1 Masers

In thermal equilibrium at any tempertaure, there are always fewer atoms in an excited state than in the ground state. Masers occur when an external agent pumps electrons into an excited state, thus inverting the population.

Detailed balance requires that the number of transitions from level 2 to level 1 equal the number of transitions from level 1 to level 2.

$$N_2 \left(A_{21} + B_{21}M + C \downarrow\right) = N_1 \left(P + B_{12}M + C \uparrow\right)$$

where A and B are the Einstein coefficients, M is the mean intensity in the maser transition, the $C \uparrow\downarrow$ are collisional excitation and de-excitation rates, and P is the pump rate.To simplify, let's take $C \uparrow = C \downarrow$, and $g_1 = g_2$ so that $B_{12} = B_{21} = B$. Then

$$\frac{N_2}{N_1} = \frac{P + BM + C}{A + BM + C} \tag{1}$$

Now let $N = N_2 + N_1$, and $N_2 = xN$. Then

$$\frac{x}{1-x} = \frac{P + BM + C}{A + BM + C} \Rightarrow x = \frac{P + BM + C}{P + 2BM + 2C + A} \tag{2}$$

and

$$\Delta N = N_2 - N_1 = N\left(2x - 1\right) = \frac{N\left(P - A\right)}{P + 2BM + 2C + A} \tag{3}$$

If the pump gets very strong, then $\Delta N \rightarrow N$.

The transfer equation is

$$\frac{dI_\nu}{d\ell} = j_\nu - \kappa I_\nu$$

where

$$j_\nu = N_2 A_{21} \frac{h\nu}{4\pi}$$

and

$$\kappa_\nu = \left(N_1 - N_2\right) B \frac{h\nu}{4\pi}$$

So

$$\frac{dI_\nu}{d\ell} = N_2 A_{21} \frac{h\nu}{4\pi} - \left(N_1 - N_2\right) B \frac{h\nu}{4\pi} I_\nu$$

and

$$J_\nu = M = \frac{1}{4\pi} \int I_\nu d \quad = f I_\nu$$

where the factor $f = 1$ if the radiation is isotropic. Now using our results (2 and3) for the populations, we have

$$\frac{dI_\nu}{d\ell} = N \frac{h\nu}{4\pi} \left(A \frac{P + BM + C}{P + 2BM + 2C + A} + \frac{\left(P - A\right) B I_\nu}{P + 2BM + 2C + A} \right)$$

Now assume that the pump is strong ($P \gg A$), and we obtain

$$\frac{dI_\nu}{d\ell} = N\frac{h\nu}{4\pi}\left(\frac{BI_\nu}{1+2C/P+2BfI/P}\right)+\epsilon \tag{4}$$

where

$$\begin{aligned}
\epsilon &= NA\frac{h\nu}{4\pi}\left[\frac{1+C/P+B\left(f-1\right)I/P}{1+\left(2BfI+2C\right)/P}\left(1-\frac{A}{1+\left(2BfI+2C\right)/P}\right)\right. \\
&\qquad \left. -\frac{BI/P}{1+2C/P+2BfI/P}\frac{A}{1+\left(2BfI+2C\right)/P}\right] \\
&= NA\frac{h\nu}{4\pi}\left[\frac{1+C/P+B\left(f-1\right)I/P}{1+\left(2BfI+2C\right)/P}-\frac{A}{P}\frac{\left[1+C/P+B\left(f-1\right)I/P+BI/P\right]}{\left[1+\left(2BfI+2C\right)/P\right]^2}\right] \\
&= NA\frac{h\nu}{4\pi}\left[\frac{1+C/P+B\left(f-1\right)I/P}{1+\left(2BfI+2C\right)/P}-\frac{A}{P}\frac{\left[1+C/P+BfI/P\right]}{\left[1+\left(2BfI+2C\right)/P\right]^2}\right]
\end{aligned}$$

and is much less than the first term, as we show below.

(I used the expansion

$$\frac{1}{1+x+y} = \frac{1}{\left(1+x\right)\left(1+\frac{y}{1+x}\right)} = \frac{1}{1+x}\left(1-\frac{y}{1+x}\right)$$

to first order in y, for $y \ll 1$)

The equation of transfer (4) is of the form

$$\frac{dI_\nu}{d\ell} = \alpha_0\frac{I_\nu}{1+\frac{I_\nu}{I_s}}+\varepsilon$$

with

$$\alpha_0 = N\frac{h\nu}{4\pi}\left(\frac{B}{1+2C/P}\right)$$

and

$$I_s = \left(\frac{P+2C}{2Bf}\right)$$

When $I \ll I_s$, the intensity grows exponentially:

$$I+\frac{\varepsilon}{\alpha_0} = I_0 e^{\alpha_0\ell}$$

so the assumption $\alpha_0 I \gg \varepsilon$ is soon satisfied. In this regime small changes in gain $\alpha_0\ell$ produce large changes in the intensity, and the intensity is independent of the pump strength P. We should expect strong variablilty as $\alpha_0\ell$ changes.

But for $I_\nu \gg I_s \gg \varepsilon/\alpha_0$, I_ν grows linearly. The maser saturates. In this regime small changes in gain produce only small changes in the intensity, and $I \propto P$.

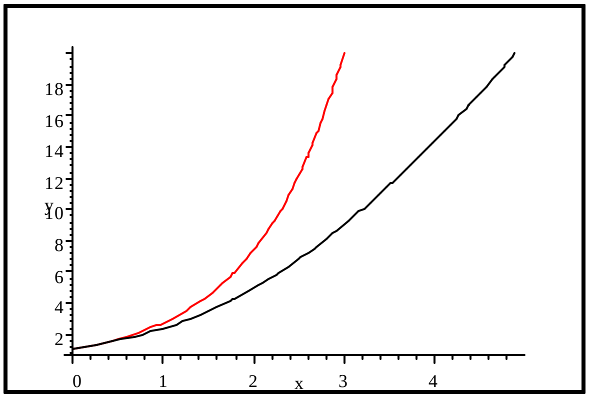
As I_ν approaches I_s, we must solve the full equation (neglecting ε), which takes the form

$$\frac{(1+y)}{y}\frac{dy}{d\ell} = \alpha_0$$

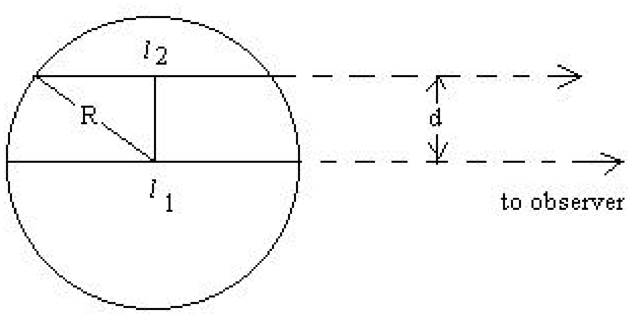
with $y = I/I_s$. Integrating, we have

$$\begin{aligned} \ln y + y|_{y_0}^{y} &= \alpha_0 \ell \\ \ln \frac{I_\nu}{I_0} + \frac{I_\nu - I_0}{I_s} &= \alpha_0 \ell \end{aligned}$$

Plot for $I_s/I_0 = 10$. The plot shows I_v/I_0 versus $\alpha_0\ell$. The red line is the exponential.



Notice that the saturation intensity I_s increases with the pump strength P.



Let's look at the emission from a spherical cloud of radius R and uniform density and temperature. Then a line of sight the through the cloud has path length

$$\ell = 2\sqrt{R^2 - d^2}$$

and the intensity is

$$I(d) \propto \exp(\alpha_0 l) = \exp\left(2\alpha_0 \sqrt{R^2 - d^2}\right)$$

Near the center of the cloud, $d/R \ll 1$,

$$\begin{aligned} I(d) &\propto \exp\left[2\alpha_0 R\left(1 - \frac{1}{2}\frac{d^2}{R^2}\right)\right] \\ &= e^{2\alpha_0 R} e^{-\alpha_0 d^2/R} \end{aligned}$$

The intensity reaches half its maximum value where

$$\alpha_0 d^2/R = \ln 2 = 0.7$$

or

$$\frac{d}{R} = \frac{\sqrt{0.7}}{\sqrt{\alpha_0 R}} = \frac{0.8}{\sqrt{\alpha_0 R}}$$

This value is $\ll 1$ if the optical depth through the center of the cloud is large. Thus maser sources appear small on the sky. Thus we may suspect a maser when (a) the intensity is very large and (b) the source is very small.