Magnetohydrodynamics

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Magetohydrodynamics (MHD) is the study of plasma motions in the low frequency approximation, $\omega \ll \omega_p, \omega_c$. That means that we use the plasma approximation ($n_e \simeq n_i$) while allowing for non-zero electric fields and also currents that depend on the small difference between n_e and n_i. MHD is a very useful tool for studying bulk motion of plasmas and MHD waves in astrophysical and lab situations.

1 The equation of motion

We may write an equation of motion for each species, including the momentum transfer due to collisions. For the ions:

$$Mn_i\left[\frac{\partial\vec{v}_i}{\partial t}+\left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i\right]=n_ie\vec{E}+en_i\vec{v}_i\times\vec{B}-\vec{\nabla}P_i+\vec{P}_{ie}+n_iM\vec{g} \qquad (1)$$

with a similar equation for the electrons:

$$mn_e\left[\frac{\partial\vec{v}_e}{\partial t}+\left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right]=-n_ee\vec{E}-en_e\vec{v}_e\times\vec{B}-\vec{\nabla}P_e+\vec{P}_{ei}+n_em\vec{g} \qquad (2)$$

The term $\vec{P}_{ei}$ is the rate of momentum transfer from ions to electrons (Diffusion notes equation 21):

$$\vec{P}_{ei}=n^2\left(\vec{v}_i-\vec{v}_e\right)e^2\eta=-\vec{P}_{ie} \qquad (3)$$

If we add the two equations, we can eliminate this term:

$$n\frac{\partial\left(M\vec{v}_i+m\vec{v}_e\right)}{\partial t}+n\left[M\left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i+m\left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right]=en\left(\vec{v}_i-\vec{v}_e\right)\times\vec{B}-\vec{\nabla}\left(P_i+P_e\right)+n\left(M+m\right)\vec{g}$$

Now we define an average velocity using the total momentum density $\vec{p}$:

$$\frac{\vec{p}}{n}=\left(M+m\right)\vec{v}=M\vec{v}_i+m\vec{v}_e \qquad (4)$$

The total pressure is the sum of the partial pressures,

$$P=P_i+P_e,$$

the current is

$$\vec{j}=ne\left(\vec{v}_i-\vec{v}_e\right), \qquad (5)$$

and the mass density is

$$\rho=n\left(M+m\right)$$

We these definitions, the equation of motion becomes:

$$\rho\frac{\partial\vec{v}}{\partial t}+n\left[M\left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i+m\left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right]=\vec{j}\times\vec{B}-\vec{\nabla}P+\rho\vec{g} \tag{6}$$

The convective derivative terms may be neglected for motions that are subsonic and sub-Alfvenic. In fact, we may replace these terms[1] with an equivalent term in $\vec{v}$ (see problem set 11)

$$\rho\frac{\partial\vec{v}}{\partial t}+\rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}=\vec{j}\times\vec{B}-\vec{\nabla}P+\rho\vec{g} \tag{7}$$

We complete the set with the two continuity equations, which add to give

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0 \tag{8}$$

and subtract to give the charge conservation equation:

$$\frac{\partial\rho_q}{\partial t}+\vec{\nabla}\cdot\vec{j}=0 \tag{9}$$

We also need an equation that gives us the time evolution of the current. So multiply equation (1) by m, equation (2) by M and subtract:

$$\begin{aligned}&Mmn\frac{\partial\left(\vec{v}_i-\vec{v}_e\right)}{\partial t}+nMm\left[\left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i-\left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right]\\ =\;&en\left(m+M\right)\vec{E}+en\left(m\vec{v}_i+M\vec{v}_e\right)\times\vec{B}-\vec{\nabla}\left(mP_i-MP_e\right)-\left(M+m\right)\vec{P}_{ei}\end{aligned} \tag{10}$$

Using the definition (5) of current, and expressing the collision term (3) in terms of resistivity, we have

$$\begin{aligned}&Mm\frac{n}{e}\frac{\partial}{\partial t}\left(\frac{\vec{j}}{n}\right)+nMm\left[\left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i-\left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right]\\ =\;&e\rho\vec{E}-e\eta\left(M+m\right)n\vec{j}+en\left(m\vec{v}_i+M\vec{v}_e\right)\times\vec{B}-\vec{\nabla}\left(mP_i-MP_e\right)\end{aligned}$$

where (using eqn 4)

$$\begin{aligned}m\vec{v}_i+M\vec{v}_e&=M\vec{v}_i+m\vec{v}_e+\left(m-M\right)\left(\vec{v}_i-\vec{v}_e\right)\\ &=\frac{\rho}{n}\vec{v}+\frac{m-M}{ne}\vec{j}\end{aligned}$$

To proceed, we make use of the fact that $m\ll M$, so that $\rho\simeq nM$, for example, and assume T_i is not $\gg T_e$.

$$\begin{aligned}\rho\frac{m}{e}\frac{\partial}{\partial t}\left(\frac{\vec{j}}{n}\right)+\rho m\left[\left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i-\left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right]&=e\rho\vec{E}-e\eta\rho\vec{j}+en\left(\frac{\rho}{n}\vec{v}+\frac{m-M}{ne}\vec{j}\right)\times\vec{B}+\vec{\nabla}\left(MP_e\right)\\ &=e\rho\left(\vec{E}+\vec{v}\times\vec{B}\right)-e\eta\rho\vec{j}-M\vec{j}\times\vec{B}+M\vec{\nabla}P_e\end{aligned}$$

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[1] Strictly we must also re-evaluate the pressure in terms of random motions about the MHD velocity $\vec{v}$. P_e and P_i are evaluated with respect to the electron and ion mean velocities $\vec{v}_e$ and $\vec{v}_i$. Of course these three velocities are almost the same in the plasma approximation.

Divide out the density:

$$\frac{m}{e}\frac{\partial}{\partial t}\left(\frac{\vec{j}}{n}\right)+m\left[\left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i-\left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right]=e\left(\vec{E}+\vec{v}\times\vec{B}\right)-e\eta\vec{j}-\frac{1}{n}\vec{j}\times\vec{B}+\frac{\vec{\nabla}P_e}{n} \tag{11}$$

Now compare the magnitude of terms on the left with those on the right:

$$\frac{m}{e}\frac{\partial}{\partial t}\left(\frac{\vec{j}}{n}\right)/\left(\frac{1}{n}\vec{j}\times\vec{B}\right)\sim\frac{mB}{e\tau}\sim\frac{1}{\omega_c\tau}$$

where τ is the time-scale over which physical properties such as $\vec{j}$ and n are changing. This ratio is small provided that time scales of interest are long compared with the electron cyclotron period.

$$\begin{aligned}\frac{m}{e}\frac{\partial}{\partial t}\left(\frac{\vec{j}}{n}\right)/\left(e\eta\vec{j}\right) \quad &\sim \quad \frac{m}{e^2\eta n\tau}\sim\frac{m}{e^2n\tau}\frac{16\pi\varepsilon_0^2\left(kT_e\right)^{3/2}}{Z_ie^2m^{1/2}}\sim\frac{\varepsilon_0^{1/2}m^{1/2}n^{1/2}}{e\tau}\frac{\left(\varepsilon_0kT_e\right)^{3/2}}{e^3n^{3/2}} \\ &\sim \quad \frac{n\lambda_D^3}{\omega_p\tau}\sim\frac{N_D}{\omega_p\tau}\end{aligned}$$

This ratio is small provided the time scales we are considering are long compared with N_D plasma oscillation periods.

(For future reference note that

$$\frac{m}{e^2\eta n\tau}\sim\frac{N_D}{\tau\omega_p} \tag{12}$$

implies

$$\eta\sim\frac{\varepsilon_0 m}{e^2n\tau}\frac{\omega_p\tau}{\varepsilon_0N_D}=\frac{1}{\varepsilon_0\omega_pN_D}\ ) \tag{13}$$

The convective derivative term is of the same order as the time derivative term

$$\left[\left(\vec{v}_i\cdot\vec{\nabla}\right)\vec{v}_i-\left(\vec{v}_e\cdot\vec{\nabla}\right)\vec{v}_e\right]\frac{en\tau}{j}\sim\frac{vj}{Len}\frac{en\tau}{j}=\frac{v}{L}\tau\sim 1$$

Thus we may neglect all terms on the left hand side in the slow (MHD) approximation, which we may now state more specifically as:

$$\tau\gg\frac{N_D}{\omega_p}$$

and

$$\tau\gg\frac{1}{\omega_c}$$

Then the current equation (10, 11) becomes:

$$\vec{E}+\vec{v}\times\vec{B}=\eta\vec{j}-\frac{1}{en}\left(\vec{j}\times\vec{B}-\vec{\nabla}P_e\right) \tag{14}$$

This is *Ohm's law*. There are several changes from the version we are used to seeing. The $\vec{v}\times\vec{B}$ term merely generalizes to the total electromagnetic force driving current. The additional terms on the right side are the Hall current term $\vec{j}\times\vec{B}/en$ and a term that depends on the electron pressure gradient. This is a diffusion term.

The Hall current term may be compared with the other terms in the equation. Remember that from the Maxwell equations, $j \sim \vec{\nabla} \times \vec{B}/\mu_0 \sim B/L\mu_0$. Thus

$$\frac{\text{Hall curent term}}{\vec{v} \times \vec{B} \text{ term}} \sim \frac{B}{enLv\mu_0} = \frac{eB}{m}\frac{m\varepsilon_0}{\mu_0\varepsilon_0 e^2 nLv} = \omega_c \frac{c^2}{\omega_p^2 Lv} = \frac{\omega_c}{\omega_p}\frac{c^2}{v^2}\frac{1}{\omega_p \tau}$$

and so the Hall current term may be neglected if time scales are sufficiently long:

$$\tau \gg \frac{\omega_c}{\omega_p^2}\frac{c^2}{v^2} \tag{15}$$

This conditions is often satisfied, especially in atstronomical applications where time scales are very long and ω_c/ω_p is small. .

Now let's compare the grad P term with the resistivity term:

$$\frac{P}{Len\eta j} \sim \frac{nkTe}{Lm}\frac{m}{e^2 n\eta}\frac{L\mu_0}{B}$$

We already calculated the middle ratio (eqn 12), so:

$$\frac{P}{Len\eta j} \sim \frac{nev_{\text{th,e}}^2}{\varepsilon_0 c^2 B}\frac{N_D}{\omega_p} = \frac{ne^2}{\varepsilon_0 m}\frac{m}{eB}\frac{v_{\text{th,e}}^2}{c^2}\frac{N_D}{\omega_p} = \frac{v_{\text{th,e}}^2}{c^2}\frac{\omega_p}{\omega_c}N_D$$

This grad P term may be neglected when:

$$\frac{v_{\text{th,e}}^2}{c^2}\frac{\omega_p}{\omega_c}N_D \ll 1$$

i.e. most of the time, since $v_{\text{th,e}}/c$ is usually small..

Finally we can even neglect the resistivity when

$$\frac{\eta j}{vB} = \frac{\eta B}{Lv\mu_0 B} = \frac{1}{Lv}\frac{\eta}{\mu_0} \ll 1$$

and so the ratio

$$R_M = \frac{Lv\mu_0}{\eta} \tag{16}$$

must be very large. This number is called the magnetic Reynolds number.

Equivalently, using relation 13 and $L = v\tau$, this condition is

$$\frac{\eta j}{vB} = \frac{\eta B}{Lv\mu_0 B} = \frac{1}{vL\mu_0\varepsilon_0\omega_p N_D} = \frac{c^2}{vL\omega_p N_D} = \left(\frac{c}{v}\right)^2 \frac{1}{\omega_p \tau N_D} \ll 1$$

$$\tau \gg \left(\frac{c}{v}\right)^2 \frac{1}{\omega_p N_D} \tag{17}$$

For example, in interstellar space, with $\omega_p \sim 6 \times 10^4$ rad/s and $v \sim 10$ km/s, we would need

$$\tau \gg \left(\frac{3 \times 10^8 \text{ m/s}}{10^4 \text{ m/s}}\right)^2 \frac{1}{6 \times 10^4 \text{ /s}}\frac{1}{2 \times 10^6} = .0075 \text{ s}$$

which is always true.

Ohm's law is often used to find the plasma velocity.

2 Diffusion in fully ionized plasmas

Now we are ready to study diffusion in fully ionized plasmas. In a steady state, zero $\vec{g}$ plasma, equation (7) becomes

$$\vec{j} \times \vec{B} = \vec{\nabla} P \tag{18}$$

Here we see that the magnetic force on the (usually small) plasma currents balances the pressure gradient. From Ohm's law (14), neglecting the terms in parentheses,

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j}$$

We cross with $\vec{B}$ to solve for $\vec{v}$:

$$\begin{aligned} \vec{E} \times \vec{B} + \left(\vec{v} \times \vec{B}\right) \times \vec{B} &= \eta \vec{j} \times \vec{B} \\ \vec{E} \times \vec{B} - \vec{v}_{\perp} B^2 &= \eta \vec{\nabla} P \end{aligned}$$

and thus

$$\vec{v}_{\perp} = \frac{\vec{E} \times \vec{B}}{B^2} - \frac{\eta \vec{\nabla} P}{B^2}$$

The first term is the $\vec{E} \times \vec{B}$ drift, while the second is the diffusion term. The particle flux due to diffusion is:

$$\vec{F}_{\perp} = n\vec{v}_{\perp} = -n\frac{\eta k \left(T_i + T_e\right) \vec{\nabla} n}{B^2}$$

which is Fick's law with a diffusion coefficient

$$D_{\perp} = n\frac{\eta k \left(T_i + T_e\right)}{B^2} \tag{19}$$

The dependence on B $(D \propto 1/B^2)$ is the same as we found in partially ionized plasmas (diffusion2 notes eqn 18). The temperature dependence is different, however. Since $\eta \propto T^{-3/2}$, $D \propto T^{-1/2}$ and decreases as T increases. In partially ionized plasmas $D \propto T^{3/2}$ and increases as T increases. The difference arises from the temperature dependence of the coulomb cross section. At high temperatures there are fewer collisons and so less motion of the guiding centers. Another important difference is that this diffusion coefficient depends on the plasma density (because plasma particles are colliding with themselves). Equation (19) describes so-called *classical diffusion*.

There is no transverse mobility! Instead, we have the $\vec{E} \times \vec{B}$ drift.

2.1 Solutions to the diffusion equation

2.1.1 Time dependence

We write $D = 2nA$ where $A = \eta kT/B^2$ and we have assumed $T_e = T_i$. Then the continuity equation is

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{v}) = 0 = \frac{\partial n}{\partial t} + \vec{\nabla} \cdot \left(-2nA\vec{\nabla} n\right)$$

or, if A is constant:

$$\frac{\partial n}{\partial t} = A\nabla^2 n^2$$

a non-linear, partial differential equation for n. Let's try to find a solution by separating: $n(t,r) = T(t)\,R(r)$. Then

$$\frac{1}{T^2}\frac{dT}{dt} = \frac{A}{R}\nabla^2 R^2 = -\frac{1}{\tau}$$

where as usual we have assumed that n is decreasing in time and thus chosen a negative separation constant. We may solve the temporal part to get

$$\begin{aligned} \frac{1}{T} &= \frac{1}{T_0} + \frac{t}{\tau} \\ T &= \frac{T_0}{1 + T_0 t/\tau} \end{aligned}$$

so the density decreases as $1/t$ at long times.

2.1.2 Steady state solutions

With a source or sink we can find steady state solutions:

$$-A\nabla^2 n^2 = S$$

If the sink is recombination, then

$$-A\nabla^2 n^2 = -\alpha n^2$$

and we can solve this linear equation for n^2. In 1-D, setting $f = n^2$,

$$\frac{d^2 f}{dx^2} = \frac{\alpha}{A} f$$

with solution

$$n^2 = f = f_0 \exp\left(-\sqrt{\frac{\alpha}{A}}x\right)$$

The plasma density decreases over a scale length

$$L \sim 2\sqrt{\frac{A}{\alpha}}$$

2.2 Bohm diffusion

Experimentally we often find $D \propto 1/B$ rather than the $1/B^2$ dependence predicted by classical diffusion. An empirical formula for this *Bohm Diffusion* is

$$D_B = \frac{kT}{16eB} \tag{20}$$

Since D_B is independent of n it leads to an exponential decay in time, which is catastrophic for containment.

Bohm diffusion corresponds to having a collision frequency of order the cyclotron frequency. Recall (diffusion notes eqn 22)

$$\eta = \frac{m\nu}{ne^2} \sim \frac{m}{ne^2}\frac{eB}{m} = \frac{B}{ne}$$

where we put $\nu = \omega_c$,and thus

$$D \sim n\eta \frac{kT}{B^2} = \frac{kT}{eB}$$

The same dependence results if $\vec{E} \times \vec{B}$ drift dominates the collisional diffusion.

$$F \sim n\frac{E}{B}$$

and

$$e\Phi_{\max} \sim kT_e$$

so

$$F \sim n\frac{kT}{eBL} \sim \left(\frac{kT}{eB}\right)\frac{n}{L} \sim D_B \nabla n$$

The Bohm diffusion coefficient is often much larger than the classical diffusion coefficient.

The confinement time may be estimated as

$$\tau \sim \frac{L^2}{D}$$

where L is a characteristic length scale. The confinement time scales inversely with D. Chen calculates values of D (classical and Bohm diffusion) for a 100 eV plasma in a 1 T field, and shows that D_B is four orders of magnitude larger than the classical diffusion coefficient, leading to confinement times of the order of seconds rather than hours.

3 Astrophysical MHD

3.1 Magnetic field evolution

To make the MHD equations really useful, we need to include an equation governing the evolution of $\overrightarrow{B}$. This comes from Maxwell's equations. The Ampere-Maxwell law is

$$\overrightarrow{\nabla} \times \overrightarrow{\mathbf{B}} = \mu_0 \overrightarrow{\mathbf{j}} + \mu_0 \epsilon_0 \frac{\partial \overrightarrow{\mathbf{E}}}{\partial t} \tag{21}$$

The last term is the displacement current term, and in low-frequency motions, we can neglect this term. From Ohm's law, if the resistivity η is small, then $\overrightarrow{\mathbf{E}} \approx \overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}}$ and so :

$$\frac{\mu_0 \epsilon_0 \left(\partial \overrightarrow{\mathbf{E}} / \partial t\right)}{\overrightarrow{\nabla} \times \overrightarrow{\mathbf{B}}} \approx \frac{vB/\left(\tau c^2\right)}{B/L} \approx \left(\frac{v}{c}\right)^2 \tag{22}$$

which we expect to be small in almost all situations.

The other Maxwell equation we need is Faraday's law:

$$\overrightarrow{\nabla} \times \overrightarrow{\mathbf{E}} = -\frac{\partial \overrightarrow{\mathbf{B}}}{\partial t} \tag{23}$$

Now take the curl of Ohm's law:

$$\vec{\nabla} \times \vec{\mathbf{E}} + \vec{\nabla} \times \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) = \vec{\nabla} \times \left(\eta \vec{\mathbf{j}}\right) + \frac{1}{e}\left[\vec{\nabla} \times \left(\frac{\vec{\mathbf{j}} \times \vec{\mathbf{B}}}{n}\right) - \vec{\nabla} \times \left(\frac{\vec{\nabla} P_e}{n}\right)\right]$$

Substitute in for curl $\vec{E}$ and for $\vec{j}$:

$$-\frac{\partial \vec{\mathbf{B}}}{\partial t} + \vec{\nabla} \times \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) = \vec{\nabla} \times \left(\eta \frac{\vec{\nabla} \times \vec{\mathbf{B}}}{\mu_0}\right) + (\text{Hall term}) + \frac{1}{en^2}\vec{\nabla} n \times \vec{\nabla} P_e \quad (24)$$

If T is constant, then $\vec{\nabla} n \| \vec{\nabla} P_e$ and the last term is zero. Then if we neglect the Hall term, and use the Maxwell equation $\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$, the RHS of equation (24) becomes:

$$\frac{\eta}{\mu_0}\nabla^2 \vec{\mathbf{B}}.$$

Now let's investigate the LHS. To see what it means, integrate over a surface S with bounding curve C. Then:

$$\frac{\partial}{\partial t}\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} - \int_S \left[\vec{\nabla} \times \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)\right] \cdot d\vec{\mathbf{A}} = \frac{\partial}{\partial t}\Phi_B - \int_C \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right) \cdot d\vec{\ell}$$

The last term may be rewritten as :

$$\int_C \vec{\mathbf{B}} \cdot \left(\vec{\mathbf{v}} \times d\vec{\ell}\right)$$

which we recognize as the rate at which flux is swept up by the surface moving at velocity $\vec{\mathbf{v}}$. Thus the LHS is just $d\vec{\mathbf{B}}/dt$, allowing for the effects of a moving frame. Thus this equation (24) is a diffusion equation for $\vec{\mathbf{B}}$, with diffusion coefficient η/μ_0. The field diffuses in a characteristic time

$$\tau_{\text{diff}} \approx L^2 \mu_0/\eta. \quad (25)$$

Thus if resistivity is small, the diffusion time becomes very long (the usual case in astrophysics).

We may define the magnetic Reynolds number:

$$R_M = \frac{v\tau_{\text{diff}}}{L} \quad (26)$$

where L and v are characteristic length and velocity scales for the flow. Diffusion dominates when $R_M < 1$ (the usual lab situation) but convection of the field with the fluid dominates when $R_M > 1$ (the usual astrophysical situation).

What happens to the magnetic field energy as $\vec{\mathbf{B}}$ changes? Field lines are moving through the plasma, induced currents flow and we have ohmic heating. The current has magnitude:

$$j = \frac{\left|\vec{\nabla} \times \vec{\mathbf{B}}\right|}{\mu_0} \sim \frac{B}{\mu_0 L}$$

where L is a characteristic length scale for variations in the plasma. Thus energy is dissipated at a rate:

$$\eta j^2 = \eta \left(\frac{B}{\mu_0 L} \right)^2$$

Thus the field energy is dissipated in a time:

$$\tau_{\mathrm{diss}} = \frac{B^2/2\mu_0}{\eta \left(B/\mu_0 L\right)^2} = \frac{1}{2}\frac{\mu_0}{\eta} L^2 \approx \tau_{\mathrm{diff}} \tag{27}$$

This is also the timescale for field to diffuse into a plasma.

Let's look at the highly conductive case ($\eta \approx 0$), neglecting the Hall current term. Then eqn (24) becomes:

$$\frac{\partial \vec{\mathbf{B}}}{\partial t} - \vec{\nabla} \times \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}} \right) = 0$$

Expanding the curl, and using $\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$,

$$\frac{\partial \vec{\mathbf{B}}}{\partial t} + \left(\vec{\mathbf{v}} \cdot \vec{\nabla} \right) \vec{\mathbf{B}} + \vec{\mathbf{B}} \left(\vec{\nabla} \cdot \vec{\mathbf{v}} \right) - \left(\vec{\mathbf{B}} \cdot \vec{\nabla} \right) \vec{\mathbf{v}} = 0$$

We may combine the first two terms, since the second is the convective derivative. We use the continuity equation to eliminate $\vec{\nabla} \cdot \vec{\mathbf{v}}$ in the third term, to get:

$$\frac{d\vec{\mathbf{B}}}{dt} + \vec{\mathbf{B}} \left(-\frac{1}{\rho}\frac{d\rho}{dt} \right) - \left(\vec{\mathbf{B}} \cdot \vec{\nabla} \right) \vec{\mathbf{v}} = 0$$

Now we may combine the first two terms:

$$\rho \frac{d}{dt} \left(\frac{\vec{\mathbf{B}}}{\rho} \right) - \left(\vec{\mathbf{B}} \cdot \vec{\nabla} \right) \vec{\mathbf{v}} = 0 \tag{28}$$

Thus if the second term is zero ($\vec{\mathbf{v}}$ is constant along field lines) then $B \propto \rho$. This is an example of *flux freezing*. We may imagine the field lines being swept along with the flow.

We may now insert the simplified LHS back into equation 24, again neglecting the Hall current term:

$$\rho \frac{d}{dt} \left(\frac{\vec{\mathbf{B}}}{\rho} \right) - \left(\vec{\mathbf{B}} \cdot \vec{\nabla} \right) \vec{\mathbf{v}} = \frac{\eta}{\mu_0} \nabla^2 \vec{\mathbf{B}} \tag{29}$$

3.2 Magnetic pressure

We may now go further to eliminate the current density from the MHD equations. We use Maxwell's equations to write:

$$\vec{\mathbf{j}} \times \vec{\mathbf{B}} = \left(\frac{\vec{\nabla} \times \vec{\mathbf{B}}}{\mu_0} \right) \times \vec{\mathbf{B}} = \frac{1}{2\mu_0} \left[2 \left(\vec{\mathbf{B}} \cdot \vec{\nabla} \right) \vec{\mathbf{B}} - \vec{\nabla} B^2 \right]$$

Then the momentum equation (7) becomes:

$$\rho\frac{d\overrightarrow{\mathbf{v}}}{dt} = \frac{1}{\mu_0}\left(\overrightarrow{\mathbf{B}}\cdot\overrightarrow{\nabla}\right)\overrightarrow{\mathbf{B}} - \overrightarrow{\nabla}\left(\frac{B^2}{2\mu_0}+p\right)+\rho\overrightarrow{\mathbf{g}}$$

We may interpret the $B^2/2\mu_0$ term as magnetic pressure, while the $\frac{1}{\mu_0}\left(\overrightarrow{\mathbf{B}}\cdot\overrightarrow{\nabla}\right)\overrightarrow{\mathbf{B}}$ term is due to field line tension. Notice that the Alfven speed is then given by $v_A^2 \approx P_{\text{mag}}/\rho$, while the ordinary fluid sound speed is given by $v_s^2 \approx P_{\text{gas}}/\rho$.

We now have a consistent set of MHD equations in ρ, $\overrightarrow{\mathbf{v}}$, and $\overrightarrow{\mathbf{B}}$:

$$\rho\frac{d\overrightarrow{\mathbf{v}}}{dt} = \frac{1}{\mu_0}\left(\overrightarrow{\mathbf{B}}\cdot\overrightarrow{\nabla}\right)\overrightarrow{\mathbf{B}} - \overrightarrow{\nabla}\left(\frac{B^2}{2\mu_0}+p\right)+\rho\overrightarrow{\mathbf{g}} \tag{30}$$

$$\frac{\partial\overrightarrow{\mathbf{B}}}{\partial t} + \left(\overrightarrow{\mathbf{v}}\cdot\overrightarrow{\nabla}\right)\overrightarrow{\mathbf{B}} + \overrightarrow{\mathbf{B}}\left(\overrightarrow{\nabla}\cdot\overrightarrow{\mathbf{v}}\right) - \left(\overrightarrow{\mathbf{B}}\cdot\overrightarrow{\nabla}\right)\overrightarrow{\mathbf{v}} = 0 \tag{31}$$

$$\frac{d}{dt}\rho + \left(\overrightarrow{\mathbf{v}}\cdot\overrightarrow{\nabla}\right)\rho = 0 \tag{32}$$

An equation of state linking P and ρ completes the set.

3.3 Using the MHD equations: Alfven waves

We may linearize this set of equations in the usual way, with $\overrightarrow{\mathbf{v}_0} = 0$ and $\overrightarrow{\mathbf{B}_0} = $ constant.

Eqn 32:

$$-i\omega\rho_1 + i\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}\rho = 0 \Longrightarrow \rho_1 = \rho\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}}{\omega} \tag{33}$$

Eqn 31 :

$$-i\omega\overrightarrow{\mathbf{B}_1} + i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}\right)\overrightarrow{\mathbf{B}_0} - i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)\overrightarrow{\mathbf{v}_1} = 0 \tag{34}$$

or:

$$\overrightarrow{\mathbf{B}_1} = \left(\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}}{\omega}\right)\overrightarrow{\mathbf{B}_0} - \left(\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}}{\omega}\right)\overrightarrow{\mathbf{v}_1} \tag{35}$$

Eqn 30:

$$-i\omega\rho\overrightarrow{\mathbf{v}_1} = -i\overrightarrow{\mathbf{k}}\left(P_1 + \frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\mu_0}\right) + i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)\frac{\overrightarrow{\mathbf{B}_1}}{\mu_0} \tag{36}$$

From the equation of state:

$$P_1 = \frac{dP}{d\rho}\rho_1 = v_s^2\rho_1 \tag{37}$$

So then equation 36 becomes:

$$-i\omega\rho\overrightarrow{\mathbf{v}_1} = -i\overrightarrow{\mathbf{k}}\left(v_s^2\rho_1 + \frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\mu_0}\right) + i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)\frac{\overrightarrow{\mathbf{B}_1}}{\mu_0} \tag{38}$$

Use equation 33 to eliminate ρ_1 :

$$-i\omega\rho\overrightarrow{\mathbf{v}_1} = -i\overrightarrow{\mathbf{k}}\left(v_s^2\rho\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}}{\omega} + \frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\mu_0}\right) + i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)\frac{\overrightarrow{\mathbf{B}_1}}{\mu_0} \tag{39}$$

Now dot equations 35 and 39 with $\overrightarrow{\mathbf{k}}$:

$$-i\omega\rho\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}\right) = -ik^2\left(v_s^2\rho\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}}{\omega} + \frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\mu_0}\right) + i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_1}}{\mu_0} \tag{40}$$

and

$$\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_1} = \left(\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}}{\omega}\right)\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right) - \left(\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}}{\omega}\right)\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}\right) = 0 \tag{41}$$

Thus the perturbation to $\vec{B}$ is perpendicular to $\vec{k}$. Then equation 40 simplifies:

$$\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}\right) = \frac{k^2}{\omega\rho}\left(v_s^2\rho\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}}{\omega} + \frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\mu_0}\right) = k^2v_s^2\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}}{\omega^2} + k^2\frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\omega\rho\mu_0}$$

Gathering up terms, we have:

$$\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}\right)\left(1 - \frac{k^2v_s^2}{\omega^2}\right) = k^2\frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\omega\rho\mu_0} \tag{42}$$

Now we can put this back into equation 39:

$$\begin{aligned} -\omega\rho\overrightarrow{\mathbf{v}_1} &= -\overrightarrow{\mathbf{k}}\left(\frac{v_s^2\rho}{\omega\left(1 - k^2v_s^2/\omega^2\right)}k^2\frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\omega\rho\mu_0} + \frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\mu_0}\right) + \left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)\frac{\overrightarrow{\mathbf{B}_1}}{\mu_0} \\ &= -\frac{\overrightarrow{\mathbf{k}}}{\left(1 - k^2v_s^2/\omega^2\right)}\frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\mu_0} + \left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)\frac{\overrightarrow{\mathbf{B}_1}}{\mu_0} \end{aligned} \tag{43}$$

Now cross equations 43 and 35 with $\overrightarrow{\mathbf{k}}$:

$$-i\omega\rho\left(\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{v}_1}\right) = i\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)\frac{\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{B}_1}}{\mu_0} \tag{44}$$

and

$$\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{B}_1} = \left(\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}}{\omega}\right)\left(\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{B}_0}\right) - \left(\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}}{\omega}\right)\left(\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{v}_1}\right) \tag{45}$$

Combining (44) and (45), we have:

$$\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{B}_1} = \left(\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{v}_1}}{\omega}\right)\left(\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{B}_0}\right) + \left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)\frac{\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{B}_1}}{\omega^2\rho\mu_0} \tag{46}$$

and using equation 42

$$\left(\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{B}_1}\right)\left(1 - \frac{\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)^2}{\omega^2\rho\mu_0}\right) - \left(\frac{k^2\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\omega^2\rho\mu_0\left(1 - k^2v_s^2/\omega^2\right)}\right)\left(\overrightarrow{\mathbf{k}}\times\overrightarrow{\mathbf{B}_0}\right) = 0 \tag{47}$$

Now if we look at propagation along $\overrightarrow{\mathbf{B}_0}$, the second term is zero. Since we have already found that $\vec{k}\cdot\vec{B}_1 = 0$, $\vec{k}\times\vec{B}_1$ cannot also be zero unless $\vec{B}_1$ is zero, and so we have:

$$\omega^2 = \frac{\left(\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}\right)^2}{\rho\mu_0} = k^2 v_A^2 \tag{48}$$

which is the dispersion relation for Alfven waves. As we found in our previous discussion of Alfven waves, these waves are transverse waves, and we can liken them to waves on strings.

For propagation across $\overrightarrow{\mathbf{B}_0}$, dot equation 35 with $\overrightarrow{\mathbf{B}_0}$

$$\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0} = \left(\frac{k^2}{(1-k^2v_s^2/\omega^2)}\frac{\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}}{\omega^2\rho\mu_0}\right)B_0^2 - \left(\frac{\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0}}{\omega}\right)\left(\overrightarrow{\mathbf{B}_0}\cdot\overrightarrow{\mathbf{v}_1}\right)$$

Since $\overrightarrow{\mathbf{k}}\cdot\overrightarrow{\mathbf{B}_0} = 0$ in this case,

$$\left(\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0}\right)\left(1 - \frac{k^2}{(1-k^2v_s^2/\omega^2)}\frac{B_0^2}{\omega^2\rho\mu_0}\right) = 0$$

If $\overrightarrow{\mathbf{B}_1}\cdot\overrightarrow{\mathbf{B}_0} \neq 0$,then:

$$\frac{\omega^2}{k^2}\left(1-\frac{k^2v_s^2}{\omega^2}\right) = v_A^2$$

$$\frac{\omega^2}{k^2} = v_A^2 + v_s^2 \tag{49}$$

These are magnetosonic waves. (cf Chen eqn 4-142, p 144, in the limit $c \gg v_A$, and plaswav notes eqn 65)

3.4 MHD power generators

The MHD power generator is used to convert thermal energy to electric energy. In this device, a weakly ionized gas expands down a channel with a magnetic field across it. Its operation may be understood using Ohm's Law (14):

$$\vec{v}\times\vec{B} = \eta\vec{j} - \frac{1}{en}\left(\vec{j}\times\vec{B} - \vec{\nabla}P_e\right) - \vec{E}$$

and the equation of motion (7)

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = \rho\frac{d\vec{v}}{dt} = \vec{j}\times\vec{B} - \vec{\nabla}P$$

where we have neglected gravity.

Initially the pressure gradient drives the plasma along the generator, perpendicular to the applied magnetic field. As $\vec{v}$ increases, the $\vec{v}\times\vec{B}$ force generates current across the generator. The resulting $\vec{j}\times\vec{B}$ force opposes the pressure gradient, and a steady state flow results. In a steady state, the momentum equation gives:

$$\frac{1}{2}\rho\frac{dv^2}{dz} = \frac{d}{dz}\left(\text{kinetic energy density}\right) \sim jB$$

Thus the kinetic energy of the flow is converted to electric current. To extract most of the flow energy, we want a channel of length L where

$$L \sim \frac{1}{2}\frac{\rho v^2}{jB}$$

But from Ohm's law:

$$j \sim \frac{vB}{\eta}$$

and thus

$$L \sim \frac{1}{2}\frac{\rho v}{B^2}\eta$$

The device is efficient because the losses are surface effects (friction, heat conducted out of the region) while the energy conversion is a volume effect.

The Hall term $\frac{1}{en}\vec{j} \times \vec{B}$ is important to consider in the design of a good MHD generator.

$$\left|\frac{1}{en}\vec{j} \times \vec{B}\right| = \frac{\eta}{en\eta} jB = \eta j \frac{B}{en}\frac{e^2 n}{m}\tau$$

(equation 22 in diffusion2 notes, τ is the collision time).

$$\left|\frac{1}{en}\vec{j} \times \vec{B}\right| = \eta j \frac{eB}{m}\tau = \eta j \omega_c \tau$$

Thus the Hall term in Ohm's law is of order $\omega_c\tau$ times the resistivity term. If $\omega_c\tau$ is not too small, then the current flowing across the channel creates, through the Hall term, a Hall electric field $\vec{E}_H$ along the channel. The Hall electric field drives a current $\vec{j}_H = \vec{E}_H/\eta$ opposite $\vec{v}$, which in its turn causes a current opposite the original current. The magnitude of this current may be estimated approximately as follows:

$$E_H \sim \frac{1}{en} jB = \eta j \omega_c \tau$$

Thus

$$j_H \sim j\omega_c\tau$$
$$E_{H^2} \sim \eta\left(\omega_c\tau\right)^2 j$$

and therefore

$$j' = \left(\omega_c\tau\right)^2 j$$

When $\omega_c\tau$ is not small, it is important to prevent the Hall current from flowing.

The gas used is usually air, CO_2, or argon with a small amount of easily ionized gas to provide the ions. This might be 0.1-1% of an alkali metal vapor such as sodium or potassium. The tempertature is 2-3000 K and the field is around 1.4 T (similar to the field in a typical loudspeaker). The magnetic reynolds number R_M is typically $\ll 1$ in these devices, which means that flow dominates. Thus the physics is more astrophysical than it is like a typical fusion experiment. Fusion devices tend to have high R_M.