Motion of charged particles in EM fields

Jan 07

1 Uniform magnetic field

The basic motion of a charged particle in a magnetic field is a helix. Let the magnetic field $\vec{B}$ be uniform. Then the equation of motion for a particle of charge q and mass m is:

$$\vec{F} = q\vec{v} \times \vec{B} = m\vec{a}$$

Thus the acceleration is always perpendicular to the velocity $\vec{v}$. Compare with the relation for uniform circular motion:

$$\vec{a} = \vec{\omega} \times \vec{v} = -\frac{q}{m}\vec{B} \times \vec{v}$$

Thus the angular velocity of the particle is

$$\vec{\omega} = -\frac{q}{m}\vec{B} \tag{1}$$

The angular speed is $\omega_C = qB/m$, the cyclotron frequency, and the direction is along the magnetic field. The direction of the particle's rotation depends on the sign of the particle's charge. An electron, with $q = -e$, has $\vec{\omega}$ parallel to $\vec{B}$. Ions, with positive charge, rotate in the opposite sense.

The radius of the helix is given by

$$v_\perp = \omega r$$

where $v_\perp$ is the component of the particle's velocity perpendicular to the magnetic field, and thus

$$r_L = \frac{mv_\perp}{|q|\,B} \tag{2}$$

This radius is called the Larmor radius. A proton moving at the same speed as an electron will have a Larmor radius 2000 times bigger than the electron's

Larmor radius. Thus the electrons are more closely tied to the magnetic field lines.

The magnetic field near the surface of a neutron star is about 10^{12} Gauss, or 10^8 T. This corresponds to

$$\omega_C = \frac{1.6 \times 10^{-19}\ \mathrm{C}}{9 \times 10^{-31}\ \mathrm{kg}} 10^8\ \mathrm{T} = 1.8 \times 10^{19}\ \frac{\mathrm{C}}{\mathrm{kg}}\mathrm{T}$$

Now do the units check out? Since we know that force is qvB, then

$$\mathrm{C} \cdot \mathrm{T} = \frac{\mathrm{N}}{\mathrm{m/s}}$$

and thus

$$\frac{\mathrm{C} \cdot \mathrm{T}}{\mathrm{kg}} = \frac{\mathrm{kg} \cdot \mathrm{m/s}^2}{\mathrm{kg} \cdot \mathrm{m/s}} = \frac{1}{\mathrm{s}}$$

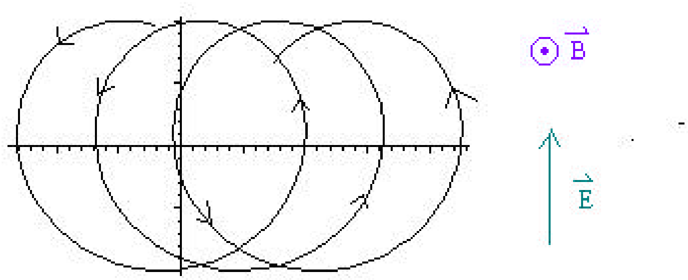
and so the units check out.

This gyrational motion is the basic motion of charged particles in a plasma. The center of the circle is called the *guiding center.* With a uniform magnetic field, and no electric field, the guiding center moves at constant speed along the field line.

The circulating charge forms a current loop, and hence has a magnetic field of its own. This field produced by the particle's motion is opposite the original, uniform field. Thus the plasma is a *diamagnetic* material.

2 Uniform magnetic field plus uniform electric field

Without a magnetic field, an electric field causes a charged particle to accelerate along the direction of $\vec{E}$. But with a magnetic field, the particle behaves more like a spinning top, and moves *perpendicular* to the applied electric field. We can understand why by noting that (for positive q) the particle speeds up as it moves parallel to $\vec{E}$, thus increasing the Larmor radius (2). As the particle moves opposite $\vec{E}$, v and r_L decrease. Thus the circles don't close, and the particle drifts as shown in the diagram. Here we show that path of an electron with $\vec{B}$ out of the page and $\vec{E}$ upward.



Now let's do the math. Let's split $\vec{E}$ into components along and perpendicular to $\vec{B}$. Then the equation of motion is:

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E}_\perp + E_\parallel \hat{b} + \vec{v} \times \vec{B}\right)$$

Dotting with $\hat{\mathbf{b}}$, we find

$$m\frac{dv_\parallel}{dt} = qE_\parallel$$

the same relation as in the absence of $\vec{B}$. The perpendicular component is:

$$\frac{d\vec{v}_\perp}{dt} = \frac{q}{m}\left(\vec{E}_\perp + \vec{v}_\perp \times \vec{B}\right) \tag{3}$$

The solution we expect is a combination of a gyration plus a constant drift. Thus, in the frame moving with the guiding center, we will see only the gyration, and the equation of motion should look like:

$$\frac{d\left(\vec{v}_\perp - \vec{v}_E\right)}{dt} = \frac{q}{m}\left(\vec{v}_\perp - \vec{v}_E\right) \times \vec{B} \tag{4}$$

Comparing equations (3) and (4), we have:

$$\vec{E}_\perp = -\vec{v}_E \times \vec{B}$$

Cross both sides with $\vec{B}$ to get:

$$\vec{E} \times \vec{B} = -\left(\vec{v}_E \times \vec{B}\right) \times \vec{B} = -\vec{B}\left(\vec{v}_E \cdot \vec{B}\right) + \vec{v}_E B^2$$

We can drop the perpendicular sign on $\vec{E}$ because the cross product selects only the perpendicular component. And since the drift velocity is perpendicular to $\vec{B}$, we have

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} \tag{5}$$

The result is independent of both the charge and mass of the particle: *all* particles have the same E-cross-B drift velocity. Changing the sign of the charge changes the direction of gyration, but the particle also accelerates in the opposite direction, so the direction of the drift is the same. A particle with smaller mass has a smaller Larmor radius, but gyrates faster. The two effects combine to create the same drift velocity. Since all particles drift at the same rate, this drift does not cause any current in the plasma, but instead causes the whole plasma to drift together.

3 Drifts due to other constant, external forces

Any constant external force perpendicular to $\vec{B}$ produces a constant drift in a similar way. In general, the equation of motion is

$$m\frac{d\vec{v}}{dt} = \vec{F} + q\vec{v} \times \vec{B}$$

and again we look for a solution in which the particle motion perpendicular to $\vec{B}$ is gyration plus drift:

$$\frac{d\left(\vec{v}_\perp - \vec{v}_D\right)}{dt} = \frac{q}{m}\left(\vec{v}_\perp - \vec{v}_D\right) \times \vec{B}$$

Thus

$$\frac{\vec{F}_\perp}{m} = -\frac{q}{m}\vec{v}_D \times \vec{B}$$

and crossing with $\vec{B}$ gives

$$\frac{1}{qB^2}\vec{F}_\perp \times \vec{B} = \vec{v}_D \tag{6}$$

The motion parallel to the field is uniformly accelerated motion with

$$a_\parallel = \frac{F_\parallel}{m}$$

One particularly important example is the *gravitational drift*. With $\vec{F} = m\vec{g}$, we get

$$\vec{v}_g = \frac{m}{qB^2}\vec{g} \times \vec{B} \tag{7}$$

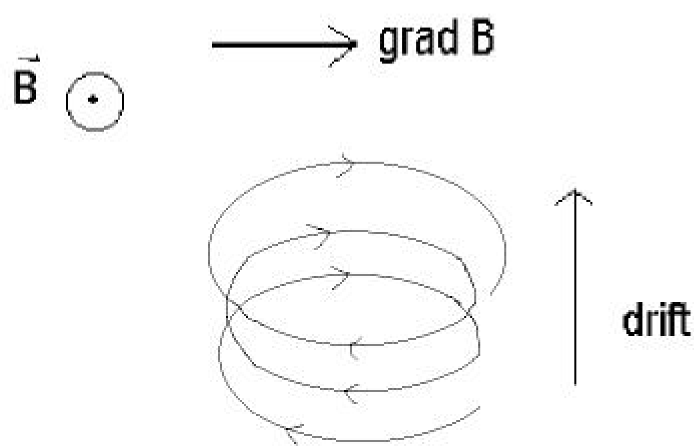
In this case the drift velocity depends on both the mass and the charge of the particle. Particles with opposite signs of charge drift in opposite directions: thus this drift creates a current in the plasma. Since the drift is proportional to mass, the current is carried primarily by the *ions*.

The result that a plasma move sideways in a gravitational field is surprising at first sight. But suppose the plasma is contained in a vessel of finite size with non-conducting walls. Then the current leads to a charge build-up on the walls, and thus creates an electric field. The resulting E-cross-B drift is downward, and so the plasma falls as expected.

4 Motion in a non-uniform field

4.1 Gradient perpendicular to $\vec{B}$.

Variations in the magnitude of $\vec{B}$ also change the Larmor radius, and so cause a drift. As B increases, the Larmor radius decreases, and so for a positively charged particle the path looks as shown in the figure.



The particle drifts in the direction of $\vec{B} \times \vec{\nabla} B$.

Now for the math. We shall consider the case in which the length scale over which B varies appreciably is large compared with the Larmor radius. Then B changes by a small amount over one orbit. Thus to first order we may neglect the change in B during one orbit.

Let $\vec{B} = B\hat{z}$ and $\vec{\nabla} B$ be in the $x-$direction. Then the particle's unperturbed path may be decribed by:

$$\vec{r} = \hat{y}\ r_L \cos \omega_c t \pm \hat{x}\ r_L \sin \omega_c t \tag{8}$$

where the choice of sign is determined by the sign of the particle's charge, and

$$\vec{v} = \pm v_\perp \cos \omega_c t\ \hat{\mathbf{x}} - v_\perp \sin \omega_c t\ \hat{\mathbf{y}}$$

The force acting on the particle has components:

$$\begin{aligned} F_x &= q v_y B = q\left(-v_\perp \sin \omega_c t\right)\left(B_0 + \vec{r} \cdot \vec{\nabla} B\Big|_0\right) \\ &= -q v_\perp \sin \omega_c t \left(B_0 + \left(\pm r_L \sin \omega_c t\right) \cdot \left.\frac{\partial B}{\partial x}\right|_0\right) \end{aligned}$$

where we used a Tylor series expansion for $\vec{B}\left(\vec{r}\right)$, and similarly

$$F_y = -q v_x B = -q\left(\pm v_\perp \cos \omega_c t\right)\left(B_0 \pm r_L \sin \omega_c t \cdot \left.\frac{\partial B}{\partial x}\right|_0\right)$$

Now we average over one orbit. All terms in $\sin \omega_c t$, $\cos \omega_c t$ and $\sin \omega_c t \cos \omega_c t$ average to zero. The term in $\sin^2 \omega_c t$ averages to $1/2$. Thus

$$< F_y > \quad = 0$$

and

$$< F_x > \quad = -\frac{1}{2} q v_\perp \left(\pm r_L \cdot \left.\frac{\partial B}{\partial x}\right|_0\right) = -\frac{1}{2} \text{sign}\left(q\right) q v_\perp r_L \left.\frac{\partial B}{\partial x}\right|_0$$

Now we can use this result in equation (6) to get

$$\begin{aligned} \vec{v}_{\text{grad B}} &= -\frac{< F_x > B}{q B^2} \hat{\mathbf{y}} \\ &= \frac{1}{2} \text{sign}\left(q\right) \frac{v_\perp r_L}{B^2} B \left.\frac{\partial B}{\partial x}\right|_0 \hat{\mathbf{y}} \\ &= \text{sign}\left(q\right) \frac{v_\perp r_L}{2 B^2} \vec{B} \times \vec{\nabla} B \end{aligned} \tag{9}$$

or, using equation (2) for the Larmor radius, we get

$$\begin{aligned}
\vec{v}_{\text{grad B}} &= \text{sign}\,(q)\,\frac{mv_\perp^2}{2\,|q|\,B}\frac{\vec{B}\times\vec{\nabla}B}{B^2} \\
&= \frac{mv_\perp^2}{2qB}\frac{\vec{B}\times\vec{\nabla}B}{B^2} \qquad (10)
\end{aligned}$$

where in this last expression q is a signed quantity, negative for electrons and positive for ions.

Since this drift again depends on the charge-to-mass ratio, we see that electrons and ions drift in opposite directions, giving rise to a current in the plasma. This current is also carried primarily by the ions.

Using equation (9), this drift is approximately:

$$v_{\text{grad B}} \sim \frac{v_\perp}{2}\frac{r_L}{L}$$

where L is the scale length over which B changes ($L = B/\left|\vec{\nabla}B\right|$). This expression shows that this is a *finite Larmor radius effect.* The drift depends on the fact that over one orbit the particle samples different values of B.

4.2 Curvature drift

When the field lines are not straight, but curved, a particle moving along the field line must be experiencing a force and thus will experience a drift. To compute the drift, we work in the reference frame moving along the field line with the particle. In this *accelerated* reference frame, the particle experiences the fictitious centrifugal force

$$\vec{F}_c = \frac{mv_\parallel^2}{R_c}\hat{\mathbf{r}} = \frac{mv_\parallel^2}{R_c^2}\vec{R}_c$$

where R_c is the radius of curvature of the field line, $v_\parallel$ is the speed along the field line, and $\hat{\mathbf{r}}$ is the unit vector directed outward. Then, from equation (6), the drift is:

$$\vec{v}_C = \frac{1}{qB^2}\vec{F}\times\vec{B} = \frac{mv_\parallel^2}{qB^2R_c^2}\vec{R}_c\times\vec{B} \qquad (11)$$

This *curvature drift* is also proportional to m/q and so gives rise to a current, carried primarily by the ions.

Now in vacuum we must have $\vec{\nabla} \times \vec{B} = 0$, and this means that curvature is always accompanied by gradients in B. To see why, use cylindrical coords and let $\vec{B}$ locally be described by $\vec{B} = B\hat{\boldsymbol{\theta}}$. Then $\vec{\nabla} \times \vec{B}$ has only an r component:

$$\vec{\nabla} \times \vec{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB)\ \hat{\mathbf{z}} = 0$$

Thus $rB =$ constant, and thus $B \propto 1/r$. Then

$$\vec{\nabla} B = -\frac{\vec{R}_c}{R_c^2} B$$

and this gives rise to a gradient drift:

$$\begin{aligned} \vec{v}_{\text{grad B}} &= \frac{mv_\perp^2}{2qB}\frac{\vec{B} \times \vec{\nabla} B}{B^2} = \frac{mv_\perp^2}{2qB}\frac{\vec{B}}{B^2} \times \left(-\frac{\vec{R}_c}{R_c^2} B\right) \\ &= \frac{mv_\perp^2}{2qB^2}\frac{\vec{R}_c}{R_c^2} \times \vec{B} \end{aligned}$$

adding this to the curvature drift, we get

$$\vec{v}_{\text{total drift}} = \frac{m}{qB^2}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\frac{\vec{R}_c}{R_c^2} \times \vec{B} \tag{12}$$

4.3 Magnetic mirrors

To round out this discussion, we consider the case where the magnetic field strength varies in the direction of $\vec{B}$. Then necessarily the field lines cannot be straight and parallel, but must converge or diverge. We put the $z-$axis along the "central" field line– the line along which the guiding center moves. Then the Lorentz force acting on a particle is:

$$\vec{F} = q\vec{v} \times \vec{B} = q\left(v_\theta B_z\ \hat{\boldsymbol{\rho}} + [v_z B_\rho - v_\rho B_z]\ \hat{\boldsymbol{\theta}} - v_\theta B_\rho\ \hat{\mathbf{z}}\right) \tag{13}$$

The $z-$component of the force causes the velocity of the particle along the field line to change. The $\rho-$component is the usual centripetal force causing the particle to gyrate around the field line. The $\theta-$component has two parts: the $v_\rho B_z$ term changes speed of gyration while the drift due to the $v_z B_\rho$ term causes the particle to track along the converging field lines.

$$\vec{v}_D \sim \frac{1}{B^2} v_z B_\rho \hat{\boldsymbol{\theta}} \times \vec{B} = v_z \frac{B_\rho}{B_z}\hat{\boldsymbol{\theta}} \times \hat{\mathbf{z}} = v_z \frac{B_\rho}{B_z}\hat{\boldsymbol{\rho}}$$

With the guiding center on the $z-$axis,

$$\vec{\nabla} \cdot \vec{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0$$

Thus

$$\rho B_\rho = -\int_0^\rho \rho \frac{\partial B_z}{\partial z} d\rho \simeq -\frac{1}{2}\rho^2 \left.\frac{\partial B_z}{\partial z}\right|_{\rho=0}$$

and thus for $\rho \ll L = B/\left|\partial B_z/\partial z\right|$

$$B_\rho \simeq -\frac{1}{2}\rho \left.\frac{\partial B_z}{\partial z}\right|_{\rho=0}$$

Then from (13)

$$F_z = \frac{q}{2} v_\theta r_L \left.\frac{\partial B_z}{\partial z}\right|_{\rho=0}$$

Now v_θ will be $\pm v_\perp$, depending on the sign of the charge (positive charges go in the $-\theta$ direction) and using equation (2) we find

$$F_z = \mp \frac{q}{2}\frac{m v_\perp^2}{|q| B} \left.\frac{\partial B_z}{\partial z}\right|_{\rho=0} = -\frac{m v_\perp^2}{2B} \left.\frac{\partial B_z}{\partial z}\right|_{\rho=0}$$

The quantity

$$\mu = \frac{m v_\perp^2}{2B} \tag{14}$$

is the *magnetic moment* of the particle. To see why, recall that the gyrating particle forms a current loop with current $I = q/T = \omega q/2\pi$ and the magnetic moment of the current loop is

$$IA = \frac{\pi r_L^2 \omega q}{2\pi} = \frac{r_L v_\perp q}{2} = \frac{m v_\perp^2}{2B} = \mu.$$

So

$$F_z = -\mu \left.\frac{\partial B_z}{\partial z}\right|_{\rho=0}$$

or, in a coordinate-free notation,

$$F_\| = -\mu \nabla_\| B$$

Thus

$$m\frac{dv_{\|}}{dt} = -\mu\nabla_{\|}B$$

Now let s be the distance travelled along the field line by the guiding center, so $v_{\|} = ds/dt$. Multiply both sides of the equation by $v_{\|}$ to get:

$$\begin{aligned} v_{\|}m\frac{dv_{\|}}{dt} &= -\mu\frac{dB}{ds}\frac{ds}{dt} \\ \frac{d}{dt}\left(\frac{1}{2}mv_{\|}^2\right) &= -\mu\frac{dB}{dt} \end{aligned} \tag{15}$$

The magnetic force does no work, so as the particle moves along the field line its energy must remain constant. As $v_{\|}$ decreases, $v_{\perp}$ must increase to compensate.

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{d}{dt}\left(\frac{1}{2}mv_{\|}^2 + \frac{1}{2}mv_{\perp}^2\right) = 0$$

Then, using the definition of μ and equation (15), we have

$$-\mu\frac{dB}{dt} + \frac{d}{dt}(\mu B) = B\frac{d\mu}{dt} = 0$$

Thus μ does not change as the particle moves along the field line: it is an *invariant* of the motion.

Strictly, μ is an *adiabatic invariant.* It remains constant only so long as the system changes slowly compared with the gyrational period of the particle. Or, equivalently, $r_L \ll L$, the length scale over which B changes.

Invariance of μ shows that the particle gyrates faster as the magnetic field strength increases: the parallel velocity decreases. Eventually, if B gets strong enough, the parallel velocity is reduced to zero. Since $F_{\|}$ does not depend on the parallel velocity, it continues to act in the same direction, and accelerates the particle back in the opposite direction: the particle is reflected. This phenomenon is called the magnetic mirror effect.

Not all particles are reflected. Obviously a particle with $v_{\perp} = 0$ is not reflected. We can find $v_{\perp}$ at the turning point from conservation of energy, and the fact that $v_{\|} = 0$ there. Suppose a particle starts at a point where the magnetic field strength is B_0 with gyrational velocity $v_{\perp 0}$, and total speed v_0. At the turning point $v_{\perp} = v_0$, the particle's initial speed. It will be reflected where

$$\frac{1}{2}m\frac{v_{\perp 0}^2}{B_0} = \frac{1}{2}m\frac{v_{\perp}^2}{B} = \frac{1}{2}m\frac{v_0^2}{B_{\mathrm{TP}}}$$

Thus

$$\frac{B_0}{B_{\rm TP}} = \frac{v_{\perp 0}^2}{v_0^2} = \sin^2\theta$$

where θ is the *pitch angle*, the angle between the particle's velocity vector and the field line at the starting point. Since $B_{\rm TP} \le B_m$, the maximum value of B, this equation shows that particles with small pitch angles will not be reflected. The continual loss of particles with small pitch angles causes the particles' velocity distribution to become anisotropic, leaving a hole called the *loss cone*. Collisions repopulate the loss cone.

Example: the Van Allen belts

The earth's field is essentially a dipole, with $B \sim B_0 (r_0/r)^3$ (ignoring the angular dependence for now). For a particle that starts at 5 earth radii and reaches the top of the atmosphere (essentially 1 earth radius) then particles that reflect have pitch angle given by

$$\begin{aligned} \sin^2\theta &\ge (r/r_0)^3 = \frac{1}{125} \\ \sin\theta &\ge 0.09 \\ \theta &\ge 5^\circ \end{aligned}$$

Thus the loss cone is very small. The particles in the loss cone are responsible for the aurora. I have asked you to investigate this system further in Problem set 3.

5 Non-uniform electric field

One of the most important examples of a non-uniform electric field is the field produced by a wave, which has the form

$$\vec{E} = \vec{E}_0 \cos kx$$

Using Fourier transforms, we can reduce a general variation to a sum of terms of this form.

Now let $\vec{B}$ be uniform, $\vec{B} = B_0 \hat{\mathbf{z}}$, and let

$$\vec{E} = E_0 \hat{\mathbf{x}} \cos kx$$

(a longitudal wave). Then the particle equation of motion is:

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

or, in components,

$$\frac{dv_x}{dt} = \frac{q}{m} E_0 \cos kx + \frac{q}{m} v_y B_0$$

$$\frac{dv_y}{dt} = -\frac{q}{m} v_x B_0$$

and

$$\frac{dv_z}{dt} = 0$$

Now we take a second derivative, to eliminate v_x from the v_y equation:

$$\frac{d^2 v_y}{dt^2} = -\frac{q}{m} B_0 \frac{dv_x}{dt} = -\omega_c \left(\frac{q}{m} E_0 \cos kx + \omega_c v_y \right)$$

and similarly

$$\begin{aligned} \frac{d^2 v_x}{dt^2} &= -k \frac{q}{m} E_0 \sin kx \frac{dx}{dt} + \omega_c \frac{dv_y}{dt} \\ &= -k \omega_c v_x \frac{E_0}{B_0} \sin kx - \omega_c^2 v_x \end{aligned}$$

Next we average over a gyrational period to isolate the drift. We assume slow variations of $\vec{E}$ ($k r_L \ll 1$) and use the unperturbed orbit position (8) in our expressions for the acceleration:

$$\sin kx = \sin k \left(x_0 \pm r_L \sin \omega t \right)$$

(cf section 4.1). Expanding the sine gives:

$$\sin kx = \sin kx_0 \cos \left(k r_L \sin \omega t \right) \pm \cos kx_0 \sin \left(k r_L \sin \omega t \right)$$

and with $k r_L \ll 1$,

$$\sin kx = \sin kx_0 \left[1 - \frac{1}{2} \left(k r_L \sin \omega t \right)^2 \right] \pm \cos kx_0 \ \ \left(k r_L \sin \omega t \right)$$

to second order in small quantities. When we time average, only the squares of $\sin \omega t$, $\cos \omega t$ do not average to zero, and so

$$< \sin kx > \ = \left(1 - \frac{1}{4} \left(k r_L \right)^2 \right) \sin kx_0$$

and similarly

$$< \cos kx > \ = \left(1 - \frac{1}{4}\left(kr_L\right)^2\right)\cos kx_0$$

so

$$< \frac{d^2 v_y}{dt^2} > \ = -\omega_c^2 < v_y > -\omega_c^2 \frac{E_0}{B_0}\left(1 - \frac{1}{4}\left(kr_L\right)^2\right)\cos kx_0$$

and

$$< \frac{d^2 v_x}{dt^2} > \ = - < v_x > \omega_c \left(k\frac{E_0}{B_0}\left(1 - \frac{1}{4}\left(kr_L\right)^2\right)\sin kx_0 + \omega_c\right)$$

From past experience, we expect the drift to be constant, so $< \frac{d^2 v_y}{dt^2} > = < \frac{d^2 v_x}{dt^2} > = 0$, and thus

$$< v_x > \ = 0$$

and

$$< v_y >= -\frac{E_0}{B_0}\left(1 - \frac{1}{4}\left(kr_L\right)^2\right)\cos kx_0$$

We'd like to write this in a coordinate-independent way. Remember that if $\vec{E}$ were constant, we would get

$$\vec{v}_E = \frac{\vec{E}\times\vec{B}}{B^2} = -\frac{E_0 \cos kx}{B_0}\hat{\mathbf{y}}$$

which is the first term of our result. The second term arises from the non-uniformity of $\vec{E}$. Thus:

$$\vec{v}_D = \left(1 + \frac{1}{4} r_L^2 \nabla^2\right)\frac{\vec{E}\times\vec{B}}{B^2}$$

This is another example of a finite Larmor radius effect. The sign of the charge does not effect the result, but the mass of the particle enters through r_L. A larger mass particle has a larger Larmor radius and thus samples more of the gradient of E.

6 Time-varying, uniform $\tilde{\mathbf{E}}$.

Here we investigate the particle motion in an electric field of the form

$$\vec{E} = E_0 e^{i\omega t}\hat{\mathbf{x}}, \qquad \vec{B} = B_0\hat{\mathbf{z}}$$

Again, this might be one Fourier component of a more general time variation. We may neglect induced magnetic fields provided that the variations are slow. ($B_{\text{ind}} \sim \mu_0 \varepsilon_0 \omega L E \sim \frac{\omega L}{c^2} E$ where L is the length scale for variation of B. Thus the magnetic force due to induced B is of order $v\omega L/c^2$ compared with the electric force.) Then the Lorentz force is

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) = m\frac{d\vec{v}}{dt}$$

Thus

$$\frac{dv_x}{dt} = \frac{q}{m}\left(E_0 e^{i\omega t} + v_y B_0\right)$$

and

$$\frac{dv_y}{dt} = -\frac{q}{m} v_x B_0 = -\omega_c v_x$$

Differentiating again, and eliminating v_x, we get

$$\frac{d^2 v_y}{dt^2} = -\omega_c \frac{dv_x}{dt} = -\omega_c^2 \left(\frac{E_0}{B_0} e^{i\omega t} + v_y\right)$$

and similarly

$$\frac{d^2 v_x}{dt^2} = \omega_c \left(i\omega \frac{E_0}{B_0} e^{i\omega t} - \omega_c v_x\right)$$

Once again we can recognize the $\vec{E} \times \vec{B}$ drift, $\vec{v}_E = -\left(E_0 e^{i\omega t}/B_0\right)\hat{\mathbf{y}} = v_E\ \hat{\mathbf{y}}$. Let's also define

$$v_P = \frac{i\omega}{\omega_c}\frac{E_0}{B_0} e^{i\omega t}$$

Then

$$\frac{d^2 v_x}{dt^2} = -\omega_c^2 \left(v_x - v_P\right)$$

and

$$\frac{d^2 v_y}{dt^2} = -\omega_c^2 \left(v_x - v_E\right)$$

Here we have to be a little more careful, because both v_E and v_P are time dependent. However,

$$\frac{d^2 v_P}{dt^2} = -\omega^2 v_P$$

and provided that $\omega \ll \omega_c$,

$$\frac{d^2\left(v_x - v_P\right)}{dt^2} = -\omega_c^2 \left(v_x - v_P\right) + \omega^2 v_P \simeq -\omega_c^2 \left(v_x - v_P\right)$$

so we have a gyration in the frame drifting with velocity

$$\vec{v}_P = \frac{1}{\omega_c B}\frac{\partial \vec{E}}{\partial t} \tag{16}$$

We also have the $\vec{E} \times \vec{B}$ drift, which is in the $\hat{\mathbf{y}}$ direction. The new drift with velocity $\vec{v}_P$ (equation 16) is called the *polarization drift.* It depends on both the charge and mass of the particle, since both appear in ω_c, and thus there is a *polarization current*:

$$\begin{aligned}\vec{j}_P &= ne\left(\vec{v}_{P+} - \vec{v}_{P-}\right) = \frac{ne}{eB^2}\left[M - (-m)\right]\frac{\partial \vec{E}}{\partial t} \\ &= \frac{\rho}{B^2}\frac{\partial \vec{E}}{\partial t} \qquad (17)\end{aligned}$$

where $\rho = n\,(M + m)$ is the plasma mass density. Once again the current is carried primarily by the ions.

This drift is a "start-up effect". As the electric field is applied, the particles begin to move along (+ charge) or opposite (− charge) $\vec{E}$, and only then start to gyrate.

7 More on adiabatic invariance

7.1 Time-varying $\tilde{\mathrm{B}}$

We know that the magnetic force does no work, and thus does not change the particle's energy. But when $\vec{B}$ changes in time there is an induced electric field that does do work. From Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

and thus the induced $\vec{E}$ is perpendicular to $\vec{B}$. The equation of motion is:

$$m\frac{d\vec{v}_\perp}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Now dot this equation with $\vec{v}_\perp$ to get:

$$\begin{aligned}m\vec{v}_\perp \cdot \frac{d\vec{v}_\perp}{dt} &= q\vec{v}_\perp \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) \\ \frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) &= q\vec{v}_\perp \cdot \vec{E}\end{aligned}$$

Now we express $v_\perp$, the gyrational velocity, as

$$\vec{v}_\perp = \frac{d\vec{\ell}}{dt}$$

where

$$d\vec{\ell} = r_L d\theta \ \hat{\boldsymbol{\theta}}$$

is a vector element along the Larmor orbit. Thus

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\vec{E}\cdot\frac{d\vec{\ell}}{dt}$$

Integrating over one period, we get

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \oint_{\text{orbit}} q\vec{E}\cdot d\vec{\ell}$$

where, provided $\vec{B}$ changes slowly compared with the gyrational period, we may integrate over the unperturbed orbit. Then applying Stokes' theorem, we get:

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = q\int\left(\vec{\nabla}\times\vec{E}\right)\cdot\hat{n}\ dA = -q\int\frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\ dA$$

Note that there is a connection between the directions of $d\vec{\ell}$ and $\hat{\mathbf{n}}$ through the right hand rule. For positively charged particles that gyrate with $\vec{\omega}$ opposite $\vec{B}$, $\hat{\mathbf{n}}$ is also opposite $\vec{B}$. For negatively charged particles, $\hat{\mathbf{n}}$ is parallel to $\vec{B}$. Thus

$$\begin{aligned}
\delta\left(\frac{1}{2}mv_\perp^2\right) &= -q\left(\mp\right)\frac{\partial B}{\partial t}\pi r_L^2 \\
&= \pm\frac{\partial B}{\partial t}\pi q\left(\frac{mv_\perp}{qB}\right)^2 \\
&= \pm\frac{\partial B}{\partial t}\frac{1}{2}\frac{mv_\perp^2}{B}\frac{2\pi m}{qB} \\
&= \frac{\partial B}{\partial t}\mu\frac{2\pi}{\omega_c}
\end{aligned}$$

where $\omega_c = |q| B/m$ is the cyclotron frequency, and μ is the magnetic moment (equation (14)). Thus the change in gyrational energy over one period is

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \delta(\mu B) = \mu\delta B$$

where δB is the change in B over one period. Thus $\delta\mu = 0$, and the magnetic moment does not change. This is the same result that we obtained in §4.3. This analysis corresponds to the situation we'd see in the reference frame moving with the guiding center.

This analysis also shows that the magnetic moment is proportional to the flux enclosed within one Larmor orbit. Thus the particle moves so as to keep the magnetic flux within its orbit a constant.

The magnetic moment is an adiabtic invariant– it remains constant only so long as the time scale for variation of B satisfies $\omega_c T > 2\pi$. (i.e.$\omega_c/\omega > 1$) "Much greater than 1" is not required (although we have not proved that here.)

7.2 The second adiabatic invariant.

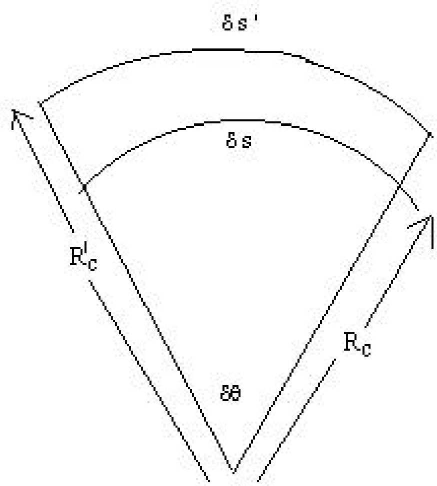
A particle trapped between two magnetic mirrors, such as a particle in the earth's radiation belts, oscillates along the field lines at the "bounce frequency". It also drifts around the earth due to grad-B and curvature drift. If P_1 and P_2 are the points at which the particle reflects at the two mirrors, the integral

$$J = \int_{P_1}^{P_2} v_\parallel \, ds$$

remains invariant as the particle drifts. This is a second adiabatic invariant. It is invariant so long as changes in $\vec{B}$ are slow compared with the bounce period. Since this period is much longer than the gyrational period, J is less likely to be invariant than μ.

Invariance of J guarantees that a drifting particle returns to the original field line after drifting through 2π radians.

Compare two neighboring field lines as shown in the diagram.



Then

$$\frac{\delta s}{R_c} = \frac{\delta s'}{R'_c} = \delta\theta$$

Thus

$$\frac{\delta s' - \delta s}{\Delta t \delta s} = \frac{R'_c - R_c}{\Delta t R_c}$$

The guiding center drift is a combination of grad-B (equation 10) and curvature drift (equation 12). Over most of the bounce period grad-B drift dominates ($v_\parallel \ll v_\perp$). The radial component is

$$\vec{v}_D \cdot \hat{r} = \frac{m v_\perp^2}{2qB} \frac{\vec{B} \times \vec{\nabla} B}{B^2} \cdot \frac{\vec{R}_c}{R_c} = \frac{R'_c - R_c}{\Delta t}$$

Thus

$$\begin{aligned}
\frac{\delta s' - \delta s}{\Delta t \delta s} R_c &= \frac{m v_\perp^2}{2qB} \frac{\vec{B} \times \vec{\nabla} B}{B^2} \cdot \frac{\vec{R}_c}{R_c} \\
\frac{1}{\delta s} \frac{d \delta s}{dt} &= \frac{m v_\perp^2}{2qB} \frac{\vec{B} \times \vec{\nabla} B}{B^2} \cdot \frac{\vec{R}_c}{R_c^2}
\end{aligned}$$

We also need to know how $v_\parallel$ varies. The total kinetic energy is

$$W = \frac{1}{2} m v_\parallel^2 + \frac{1}{2} m v_\perp^2 = \frac{1}{2} m v_\parallel^2 + \mu B$$

and thus

$$v_\parallel = \sqrt{\frac{2}{m}\left(W - \mu B\right)}$$

where both W and μ are constants in this expression. Thus

$$\frac{dv_\parallel}{dt} = \sqrt{\frac{2}{m}}\frac{1}{2}\frac{-\mu dB/dt}{\sqrt{W - \mu B}}$$

and

$$\frac{1}{v_\parallel}\frac{dv_\parallel}{dt} = \frac{1}{2}\frac{-\mu dB/dt}{(W - \mu B)} = \frac{-\mu dB/dt}{m v_\parallel^2}$$

The magnetic field B at the particle's position changes because of the guiding center motion, so

$$\frac{dB}{dt} = \vec{v}_{\mathrm{gc}} \cdot \vec{\nabla} B = \left[\frac{m}{qB^2} v_\parallel^2 \frac{\vec{R}_c}{R_c^2} \times \vec{B}\right] \cdot \vec{\nabla} B$$

and rearranging the triple scalar product,

$$\begin{aligned} \frac{1}{v_\parallel}\frac{dv_\parallel}{dt} &= \frac{-\mu}{m v_\parallel^2}\frac{m}{qB^2} v_\parallel^2 \frac{\vec{R}_c}{R_c^2} \cdot \left(\vec{B} \times \vec{\nabla} B\right) \\ &= -\frac{m v_\perp^2}{2qB^3}\frac{\vec{R}_c}{R_c^2} \cdot \left(\vec{B} \times \vec{\nabla} B\right) = -\frac{1}{\delta s}\frac{d\delta s}{dt} \end{aligned}$$

Thus

$$\frac{d}{dt}\left(v_\parallel \delta s\right) = 0$$

Can we now conclude that J is invariant? What if the turning points P_1 and P_2 are not quite at the same point on a neighboring field line? It doesn't matter because $v_\parallel \to 0$ at the turning points, and so the contribution to the integral due to a small change in P is negligible. Thus

$$J = \int_{P_1}^{P_2} v_\parallel ds \text{ is an adiabatic invariant.}$$

2nd proof of invariance of J. (Boyd andSanderson pg 29) (See also Sturrock, *Plasma Physics* Ch 4)

The invariant is

$$J = \oint v_{\|} ds = \int_{P_1}^{P_2} v_{\|} ds + \int_{P_2}^{P_1} v_{\|} ds = 2 \int_{P_1}^{P_2} v_{\|} ds$$

We begin by writing

$$v_{\|} = \sqrt{\frac{2}{m}\left(W - \mu B\right)}$$

where $W = \frac{1}{2} m \left(v_{\perp}^2 + v_{\|}\right)$ is the energy, μ (14) is the magnetic moment, and looking at

$$J\left(W, s, t\right) = \int_{s_1}^{s} \sqrt{\frac{2}{m}\left(W - \mu B\right)} ds$$

where s is a coordinate measured along the field line. Then

$$\frac{dJ}{dt} = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial W}\frac{dW}{dt} + \frac{\partial J}{\partial s}\frac{ds}{dt}$$

where

$$\frac{\partial J}{\partial t} = -\int_{s_1}^{s} \frac{1}{\sqrt{\frac{2}{m}\left(W - \mu B\right)}} \frac{\mu}{m}\frac{\partial B}{\partial t} ds$$

$$\frac{\partial J}{\partial W}\frac{dW}{dt} = -\int_{s_1}^{s} \frac{1}{\sqrt{\frac{2}{m}\left(W - \mu B\right)}} ds \left\{ v_{\|}\frac{\partial v_{\|}}{\partial t} + \frac{\mu}{m}\frac{\partial B}{\partial t} + \frac{\mu}{m} v_{\|} \frac{\partial B}{\partial s} \right\}$$

and

$$\frac{\partial J}{\partial s}\frac{ds}{dt} = \sqrt{\frac{2}{m}\left(W - \mu B\right)} v_{\|} - v_{\|} \int_{s_1}^{s} \frac{1}{\sqrt{\frac{2}{m}\left(W - \mu B\right)}} \frac{\mu}{m}\frac{\partial B}{\partial s} ds$$

Now we want to evaluate

$$\frac{dJ\left(W, s_1, t\right)}{dt}$$

and we note that at $s = s_1$, $v_{\|} = 0$ since this is a turning point. Thus many of these terms vanish. We are left with

$$\begin{aligned}\frac{dJ(W,s_1,t)}{dt} &= -\int_{s_1}^{s} \frac{1}{\sqrt{\frac{2}{m}(W-\mu B)}} \frac{\mu}{m}\frac{\partial B}{\partial t} ds \\ &\quad -\int_{s_1}^{s} \frac{1}{\sqrt{\frac{2}{m}(W-\mu B)}} ds \frac{\mu}{m}\frac{\partial B}{\partial t} \\ &= -\int_0^{T/2} \frac{\mu}{m}\frac{\partial B}{\partial t} dt - \frac{\mu}{m}\frac{\partial B}{\partial t}(s_1)\int_0^{T/2} dt\end{aligned}$$

where we used the fact that $ds/v_\parallel = dt$, and T is the bounce period. Evaluating the integrals, we get

$$\frac{dJ(W,s_1,t)}{dt} = \frac{\mu}{m}\left[B(0) - B\left(\frac{T}{2}\right) - \frac{T}{2}\frac{\partial B}{\partial t}\left(\frac{T}{2}\right)\right]$$

Now we may expand B in a Taylor series, to get

$$B(0) = B\left(\frac{T}{2}\right) - \frac{T}{2}\frac{\partial B}{\partial t}\left(\frac{T}{2}\right) + \frac{1}{2}\left(\frac{T}{2}\right)^2 \frac{\partial^2 B}{\partial t^2} + \cdots$$

Thus

$$\frac{dJ(W,s_1,t)}{dt} = \frac{\mu}{m}\left[-T\frac{\partial B}{\partial t}\left(\frac{T}{2}\right) + \cdots\right]$$

and is of order

$$\frac{\mu B}{m}\left(\frac{T}{\tau}\right)$$

where τ is the time-scale for change in B. Thus J is invariant provided that $T \ll \tau$.

7.3 The third adiabatic invariant

There are three adiabatic invariants because the particle has three degrees of freedom. These are the integrals of the motion (*cf* your classical mechanics course). The third invariant is Φ, the flux enclosed by the orbit of the guiding center as it drifts. This is invariant when system changes are slow compared with the guiding center orbit period. Since this is quite long, this third invariant is much less useful than the first two.