Non-linear effects

S.M.Lea

January 2007

Now it is time to take a brief look at some phenomena that arise when instabilities lead to large amplitude fields and density perturbations.

1 The ponderomotive force

The ponderomotive force is akin to radiation pressure. It is a force on the plasma that arises from the presence of a large-amplitude wave. The electron equation of motion is

$$m\frac{d\vec{v}\,(\vec{r},t)}{dt} = -e\left[\vec{E}\,(\vec{r},t) + \vec{v}\,(\vec{r},t)\times\vec{B}\,(\vec{r},t)\right] \tag{1}$$

where the position of the electron is $\vec{r}\,(t)$. There are non-linear effects due to

1. the $\vec{v}\times\vec{B}$ term
2. the fact that we must use $\vec{E}\,(\vec{r},t)$, the field at the perturbed position of the electron, not $\vec{E}\,(\vec{r}_0)$.

Now let us assume that the electric field is due to a plasma wave and has the form

$$\vec{E}\,(\vec{r},t) = \vec{E}_s\,(\vec{r})\cos\omega t$$

where the function $\vec{E}_s\,(\vec{r})$ contains all the information about the spatial dependence of the electric field, and the field is oscillating in time as expected for a wave. We cannot use the exponential form $e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$ here because of the non-linearity. (At least, we cannot do it easily, because of all the products that occur. Extreme care is needed with taking the real part.) We use a "bootstrap" method to build the solution, as follows.

1st order

The first order response is the response due to the electric field at the unperturbed position:

$$m\frac{d\vec{v}_1\,(\vec{r},t)}{dt} = -e\vec{E}\,(\vec{r}_0) = -e\vec{E}_s\,(\vec{r}_0)\cos\omega t$$

(Note that the time average of this force is zero.) We can integrate this equation with respect to time to get the velocity

$$\vec{v}_1\,(\vec{r},t) - v_1\,(\vec{r},0) = -\frac{e}{m}\vec{E}_s\,(\vec{r}_0)\int_0^t \cos\omega t dt = -\frac{e}{m\omega}\vec{E}_s\,(\vec{r}_0)\sin\omega t \tag{2}$$

and integrate again to get the first order displacement. Take $\vec{v}_1(\vec{r},0) \equiv 0$. Then

$$\delta\vec{r}_1(\vec{r},t) = -\frac{e}{m}\vec{E}_s(\vec{r}_0)\int_0^t \sin\omega t dt = \frac{e}{m\omega^2}\vec{E}_s(\vec{r}_0)(\cos\omega t - 1) \tag{3}$$

Now we can use the dieplacement to calculate the electric field at the perturbed position

$$\vec{E}_s(\vec{r}) = \vec{E}_s(\vec{r}_0) + \left(\delta\vec{r}_1 \cdot \vec{\nabla}\right)\vec{E}_s\Big|_{\vec{r}_0} + \cdots \tag{4}$$

Next we use Faraday's Law to find the corresponding $\vec{B}$.

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Assume that $\vec{B}$ has the same form as $\vec{E}$

$$\vec{B}(\vec{r},t) = \vec{B}_s(\vec{r})\sin\omega t$$

(The reason for the sine rather than the cosine is made obvious in the line below.) Then, substituting into Faraday's law, we have

$$\vec{\nabla}\times\vec{E}_s(\vec{r})\cos\omega t = -\omega\vec{B}_s(\vec{r})\cos\omega t$$

and so the first order term is:

$$\vec{B}_1(\vec{r},t) = -\frac{1}{\omega}\vec{\nabla}\times\vec{E}_s(\vec{r}_0)\sin\omega t \tag{5}$$

We now have the complete set of first order quantities that we need, and we can use these to find the next order.

2nd order

We look for corrections to the first order terms. The second order terms in the equation of motion (1) are:

$$\begin{aligned} m\frac{d\vec{v}_2(\vec{r},t)}{dt} &= -e\left(\vec{E}_1 + \vec{v}_1 \times B_1\right) \\ &= -e\left[\left(\delta\vec{r}_1\cdot\vec{\nabla}\right)\vec{E}_s\Big|_{\vec{r}_0}\cos\omega t + \left(-\frac{e}{m\omega}\vec{E}_s(\vec{r}_0)\sin\omega t\right)\times\left(-\frac{1}{\omega}\vec{\nabla}\times\vec{E}_s(\vec{r}_0)\sin\omega t\right)\right] \\ &= -\frac{e^2}{m\omega^2}\left[\vec{E}_s(\vec{r}_0)(\cos\omega t - 1)\cdot\vec{\nabla}\right]\vec{E}_s(\vec{r}_0)\cos\omega t - \frac{e^2}{m\omega^2}\vec{E}_s(\vec{r}_0)\times\left(\vec{\nabla}\times\vec{E}_s(\vec{r}_0)\right)\sin^2\omega t \end{aligned}$$

Now we time average. The term in $\cos\omega t$ averages away, and the terms in $\sin^2$ and $\cos^2$ average to 1/2. Thus

$$m < \frac{d\vec{v}_2(\vec{r})}{dt} > = -\frac{e^2}{2m\omega^2}\left[\left(\vec{E}_s(\vec{r}_0)\cdot\vec{\nabla}\right)\vec{E}_s(\vec{r}_0) + \vec{E}_s(\vec{r}_0)\times\left(\vec{\nabla}\times\vec{E}_s(\vec{r}_0)\right)\right]$$

Expand the second term:

$$\begin{aligned} \varepsilon_{ijk}E_j\left(\vec{\nabla}\times\vec{E}_s(\vec{r}_0)\right)_k &= \varepsilon_{ijk}E_j\varepsilon_{klm}\partial_l E_m \\ &= \left(\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}\right)E_j\partial_l E_m \\ &= E_m\partial_i E_m - E_j\partial_j E_i \\ &= \frac{1}{2}\partial_i E^2 - \left(\vec{E}\cdot\vec{\nabla}\right)E_i \end{aligned}$$

Thus

$$
\begin{aligned}
m \quad < \quad \frac{d\vec{v}_2(\vec{r})}{dt} >&= -\frac{e^2}{2m\omega^2}\left[\left(\vec{E}_s(\vec{r}_0)\cdot\vec{\nabla}\right)\vec{E}_s(\vec{r}_0)+\frac{1}{2}\vec{\nabla}E_s^2-\left(\vec{E}\cdot\vec{\nabla}\right)\vec{E}_s(\vec{r}_0)\right] \\
&= -\frac{e^2}{4m\omega^2}\vec{\nabla}E_s^2
\end{aligned}
$$

Since the time average of the first-order force is zero, the time averaged non-linear force on the electron is

$$
\vec{F} = -\frac{e^2}{4m\omega^2}\vec{\nabla}E_s^2 \tag{6}
$$

The force on a unit volume of the plasma is then the force exerted on all the electrons in the plasma:

$$
\begin{aligned}
\vec{F}_{NL} &= -\frac{n_0 e^2}{4m\omega^2}\frac{\varepsilon_0}{\varepsilon_0}\vec{\nabla}E_s^2 = -\frac{\omega_p^2}{\omega^2}\vec{\nabla}\frac{\varepsilon_0 E_s^2}{4} \\
&= -\frac{\omega_p^2}{\omega^2}\vec{\nabla}\frac{\varepsilon_0 < E^2 >}{2} = -\frac{\omega_p^2}{\omega^2}\vec{\nabla} < u_E >
\end{aligned} \tag{7}
$$

where

$$
u_E = \frac{\varepsilon_0 E^2}{2}
$$

is the electric energy density and $<>$ denotes the time average.

We may now apply this result to some of the waves we have studied.

Electrostatic waves

The electric field accelerates the electron, which moves farther in the half cycle in which the field is decreasing than it does in the half cycle with increasing field. There is a net drift. Since $\vec{E}$ is along $\vec{k}$, so is the force and the drift.

Electromagnetic waves

The wave magnetic field pushes electrons in the direction of $\vec{k}$ ($\vec{v}$ is parallel to $\vec{E}$ and $\vec{E}\times\vec{B}$ is parallel to $\vec{k}$). So there is a drift along $\vec{k}$.

The ponderomotive force acts primarily on electrons (notice the mass in the denominator of equation (6). The resulting charge separation generates additional electric fields that transmit the force to the ions. This field also acts on the electrons, of course. The net result is to slow the electrons and speed the ions. (We saw a similar effect in ambipolar diffusion.) The net force is given by equation (7).

2 Parametric instabilities

These instabilities are also known as wave-wave interactions. The ponderomtive force plays an important role in these instabilities.

From a QM point of view, we can regard the plasma waves as particles– plasmons– with energy $\hbar\omega$ and momentum $\hbar\vec{k}$. When one wave (called the pump wave) interacts with a plasma, it can generate two "daughter" waves, provided that energy and momentum are conserved. That is, we need

$$
\omega_0 = \omega_1 + \omega_2
$$

and

$$\vec{k}_0 = \vec{k}_1 + \vec{k}_2$$

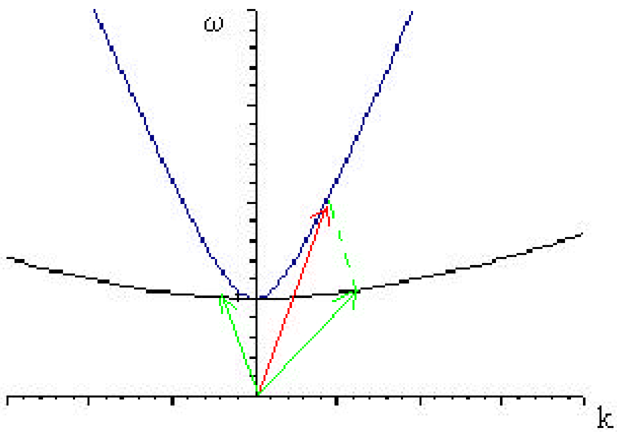
We can see how this works for Langmuir waves and EM waves by looking at the ω versus k plot. Remember

$$\omega_0^2 = \omega_p^2 + k^2 c^2$$

for EM waves (blue curve) and

$$\omega_{1,2}^2 = \omega_p^2 + k_{1,2}^2 v_{\text{th}}^2$$

for Langmuir waves (black curve). The plot shows an EM wave ($\vec{k}_0$, red arrow) with two Langmuir wave daughters (green arrows).



Mechanism of the instability:

Suppose we have a pump wave with electric field amplitude $\vec{E}_0$ and a perturbation with wave number k_1 forms in the plasma, giving rise to $\vec{E}_1$. Then there will be a force on the plasma

$$\vec{F}_{NL} = -\frac{\omega_p^2}{\omega^2}\vec{\nabla}\frac{\varepsilon_0}{4}\left(\vec{E}_0 + \vec{E}_1\right)^2$$

If the pump wave has wavelength long compared with the perturbation ($k_0 \ll k_1$), and $E_1 \ll E_0$ initially, then

$$\begin{aligned}
\vec{F}_{NL} &\simeq -\frac{\omega_p^2}{\omega^2}\vec{\nabla}\frac{\varepsilon_0}{4}\left(2\vec{E}_0 \cdot \vec{E}_1\right) \\
&= -\frac{\omega_p^2}{2\omega^2}\varepsilon_0\vec{\nabla}\left[\vec{E}_{0a} \cdot \vec{E}_{1a}\cos\left(\vec{k}_0 \cdot \vec{x} - \omega_0 t\right)\cos\left(\vec{k}_1 \cdot \vec{x} - \omega_1 t\right)\right] \\
&= -\frac{\omega_p^2}{4\omega^2}\varepsilon_0\vec{\nabla}\left[\vec{E}_{0a} \cdot \vec{E}_{1a}\left\{\cos\left[\left(\vec{k}_0 + \vec{k}_1\right) \cdot \vec{x} - (\omega_0 + \omega_1)\, t\right] + \cos\left[\left(\vec{k}_0 - \vec{k}_1\right) \cdot x - (\omega_0 - \omega_1)\, t\right]\right\}\right]
\end{aligned}$$

Thus this force drives disturbances at the two frequencies

$$\omega = \omega_0 \pm \omega_1$$

and also

$$\vec{k} = \vec{k}_0 \pm \vec{k}_1$$

This is the frequency matching condition described above. We also need

$$\vec{E}_{0a} \cdot \vec{E}_{1a} \neq 0$$

So if the pump wave is an EM wave and the daughters are Langmuir waves, the Langmuir waves cannot propagate in the same direction as the pump.

If there is no damping, a pump wave of any amplitude can drive a parametric instability. But usually there is some form of damping, and then there is a threshold amplitude below which the instability does not go. If Γ_1 and Γ_2 are the damping rates for the two daughters, then we find

$$E_0^2 > C\omega_1\omega_2\Gamma_1\Gamma_2$$

where C is a constant. (See Chen sec 8.5.3 for the details.)

3 Shock waves and solitons

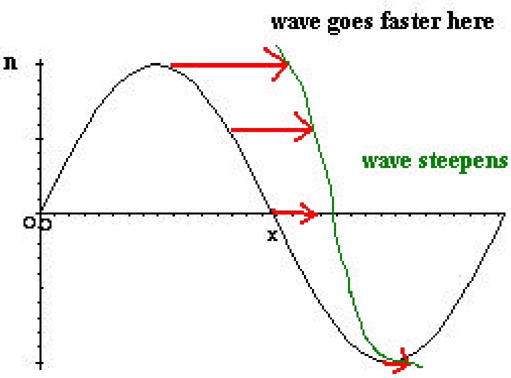
These are intrinsically non-linear phenomena. When a wave grows to large amplitude, different parts of the wave travel with different speeds. This leads to steepening of the wave form. In a soliton, the wave has a profile that moves without change of shape as the non-linear effects of steepening and dispersion balance each other. To see how this can work, look at an ion sound wave with dispersion relation (waves notes page 10)

$$\frac{\omega^2}{k^2} = \frac{v_s^2}{1 + k^2\lambda_D^2}$$

Since

$$\lambda_D^2 = \frac{v_s^2}{{}_p^2} \propto \frac{1}{n}$$

the phase speed $v_\phi = \omega/k$ increases with n. Thus a wave profile that is initially sinusoidal will steepen.



Clearly the effect is negligible unless $k\lambda_D$ is not much less than one. Thus shock waves and solitons have structure on the order of λ_D.

Now for the details. The ion-sound wave is a longitudinal wave, so everything is one-dimensional. We choose x as a coordinate measured along the direction of wave propagation. From energy conservation, if the ion speed is u_0 at a point where the potential is zero, then

$$\frac{1}{2}Mu_0^2 = \frac{1}{2}Mu^2 + e\phi$$

and thus

$$u = \sqrt{u_0^2 - 2\frac{e\phi}{M}}$$

From the continuity equation for the ions,

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{u}) = 0$$

in a steady state ($\partial/\partial t \equiv 0$), nu is constant:

$$n_i = n_0\frac{u_0}{u} = n_0\left(1 - 2\frac{e\phi}{Mu_0^2}\right)^{-1/2}$$

The electron density is given by the Boltzmann relation

$$n_e = n_0 \exp\left(\frac{e\phi}{kT_e}\right)$$

Then Poisson's equation gives

$$\frac{d^2\phi}{dx^2} = \frac{e}{\varepsilon_0}(n_e - n_i) = \frac{en_0}{\varepsilon_0}\left[\exp\left(\frac{e\phi}{kT_e}\right) - \left(1 - 2\frac{e\phi}{Mu_0^2}\right)^{-1/2}\right] \tag{8}$$

To simplify this, let's introduce the dimensionless variables

$$\chi = \frac{e\phi}{kT_e} = \frac{e}{Mv_s^2}\phi \tag{9}$$

and, since we expect structure on a scale λ_D,

$$\xi = \frac{x}{\lambda_D} = x\frac{{}_p}{v_s} = \frac{x}{v_s}\sqrt{\frac{ne^2}{\varepsilon_0 M}} \tag{10}$$

Finally, we introduce the Mach number

$$\mathcal{M} = \frac{u_0}{v_s} = \frac{u_0\sqrt{M}}{\sqrt{kT_e}} \tag{11}$$

Then equation (8) becomes:

$$\begin{aligned} \frac{Mv_s^2}{e}\frac{d^2\chi}{d\xi^2}\frac{{}^2_p}{v_s^2} &= \frac{en_0}{\varepsilon_0}\left[\exp\chi - \left(1 - 2\frac{\chi kT_e}{Mu_0^2}\right)^{-1/2}\right] \\ \frac{d^2\chi}{d\xi^2} &= e^{\chi} - \left(1 - 2\frac{\chi}{\mathcal{M}^2}\right)^{-1/2} \end{aligned} \tag{12}$$

Now if we define the pseudo-potential function

$$V(\chi) = 1 - e^{\chi} + \mathcal{M}^2\left(1 - \sqrt{1 - 2\frac{\chi}{\mathcal{M}^2}}\right)$$

which has the property that

$$V(0) = 0$$

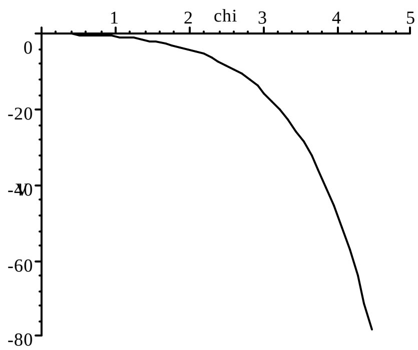
then

$$\frac{dV}{d\chi} = -e^{\chi} + \mathcal{M}^2\left(-\frac{1}{2}\left(-\frac{2}{\mathcal{M}^2}\right)\left(1 - 2\frac{\chi}{\mathcal{M}^2}\right)^{-1/2}\right) = -e^{\chi} + \left(1 - 2\frac{\chi}{\mathcal{M}^2}\right)^{-1/2}$$

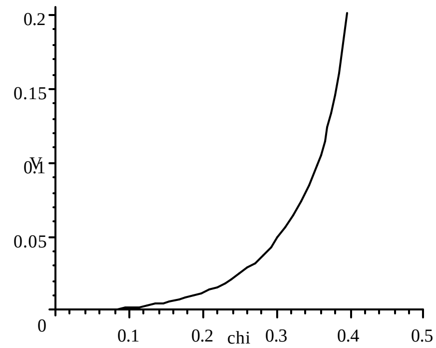
and thus Possion's equation (12) takes the form:

$$\frac{d^2\chi}{d\xi^2} = -\frac{dV(\chi)}{d\chi} \tag{13}$$

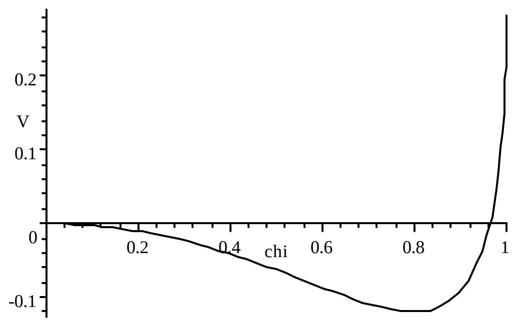
We can solve this by analogy to the problem of a particle moving in a potential well. The potential χ is like a coordinate x and the spatial coordinate ξ is like t. Let $y = d\chi/d\xi$. (This is the electric field and acts like the velocity in our comparison problem.) The diagram below shows what the potential well looks like for three values of the Mach number. The vertical axis is V and the horizontal axis is χ.



$\mathcal{M} = 3$

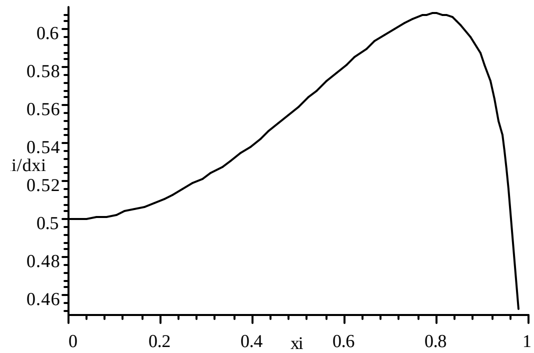


$\mathcal{M} = .9$



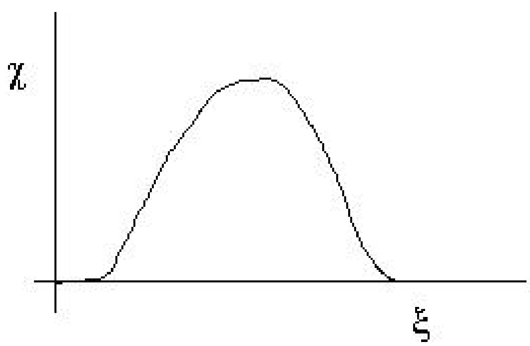
$\mathcal{M} = \sqrt{2}$

Looking at the bottom plot for a moment, a particle entering at $\chi = 0$ will find its velocity increasing for a while, and then decreasing again as it reaches the wall at the far side of the well. Its "velocity" will become zero, so it will reflect and return to the origin, repeating its motion (but in reverse). The other plot shows that if the Mach number is too large or too small, the "potential" does not have a well, and we do not get this behavior.



"velocity" versus ξ

In our case this "velocity" is $d\chi/d\xi$, so we may "integrate" to get the resulting χ. $d\chi/d\xi$ is zero (zero slope) at $\xi = 0$, then positive, then zero, then negative as the particle returns to the origin.. We get a graph like this:



If the particle loses speed for some reason, it can get trapped in the well. This leads to an oscillating velocity, and thus an oscillating χ.

Multiplying both sides of equation (13) by y, we get:

$$\begin{aligned} y\frac{dy}{d\xi} &= -\frac{dV(\chi)}{d\chi}\frac{d\chi}{d\xi} \\ \frac{d}{d\xi}\left(\frac{1}{2}y^2\right) &= -\frac{dV}{d\xi} \end{aligned}$$

which we can integrate immediately to get

$$y = \frac{d\chi}{d\xi} = \pm\sqrt{-2V} = \pm\sqrt{-2\left[1 - e^{\chi} + \mathcal{M}^2\left(1 - \sqrt{1 - 2\frac{\chi}{\mathcal{M}^2}}\right)\right]} \qquad (14)$$

This is not easily integrable, so now we approximate.

Let χ be small (but not *very* small), so we expand the functions in equation (12) to second order:

$$\frac{d^2\chi}{d\xi^2} = 1+\chi+\frac{\chi^2}{2}+\cdots-\left(1+\frac{\chi}{\mathcal{M}^2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(\frac{2\chi}{\mathcal{M}^2}\right)^2+\cdots\right)$$
$$= \chi\left(1-\frac{1}{\mathcal{M}^2}\right)+\frac{\chi^2}{2}\left(1-\frac{3}{\mathcal{M}^4}\right)+\cdots \tag{15}$$

Now with 20-20 hindsight, we let

$$\chi = \chi_m\left[\cosh\left(\frac{\xi}{\Delta}\right)\right]^{-l} \tag{16}$$

Then

$$\chi' = \frac{-l}{\Delta}\chi_m \sinh\frac{\xi}{\Delta}\left[\cosh\left(\frac{\xi}{\Delta}\right)\right]^{-(l+1)}$$

and

$$\chi'' = \frac{-l}{\Delta}\chi_m\left\{\frac{1}{\Delta}\left[\cosh\left(\frac{\xi}{\Delta}\right)\right]^{-l}-\frac{(l+1)}{\Delta}\sinh^2\frac{\xi}{\Delta}\left[\cosh\left(\frac{\xi}{\Delta}\right)\right]^{-(l+2)}\right\}$$
$$= \frac{-l}{\Delta^2}\chi_m\left[\cosh\left(\frac{\xi}{\Delta}\right)\right]^{-l}\left\{1-(l+1)\left(1-\frac{1}{\cosh^2\xi/\Delta}\right)\right\}$$
$$= \frac{-l}{\Delta^2}\chi\left(-l+\frac{l+1}{\cosh^2\xi/\Delta}\right)=\frac{l^2}{\Delta^2}\chi-\frac{l\,(l+1)}{\Delta^2}\chi\left(\frac{\chi}{\chi_m}\right)^{2/l} \tag{17}$$

Now we compare this result (17) with equation (15) and ask what it will take to make them the same.

1. To get the coefficient of χ right we must have

$$\frac{l^2}{\Delta^2} = 1-\frac{1}{\mathcal{M}^2} \tag{18}$$

Since the left hand side is a square, and thus positive, it is necessary that $\mathcal{M} > 1$.

2. To make the second term be a square of χ we need

$$l = 2 \tag{19}$$

3. To get the right coefficient of χ^2 we need

$$-\frac{l\,(l+1)}{\Delta^2\chi_m} = \frac{1}{2}\left(1-\frac{3}{\mathcal{M}^4}\right) \tag{20}$$

Soving for χ_m, using (18) and (19), we get

$$\chi_m = \frac{-2l\,(l+1)}{\Delta^2\left(1-\frac{3}{\mathcal{M}^4}\right)} = -\left(1-\frac{1}{\mathcal{M}^2}\right)\frac{3}{\left(1-\frac{3}{\mathcal{M}^4}\right)}$$
$$= -3\frac{\left(\mathcal{M}^2-1\right)\mathcal{M}^2}{\left(\mathcal{M}^4-3\right)} \tag{21}$$

For χ_m to be positive, we must have$1 < \mathcal{M} < 3^{1/4} = 1.3$. (A more exact treatment without the approximations we have made gives $\mathcal{M}_{\text{max}} = 1.6$).

Finally, from (18) and (19) we obtain

$$\Delta = \frac{2}{\sqrt{1 - 1/\mathcal{M}^2}} = \frac{2\mathcal{M}}{\sqrt{\mathcal{M}^2 - 1}} \tag{22}$$

Thus the solution we have found is (16, 21, 22 and 19)

$$\chi = 3\frac{\left(\mathcal{M}^2 - 1\right)\mathcal{M}^2}{(3 - \mathcal{M}^4)\cosh^2\left[\frac{\xi}{2\mathcal{M}}\sqrt{\mathcal{M}^2 - 1}\right]} \tag{23}$$

The amplitude is < 1, justifying our approximations, if

$$\begin{aligned} 3\mathcal{M}^4 - 3\mathcal{M}^2 &< 3 - \mathcal{M}^4 \\ 4\mathcal{M}^4 - 3\mathcal{M}^2 - 3 &< 0 \\ \left(\mathcal{M}^2 - \frac{3 + \sqrt{57}}{8}\right)\left(\mathcal{M}^2 - \frac{3 - \sqrt{57}}{8}\right) &< 0 \end{aligned}$$

which means

$$\begin{aligned} \frac{3 - \sqrt{57}}{8} &< \mathcal{M}^2 < \frac{3 + \sqrt{57}}{8} \\ -.56873 &< \mathcal{M}^2 < 1.3187 \end{aligned}$$

and thus

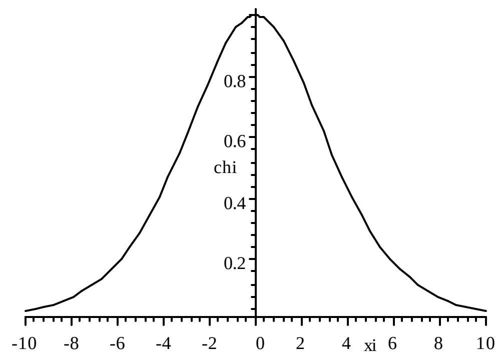
$$\mathcal{M} < \sqrt{1.3187} = 1.1483$$

With $\mathcal{M} = 1.125$, the solution looks like:

$$\begin{aligned} \chi &= 3\frac{(1.3187 - 1)\,1.3187}{\left(3 - (1.3187)^2\right)\cosh^2\left[\frac{\xi}{2\sqrt{1.3187}}\sqrt{1.3187 - 1}\right]} \\ &= \frac{.99982}{\cosh^2 .2458\xi} \end{aligned}$$

Or, In the original variables,

$$\frac{e\phi}{kT_e} = \frac{.99982}{\cosh^2 .2458x/\lambda_D}$$

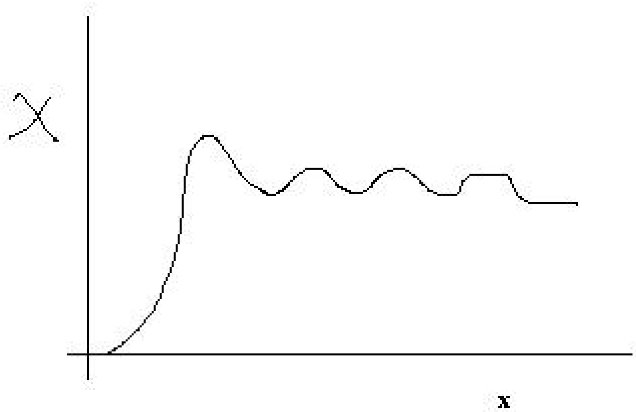


This is a soliton profile. The width is of order $10\lambda_D$. The pulse gets narrower as $\mathcal{M}$ increases.

Remembering that the "velocity" in our analogy is essentially the electric field, we can determine it from Poisson's equation (8 or 12). Integrating once,

$$\frac{d\chi}{d\xi} = \frac{\lambda_D^2}{n_0} \int_0^\xi (n_e - n_i)\, d\xi$$

Thus the "velocity" decreases if the density difference decreases. This may occur due to reflection of ions from the front of the shock, leading to an increased ion density upstream. If the ions are not completely cold, some of them will not have enough kinetic energy to get over the initial potential hill. Then the picture looks like this:



The thickness of the transition region is again a few λ_D. This an ion shock wave.

4 Plasma sheaths

A sheath is basically a boundary layer between the plasma proper and, for example, a container wall. With no magnetic field, the electrons diffuse to the wall faster than the ions, leaving a net positive charge in the plasma. An electric field is set up by the resulting charge distribution, and this field slows the electron motion. Choosing the potential in the bulk plasma to be zero, the potential at the wall will become negative, forming a potential barrier that repels electrons.

To analyze the sheath, we assume that the ions are cold ($T_i \equiv 0$). Ions drift toward the wall with speed u_0 at $x = 0$. Neglect collisions. (This is OK. The electrons still reach the wall faster because their thermal speed is greater.) Then from energy conservation, the Boltzmann relation, the continuity equation and Poisson's equation, we retrieve the same set of equations that we had for solitons. In particular, from (14), we have

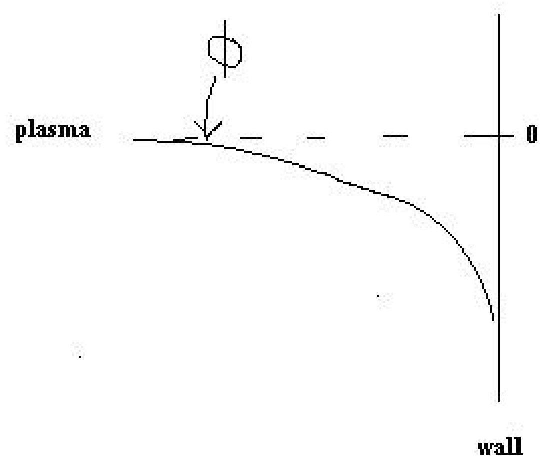
$$\frac{1}{2}(\chi')^2 = -V = -\left[1 - e^{\chi} + \mathcal{M}^2\left(1 - \sqrt{1 - 2\frac{\chi}{\mathcal{M}^2}}\right)\right]$$

Again we chose the constant to make $V = 0$ and thus $E = 0$ in the bulk of the plasma, (at $\xi = 0$).

The right hand side must be positive since the left is a square, and, as we found before, this means that $\mathcal{M} > 1$. This is the Bohm sheath criterion. (For small χ, the RHS is

$$-1 + \left(1 + \chi + \frac{\chi^2}{2}\right) - \mathcal{M}^2\left(1 - 1 + \frac{\chi}{\mathcal{M}^2} - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(2\frac{\chi}{\mathcal{M}^2}\right)^2\right) = \frac{\chi^2}{2}\left(1 - \frac{1}{\mathcal{M}^2}\right)$$

) The ions must be accelerated by the electric field before entering the sheath region. This means that E cannot be zero at $x = 0$, as we have assumed, but it can be very small. The picture now looks like this:



The potential slope is negative and the curvature is also negative. This means that $n_i > n_e$

throughout this region.

Since χ is negative, it is convenient to introduce a new variable $\Psi = -\chi$. (My Ψ =Chen's χ in section 8.2) Then Poisson's equation (12) becomes:

$$\frac{d^2\Psi}{d\xi^2} = -e^{-\Psi} + \left(1 + 2\frac{\Psi}{\mathcal{M}^2}\right)^{-1/2}$$

When $\Psi \gg 1$ (close to the wall), this equation for Ψ becomes approximately

$$\Psi'' \simeq \left(1 + 2\frac{\Psi}{\mathcal{M}^2}\right)^{-1/2} \simeq \frac{\mathcal{M}}{\sqrt{2\Psi}}$$

Multiply both sides by Ψ' and integrate to get

$$\frac{1}{2}\left(\Psi'\right)^2 = \mathcal{M}\sqrt{2\Psi} + \text{ constant}$$

Now it is convenient to redefine the zero level for potential so that $\Psi = 0$ at some value ξ_s of ξ where Ψ' is very small (essentially zero). Then we may take the constant to be zero. Taking the square root

$$\Psi' = 2^{3/4}\mathcal{M}^{1/2}\Psi^{1/4}$$

and integrating again, we get

$$\frac{4}{3}\Psi^{3/4} = 2^{3/4}\mathcal{M}^{1/2}\xi + \text{ constant}$$

Putting in the boundary conditions, $\Psi = 0$ at ξ_s

$$\frac{4}{3}\Psi^{3/4} = 2^{3/4}\mathcal{M}^{1/2}\left(\xi - \xi_s\right)$$

We may now rewrite all this in terms of the physical variables ϕ, x, v_s etc.

$$\frac{4}{3}\left(\frac{-e\phi}{kT_e}\right)^{3/4} = 2^{3/4}\mathcal{M}^{1/2}\left(\frac{x - x_s}{\lambda_D}\right) \tag{24}$$

We are interested in the current toward the wall in terms of the potential at the wall, where the ion current is

$$j = n_0 e u_0 = n_0 e \mathcal{M} v_s$$

Note that $x_{\text{wall}} \simeq x_s + d$ where d is the thickness of the sheath. We find $\mathcal{M}$ from (24),

$$\begin{aligned}
\mathcal{M}^{1/2} &= \frac{2^2}{3 \times 2^{3/4}}\left(\frac{-e\phi}{kT_e}\right)^{3/4}\frac{\lambda_D}{d} \\
\mathcal{M} &= \frac{2^{5/2}}{9}\left(-e\phi_w\right)^{3/2}\frac{v_s^2}{{}^2_p\left(kT_e\right)^{3/2}d^2} \\
&= \frac{4\sqrt{2}}{9}\left(-e\phi_w\right)^{3/2}\frac{\varepsilon_0 M}{n_0 e^2\left(kT_e\right)^{1/2}M d^2} \\
&= \frac{4\sqrt{2}}{9}\left(-e\phi_w\right)^{3/2}\frac{\varepsilon_0}{n_0 e^2 v_s\sqrt{M}d^2}
\end{aligned}$$

and thus

$$\begin{aligned} j &= n_0 e \frac{4\sqrt{2}}{9} \left(-e\phi_w\right)^{3/2} \frac{\varepsilon_0}{n_0 e^2 v_s \sqrt{M} d^2} v_s \\ &= \frac{4}{9}\sqrt{\frac{2e}{M}}\varepsilon_0 \frac{\left(-\phi_w\right)^{3/2}}{d^2} \end{aligned} \tag{25}$$

This is the Child-Langmuir Law. Notice that the current is independent of the plasma density n_0. The sheath thickness d is determined by equation (25) once the current and the potential are measured.