Plasmas as fluids

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So far we have considered a plasma as a set of non-intereacting particles, each following its own path in the electric and magnetic fields. Now we want to consider what other views of a plasma might be useful.

1 Plasma as a material medium

1.1 Permeability

In a magnetic medium, the individual particles of the medium (atoms, molecules etc) contribute magnetic moments $\vec{m}$ and the integral over all the individual magnetic moments gives rise to a magnetization $\vec{M}$. This can be viewed as arising from a "bound current" $\vec{j}_B$ where

$$\vec{\nabla} \times \vec{M} = \vec{j}_B$$

Ampere's law takes the form

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} = \mu_0 \left(\vec{j}_f + \vec{j}_B \right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

or equivalently

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \mu_0 \left(\vec{H} + \vec{M} \right) = \mu_0 \left(\vec{j}_f + \vec{j}_B \right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

or

$$\vec{\nabla} \times \vec{H} = \vec{j}_f + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

In a LIH material, we assume that $\vec{M} = \chi_m \vec{H}$ so that

$$\vec{B} = \vec{H} \left(1 + \chi_m \right) = \mu \vec{H}$$

But in a plasma, the magetization is

$$\vec{M} = \sum_i \vec{\mu}_i$$

where μ is the magetic moment of a gyrating particle,

$$\mu = \frac{1}{2} \frac{m v_\perp^2}{B}$$

and thus we would have

$$M \propto \frac{1}{B}$$

Thus both χ_m and μ_m would have a nasty dependence on B. Thus the plasma is not a linear material (nor is it isotropic) and thus it not useful to view a plasma as a magnetic material.

1.2 Dielectric properties

In general the relation between $\vec{D}$ and $\vec{E}$ for a material is a frequency dependent relation. The material's response depends on the time-dependence of the applied electric field. This is true for plasmas too,

First we'll look at the "almost static limit" ($\omega \ll \omega_c$). The polarization is the sum of all the electric dipoles in a medium:

$$\vec{P} = \sum_i \vec{p}_i$$

and the polarization gives rise to a "bound charge density"

$$\rho_B = -\vec{\nabla} \cdot \vec{P}$$

Gauss's law is

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_f + \rho_B}{\varepsilon_0}$$

Now let

$$\vec{D} = \varepsilon_0 \left(\vec{E} + \vec{P} \right)$$

so that

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot \varepsilon_0 \left(\vec{E} + \vec{P} \right) = \rho_f$$

Now again in an LIH material, we expect

$$\vec{P} = \chi_e \vec{E}$$

so that

$$\vec{D} = \varepsilon_0 \left(1 + \chi_e\right) \vec{E} = \varepsilon \vec{E}$$

In a plasma we have seen that a time-dependent applied field (with $\omega \ll \omega_c$) gives rise to a polarization current: ("Motion" notes equation 17)

$$\vec{j}_p = \frac{\rho}{B^2} \frac{\partial \vec{E}}{\partial t}$$

which will affect $\vec{B}$ through Ampere's law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{j}_f + \frac{\rho}{B^2} \frac{\partial \vec{E}}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Thus this bound current can be combined with the displacement current to give:

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \mu_0 \vec{j}_f + \mu_0 \left(\frac{\rho}{B^2} \frac{\partial \vec{E}}{\partial t} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \\ &= \mu_0 \left(\vec{j}_f + \varepsilon \frac{\partial \vec{E}}{\partial t} \right) \end{aligned}$$

with dielectric constant

$$\varepsilon = \varepsilon_0 + \frac{\rho}{B^2} \tag{1}$$

Since ε is independent of $\vec{E}$, it appears that this could be a useful description of a plasma.

Convince yourself that charge conservation allows us to write ρ_B in terms of the polarization current, and hence we also get the correct form for Gauss' law as well.

When we study plasma waves we will be able to find additional expressions for ε valid in different frequency regimes.

2 The fluid picture

When collisions between the plasma particles become important, we can no longer regard them as independent particles, but instead we find it useful to look at small volumes containing many particles and consider average properties of the particles in each small volume– this is the *fluid* point of view. When dealing with a plasma, we must recognize that particles can influence each other through the long-range electromagnetic forces in the system, and so we must regard the term "collision" rather generally.

The fluid properties of interest are: the density ρ, pressure P, and velocity $\vec{v}$. The equations governing the motion of a fluid are:

1. **Conservation of mass**

The mass within an arbitrary, fixed volume of the plasma can change only by flow of plasma into or out of that volume:

$$\frac{d}{dt}\int_V \rho dV = -\int_S \rho\vec{v}\cdot\hat{\mathbf{n}}\, dA = -\int_V \vec{\nabla}\cdot(\rho\vec{v})\ dV$$

and since this must be true for an arbitrary volume V, we have the differential equation:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{2}$$

Equation (2) is called the continuity equation.

2**Momentum equation**

This equation is just Newton's second law applied to the plasma. The relevant forces are the Lorentz force, and pressure forces. (We can add gravity when appropriate.) The acceleration of a mass element $dm = \rho\ dV$ is given by

$$dm\frac{d\vec{v}}{dt} = dq\left(\vec{E} + \vec{v}\times\vec{B}\right) - \int_S P\hat{\mathbf{n}}\, dA$$

where the last term is an integral over the differential volume, and represents the effect of neighboring fluid on our element. (Recall that pressure is the normal force per unit area.) The charge $dq = nedV$. Thus, applying the divergence theorem to the last term, we have

$$\rho\frac{d\vec{v}}{dt}dV = ne\left(\vec{E} + \vec{v}\times\vec{B}\right)dV - \int \vec{\nabla}P\ dV$$

and so we obtain the differential equation:

$$\rho\frac{d\vec{v}}{dt} = ne\left(\vec{E} + \vec{v} \times \vec{B}\right) - \vec{\nabla}P \tag{3}$$

Because the charge of the particles appears in this equation, we will need one equation for the ions and a different equation for the electrons.

3**Energy equation**

The equations involve the five variables ρ, P and $\vec{v}$, but with one vector and one scalar equation (for a total of 4) we are one short of a complete set. The third equation we need is an energy equation. Often we can avoid a full energy equation by using an equation of state, that is, a known relation between the pressure P and the density ρ. Two common choices are:

The ideal gas law with a constant temperature T (isothermal plasma)

$$P = nkT \tag{4}$$

or the adiabatic equation of state

$$P \propto \rho^{\gamma} \tag{5}$$

where $\gamma = (2 + N)/N$ and N is the number of degrees of freedom of our system. For an unmagnetized gas of ions and electrons, $N = 3$ and $\gamma = 5/3$.

2.1 Formal derivation of the fluid equations

2.1.1 More about the distribution function

An important link in the jump from considering individual particles to viewing the plasma as a fluid is the distribution function $f(\vec{r}, \vec{v}, t)$. It tells us how many particles are where, going how fast, in which direction.

number of particles with position vectors $\vec{r}$ to $\vec{r}+d\vec{r}$ and velocities in range $\vec{v}$ to $\vec{v}+d\vec{v}$ is $f(\vec{r}, \vec{v}, t)\ d^3\vec{r}d^3\vec{v}$

For many purposes the distribution function gives more information than we need. Thus we obtain desired quantities as averages. For example, we may obtain the density by summing up over all possible velocities:

$$n(\vec{r}, t) = \int f(\vec{r}, \vec{v}, t)\, d^3\vec{v} \tag{6}$$

where the integral is over the complete velocity space, and the average velocity of the particles at position $\vec{r}$ is

$$\vec{u}(\vec{r}, t) = \frac{\int \vec{v} f(\vec{r}, \vec{v}, t)\ d^3\vec{v}}{\int f(\vec{r}, \vec{v}, t)\ d^3\vec{v}} = \frac{1}{n}\int \vec{v} f(\vec{r}, \vec{v}, t)\ d^3\vec{v} \tag{7}$$

Sometimes the distribution function is normalized by dividing out the density:

$$\hat{f}(\vec{r}, \vec{v}) = \frac{1}{n} f(\vec{r}, \vec{v})$$

so that

$$\int \hat{f}(\vec{r}, \vec{v})\ d^3\vec{v} = 1$$

We have already seen one example of a distribution function: the Maxwellian:

$$f(\vec{r}, \vec{v}) = n(\vec{r}) \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) \tag{8}$$

where here the dependence on the space and velocity variables is separable. This is an isotropic velocity distribution, since f does not depend on the direction of the vector $\vec{v}$ but only on its length. Particles in a magnetic mirror may have a loss-cone distribution, which has a deficiency of particles with velocity vectors pointing along the direction of $\vec{B}$. This is an anisotropic distribution. (See Figure 7.7, pg 232). Additional examples are shown on pages 230-232 of the text.

2.1.2 The Boltzmann and Vlasov equations

The Boltzmann equation is a mathematical statement of the fact that particles cannot disappear. The number of particles in a region of space can change only if they move somewhere else. The number of particles in a given region of velocity space can change only if (*a*) they move to a new velocity (they are accelerated) or (*b*) a collision knocks them into a new region of velocity space. This physical principle is stated mathematically as

$$\frac{\partial f}{\partial t} = -\vec{v} \cdot \vec{\nabla} f - \vec{a} \cdot \frac{\partial}{\partial \vec{v}} f + \left(\frac{df}{dt}\right)_{\text{due to collisions}}$$

where

$$\vec{v} \cdot \vec{\nabla} f = \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z}$$

and similarly for $\vec{a} \cdot \frac{\partial}{\partial \vec{v}} f$, or equivalently

$$\frac{df}{dt} = \left(\frac{df}{dt}\right)_{\text{due to collisions}}$$

where the total time derivative is

$$\frac{df(\vec{r}, \vec{v}, t)}{dt} = \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \vec{a} \cdot \frac{\partial}{\partial \vec{v}} f$$

and we have written the shorthand expression

$$\frac{\partial}{\partial \vec{v}} \equiv \vec{\nabla}_v$$

as the gradient in the velocity space. Thus Boltzmann's equation says that the distribution function can be changed only by collisions.

Next we replace $\vec{a}$ with the force acting:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{\vec{F}}{m} \cdot \frac{\partial}{\partial \vec{v}} f = \left(\frac{df}{dt}\right)_{\text{due to collisions}}$$

In a plasma, the relevant force is the Lorentz force:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{q}{m}\left(\vec{E} + \vec{v} \times B\right) \cdot \frac{\partial}{\partial \vec{v}} f = \left(\frac{df}{dt}\right)_{\text{due to collisions}} \tag{9}$$

and in this form, equation (9) is called the Vlasov equation. Setting the right side to zero, we obtain the *collisionless* Vlasov equation. This equation is very powerful in predicting the plasma behavior – it contains a lot of information, and we will look at some of its conseqeunces later. But often we don't need that much power, so we average over the velocity distribution.

2.1.3 The zeroth moment

To obtain the nth moment, we multiply equation (9) by $\vec{v}^n$ and integrate over the velocity space. We start with $n = 0$:

$$\int \frac{\partial f}{\partial t} \, dV + \int \vec{v} \cdot \vec{\nabla} f \, dV + \frac{q}{m} \int \left(\vec{E} + \vec{v} \times B\right) \cdot \frac{\partial}{\partial \vec{v}} f \, dV = \int \left(\frac{df}{dt}\right)_{\text{due to collisions}} dV$$

Immediately we see that the right side is zero, since every particle knocked out of a volume element dV by a collision must be knocked into another, and both dVs are in the integrated volume.

On the left side, the first term is:

$$\frac{\partial}{\partial t} \int f \, dV = \frac{\partial n}{\partial t}$$

where we used equation (6). To evaluate the second term, note that $\vec{r}$ and $\vec{v}$ are independent variables. We are integrating over $\vec{v}$ but differentiating with respect to $\vec{r}$. Thus, using equation (7):

$$\int \vec{v} \cdot \vec{\nabla} f \, dV = \int \left[\vec{\nabla} \cdot (\vec{v} f) - f \vec{\nabla} \cdot \vec{v}\right] \, dV = \vec{\nabla} \cdot \int \vec{v} f \, dV - 0 = \vec{\nabla} \cdot (n \vec{u})$$

To evaluate the third term we use the divergence theorem in the velocity space,

$$\int \vec{E} \cdot \frac{\partial}{\partial \vec{v}} f \, dV = \vec{E} \cdot \int \frac{\partial}{\partial \vec{v}} f \, dV = \vec{E} \cdot \int_{S_\infty} \hat{n}_v f \, dA_v$$

The surface integral is at infinity in the velocity space, and $f \to 0$ there. In fact f *must* go to zero at least as fast as $1/v^4$ to ensure that the total plasma energy is finite. ($\int v^2 f v^2 dvd \quad _v$ is finite.) Thus the third term is zero.

The fourth term is:

$$\begin{aligned} \int (\vec{v} \times B) \cdot \frac{\partial}{\partial \vec{v}} f \, dV &= \int \left\{ \frac{\partial}{\partial \vec{v}} \cdot \left[(\vec{v} \times B) \, f\right] - f \frac{\partial}{\partial \vec{v}} \cdot (\vec{v} \times B) \right\} \, dV \\ &= \int f \, (\vec{v} \times B) \cdot \hat{\mathbf{n}} \, dA_v \end{aligned}$$

$\frac{\partial}{\partial \vec{v}} \cdot (\vec{v} \times B)$ is identically zero, since the first component of $\vec{v} \times \vec{B}$ contains only v_y and v_z, and not v_x, and similarly for the other terms. The surface integral is zero because $f \to 0$

sufficiently fast at infinity. Thus we have

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{u}) = 0 \tag{10}$$

and we have derived equation (2).

2.1.4 The first moment.

Now we multiply the Vlasov equation (9) by $m\vec{v}$ before integrating:

$$m \int \vec{v}\frac{\partial f}{\partial t}\, dV + m \int \vec{v}\left(\vec{v}\cdot\vec{\nabla}\right) f\, dV + q \int \vec{v}\left[\left(\vec{E} + \vec{v}\times B\right)\cdot \frac{\partial}{\partial \vec{v}} f\right] dV = \int m\vec{v}\left(\frac{df}{dt}\right)_{\text{due to collisions}} dV$$

The term on the right is the total change of momentum due to collisions, which is zero, provided that we are integrating over all the colliding particles. We can include more than one species of particle by writing this term as $P_{ab} =$ momentum transferred to species a by species b.

Now for the left side. The first term is:

$$m\frac{\partial}{\partial t}\int \vec{v} f\, dV = m\frac{\partial}{\partial t}(n\vec{u})$$

The second term is hardest, so let's leave it aside for the moment. The third term is, in index notation:

$$\int \left\{ \frac{\partial}{\partial v_j}\left\{ v_i\left[E_j + \left(\vec{v}\times\vec{B}\right)_j\right] f\right\} - f\frac{\partial}{\partial v_j}\left(v_i\left[E_j + \left(\vec{v}\times\vec{B}\right)_j\right]\right)\right\} dV$$

We convert the first term to a surface integral, as before, and it vanishes, leaving

$$\begin{aligned} & -\int\left\{ f\frac{\partial v_i}{\partial v_j}\left[E_j + \left(\vec{v}\times\vec{B}\right)_j\right] + f v_i \frac{\partial}{\partial v_j}\left[E_j + \left(\vec{v}\times\vec{B}\right)_j\right]\right\} dV \\ = & \ -\int\left\{ f\delta_{ij}\left[E_j + \left(\vec{v}\times\vec{B}\right)_j\right] + 0\right\} dV \end{aligned}$$

where E_j is independent of v_j and $\left(\vec{v}\times\vec{B}\right)_j$ contains the two components of $\vec{v}$ *other than* v_j. Finally we have

$$-\int f\left[E_i + \left(\vec{v}\times\vec{B}\right)_i\right] dV = -n\left[E_i + \left(\vec{u}\times\vec{B}\right)_i\right]$$

Now we tackle the second term, again using index notation:

$$\int v_i v_j \frac{\partial f}{\partial x_j}\, dV = \frac{\partial}{\partial x_j}\int v_i v_j f\, dV$$

where the integral is over the velocity space. Now we write each v_i in terms of the average fluid velocity $\vec{u}$ (which is a function of $\vec{r}$ but not of $\vec{v}$) and the difference $\vec{w}$ between $\vec{v}$ and $\vec{u}$:

$$v_i = u_i + w_i$$

so that

$$v_i v_j = u_i u_j + u_i w_j + u_j w_i + w_i w_j$$

and

$$\frac{\partial}{\partial x_j}\int v_i v_j f\, dV = \frac{\partial}{\partial x_j}\int (u_i u_j + u_i w_j + u_j w_i + w_i w_j) f\, dV$$
$$= \frac{\partial}{\partial x_j}(u_i u_j n) + \frac{\partial}{\partial x_j}\left[u_i \int w_j f\, dV + u_j \int w_i f\, dV\right] + \frac{\partial}{\partial x_j}\int w_i w_j f\, dV$$

The second term is zero by definition of $\vec{w}$. $\left(\int f w_j dV = \int f\,(v_j - u_j)\, dV = n u_j - n u_j = 0\right)$
We express the last term using the stress tensor

$$P_{ij} \equiv m \int w_i w_j f\, dV$$

so that

$$m \int v_i v_j \frac{\partial f}{\partial x_j}\, dV = m\frac{\partial}{\partial x_j}(u_i u_j n) + \frac{\partial}{\partial x_j} P_{ij}$$
$$= m u_i \frac{\partial}{\partial x_j}(u_j n) + m n u_j \frac{\partial}{\partial x_j} u_i + \frac{\partial}{\partial x_j} P_{ij}$$

Thus the first moment equation is:

$$m\frac{\partial}{\partial t}(n\vec{u}) + m\vec{u}\left(\vec{\nabla}\cdot n\vec{u}\right) + mn\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} + m\vec{\nabla}\widetilde{P} - qn\left(\vec{E} + \vec{u}\times\vec{B}\right) = P_{ab}$$

We can simplify this using our first relation (10):

$$m\left(\vec{u}\frac{\partial n}{\partial t} + n\frac{\partial \vec{u}}{\partial t}\right) + m\vec{u}\left(-\frac{\partial n}{\partial t}\right) + mn\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} + \vec{\nabla}\widetilde{P} - qn\left(\vec{E} + \vec{u}\times\vec{B}\right) = P_{ab}$$
$$mn\left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right] + \vec{\nabla}\widetilde{P} - qn\left(\vec{E} + \vec{u}\times\vec{B}\right) = P_{ab}$$

or

$$mn\frac{d\vec{u}}{dt} = qn\left(\vec{E} + \vec{u}\times\vec{B}\right) - \vec{\nabla}\widetilde{P} + P_{ab} \tag{11}$$

where the total time derivative

$$\frac{d\vec{u}}{dt} = \frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}, \tag{12}$$

the second term being the convective derivative.

When the velocity distribution is isotropic, P_{ij} is diagonal:

$$\int f w_i w_j\, d^3\vec{v} = P\delta_{ij}$$

and the temperature is defined by

$$3nkT = m\int w_i w_i f\, d^3\vec{v} = mn < w^2 >$$

Then the rms velocity about the mean is

$$\sqrt{< w^2 >} = \frac{3kT}{m}$$

a familiar result. Then we write

$$P_{ij} = \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$$

where $P = nkT$. This reduces our equation to the form:

$$mn\frac{d\vec{u}}{dt} = qn\left(\vec{E} + \vec{u} \times \vec{B}\right) - \vec{\nabla}P + P_{ab} \tag{13}$$

and we have derived equation (3).

The second moment gives the energy equation. We won't need it, so we won't derive it here. (See http://www.physics.sfsu.edu/~lea/courses/grad/fluids.PDF page 2 if you are interested.)

3 Fluids, plasmas and distribution functions

We'll be using equations (10) and (11) extensively, so let's review how we got them.

We needed to assume some properties of the distribution function, specifically that $f(\vec{v}) \to 0$ as $v \to \infty$ at least as fast as v^4. This is necessary to ensure that the total energy is finite. The Maxwellian distribution satisfies this, since it goes to zero *exponentially,* i.e. faster than any power.

The Maxwellian distribution is the solution to the equation:

$$\left(\frac{df}{dt}\right)_{\text{due to collisions}} = 0$$

i.e. a Boltzmann equation where the right side dominates. That means that collisions are very important.

How important are collisions in plasmas? The mean free path ("mfp") between collisions, computed using the Coulomb force (see e.g. Spitzer's lovely book "Physics of fully ionized gases") can often be very large, larger than the length scale of the plasma system. This is especially true in Astrophysics. Yet we see phenomena that are intrinsically fluid phenomena, like shock waves. Satellites in far Earth orbit have obtained very clear evidence of a shock wave where the solar wind meets the Earth's magnetic field, for example. In a plasma the mfp is restricted because the charged particles are forced to gyrate around $\vec{B}$, and this usually makes them good fluids. Additional collisional-type interactions occur when the plasma particles interact with plasma waves. Thus we have good reason to believe that we can successfully apply the fluid equations to most plasmas.

4 Using the equations to compute plasma drifts

We will begin by using the plasma fluid equations to compute drifts under the assumptions:

1. The drifts are *slow.* We'll make this assumption quantitative in a minute, and

2. The drifts are constant in time.

These assumptions are consistent with the kind of behavior we've come to expect from the individual particle motions.

First let's look at what we mean by "slow". The momentum equation, (3) is

$$\rho\frac{d\vec{u}}{dt} = \rho\left(\frac{\partial\vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right) = nq\left(\vec{E}+\vec{u}\times\vec{B}\right) - \vec{\nabla}P$$

and if the drift is time-independent, the partial derivative with respect to time vanishes, leaving

$$\begin{aligned}\rho\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} &= nq\left(\vec{E}+\vec{u}\times\vec{B}\right) - \vec{\nabla}P \\ \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} &= \frac{q}{m}\left(\vec{E}+\vec{u}\times\vec{B}\right) - \frac{\vec{\nabla}P}{\rho}\end{aligned}$$

With L being a length scale for our system, the order of the terms is

$$\text{LHS } \frac{v^2}{L}$$

$$\text{RHS, first term, second term } \omega_c\left(\frac{E}{B}, v\right)$$

and

$$\text{RHS, third term } \frac{c_s^2}{L}$$

where c_s is the sound speed in the fluid (see below). Thus the LHS is much less than the third term on the right provided the drift is highly subsonic, $v \ll c_s$. The LHS is much less than the second term on the right provided that the drift speed is much less than the particle's orbital speed times L/r_L. This suggests that we can safely neglect the quadratic term in v on the left side. Then in a steady state we have

$$0 = \frac{q}{m}\left(\vec{E}+\vec{u}\times\vec{B}\right) - \frac{\vec{\nabla}P}{\rho}$$

where only components of $\vec{u}$ perpendicular to $\vec{B}$ contribute. Now dot with $\vec{B}$ to get

$$0 = \frac{q}{m}\vec{E}\cdot\vec{B} - \vec{B}\cdot\frac{\vec{\nabla}P}{\rho} \tag{14}$$

and cross with $\vec{B}$ to get

$$\begin{aligned}0 &= \frac{q}{m}\left(\vec{E}\times\vec{B} + \left(\vec{u}\times\vec{B}\right)\times\vec{B}\right) - \frac{\vec{\nabla}P}{\rho}\times\vec{B} \\ 0 &= \frac{q}{m}\left(\vec{E}\times\vec{B} - \vec{u}B^2\right) - \frac{\vec{\nabla}P}{\rho}\times\vec{B}\end{aligned}$$

where we took $\vec{u}\cdot\vec{B} = 0$ since only perpendicular components of $\vec{u}$ appear in this relation.

Next solve for $\vec{u}$:

$$\vec{u} = \frac{\vec{E} \times \vec{B}}{B^2} - \frac{m}{q}\frac{\vec{\nabla} P}{\rho} \times \frac{\vec{B}}{B^2}$$

The first term is our old friend the $\vec{E} \times \vec{B}$ drift. The second term is new. It is called the diamagnetic drift.

$$\vec{v}_{\text{diamagnetic}} = -\frac{\vec{\nabla} P}{nq} \times \frac{\vec{B}}{B^2} \tag{15}$$

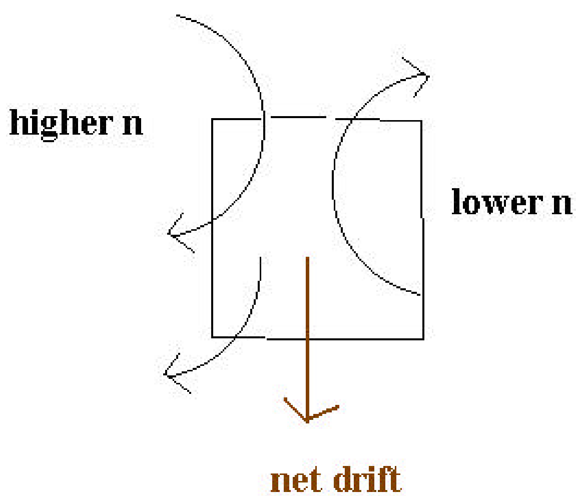
or, for an isothermal plasma:

$$\vec{v}_{\text{diamagnetic}} = -\frac{kT}{q}\frac{\vec{\nabla} n}{n} \times \frac{\vec{B}}{B^2}$$

Since the charge appears explicitly, ions and electrons go in opposite directions, and so there is a *diamagnetic current*:

$$\begin{aligned} \vec{j}_D &= ne\,(\vec{v}_i - \vec{v}_e) = -k\,(T_i + T_e)\,\vec{\nabla} n \times \frac{\vec{B}}{B^2} \\ &= k\,(T_i + T_e)\,\frac{\vec{B}}{B^2} \times \vec{\nabla} n \end{aligned} \tag{16}$$

In the fluid picture we have lost a lot of detail. We did not get any of the "finite Larmor radius effects", but we also get something new. The diamagnetic drift is a purely fluid phenomena that arises because in a given region, more particles move in one direction than the opposite direction if there are gradients in the density, as shown in the picture below. The drift is independent of the particle's mass, because the particle's thermal speed does depend on m ($v \propto m^{-1/2}$) but the Larmor radius is proportional to $m^{+1/2}$, so the slower particle samples more of the density gradient. The two effects exactly cancel.



Now let's go back and see what we can learn from the parallel component of the momentum equation (14). Again let's specialize to an isothermal plasma. Then:

$$\begin{aligned} 0 &= \frac{q}{m}\vec{E}\cdot\vec{B} - \vec{B}\cdot\frac{\vec{\nabla}P}{\rho} = \vec{B}\cdot\left(-\frac{q}{m}\vec{\nabla}\phi - \frac{kT}{m}\frac{\vec{\nabla}n}{n}\right) \\ &= -\frac{\vec{B}}{m}\cdot\vec{\nabla}\left(q\phi + kT\ln n\right) \end{aligned}$$

and thus, for electrons with $q = -e$,

$$e\phi - kT\ln n = \text{constant along a field line}$$

or, taking the exponential of both sides:

$$n = n_0 \exp\left(\frac{e\phi}{kT}\right) \tag{17}$$

which is the Boltzmann relation.

Remember that we obtained equation (14) by neglecting the acceleration in the momentum equation. Because of the mass dependence on the right, the acceleration of the electrons is much larger than the acceleration of the ions. The electrons move rapidly until equation (17) is satisfied. The ions do not have time to move.

5 The plasma approximation

The plasma is quasi-neutral: that means that the electron and ion densities are almost equal everywhere in the plasma (assuming the ions are protons, or, at least, singly ionized). When an electric field exists in the plasma, the electrons move rapidly to neutralize the field. The ions follow, more slowly. Generally, for low frequency motions, we do not use Poisson's equation to get $\vec{E}$. Rather, we use the equation of motion to get $\vec{E}$, and then use Poisson's equation to compute the small difference between n_i and n_e. Our derivation of the Boltzmann relation (17) is an example of this procedure.

The assumption that $n_e \approx n_i$ (and $|n_e - n_i| \ll n_e$) with $\vec{E} \neq 0$ is called the *plasma approximation.* It is valid for frequencies that are low in a sense that we shall describe more precisely later.