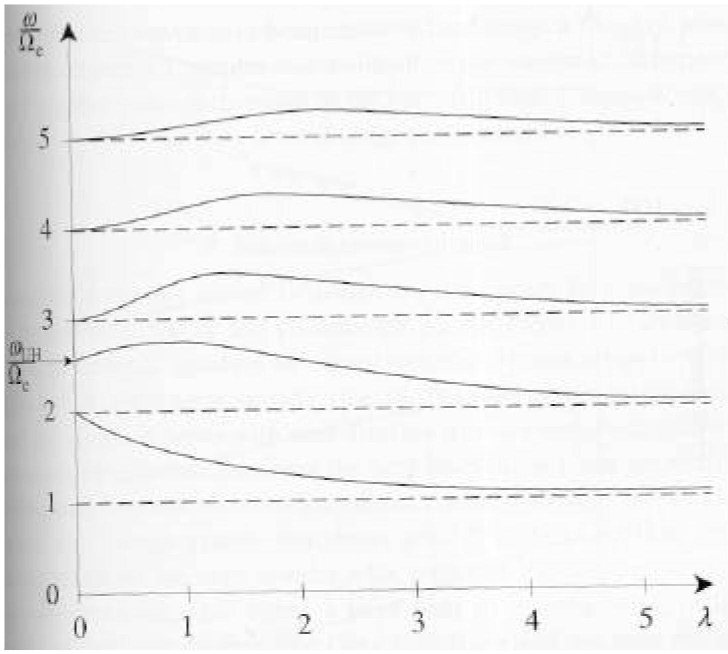
which does not go to zero as $k\to 0$. So there is no limit at $\omega=\omega_c$ in this case. However,

$$\begin{aligned}
&= \alpha^2\omega_c^2k^2r_{\text{th}}^2\left[\frac{3\omega_c^2}{\omega^2-4\omega_c^2}+\frac{k^2r_{\text{th}}^2}{2}+\frac{3\left(\omega^2-\omega_c^2\right)}{\omega^2-9\omega_c^2}\frac{1}{2!}\frac{k^2r_{\text{th}}^2}{4}\right]\\
\frac{3\omega_c^2}{\omega^2-4\omega_c^2} &= \frac{\omega^2-\gamma^2\omega_c^2}{\alpha^2\omega_c^2k^2r_{\text{th}}^2}-\frac{k^2r_{\text{th}}^2}{2}+\frac{3\left(\omega^2-\omega_c^2\right)}{\omega^2-9\omega_c^2}\frac{1}{2!}\frac{k^2r_{\text{th}}^2}{4}\\
&\to \infty \text{ as } k\to 0
\end{aligned}$$

Thus, since $\gamma>2$,

$$\frac{\omega^2-4\omega_c^2}{3\omega_c^2}\to\frac{\alpha^2\omega_c^2k^2r_{\text{th}}^2}{\omega^2-\gamma^2\omega_c^2}\to 0 \text{ from below}$$

Thus the plot now looks like this:



Reference: Boyd and Sanderson pg 281

See Chen pg 281 for the curves for $3 < \gamma < 4$.