Plasma Waves

S.M.Lea

January 2007

1 General considerations

To consider the different possible normal modes of a plasma, we will usually begin by assuming that there is an equilibrium in which the plasma parameters such as density and magnetic field are uniform and constant in time. We will then look at small perturbations away from this equilibrium, and investigate the time and space dependence of those perturbations.

The usual notation is to label the equilibrium quantities with a subscript 0, e.g. n_0, and the pertrubed quantities with a subscript 1, eg n_1. Then the assumption of small perturbations is $|n_1/n_0| \ll 1$. When the perturbations are small, we can generally ignore squares and higher powers of these quantities, thus obtaining a set of linear equations for the unknowns.

These linear equations may be Fourier transformed in both space and time, thus reducing the differential equations to a set of algebraic equations. Equivalently, we may assume that each perturbed quantity has the mathematical form

$$n_1 = n \exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right) \tag{1}$$

where the real part is implicitly assumed. This form describes a wave. The amplitude n is in general complex, allowing for a non-zero phase constant ϕ_0. The vector $\vec{k}$, called the wave vector, gives both the direction of propagation of the wave and the wavelength: $k = 2\pi/\lambda$; ω is the angular frequency. There is a relation between ω and $\vec{k}$ that is determined by the physical properties of the system. The function $\omega\left(\vec{k}\right)$ is called the *dispersion relation* for the wave.

A point of constant phase on the wave form moves so that

$$\frac{d\phi}{dt} = \vec{k}\cdot\vec{v}_\phi - \omega = 0$$

where the wave phase velocity is

$$\vec{v}_\phi = \frac{\omega}{k}\hat{\mathbf{k}} \tag{2}$$

The phase speed of a wave may (and often does) exceed the speed of light, since no information is carried at v_ϕ. Information is carried by a modulation of the wave, in either amplitude or frequency. For example, a wave pulse may have a Gaussian envelope. For any physical quantity u,

$$u\left(x,t\right)=Ae^{-x^{2}/a^{2}}e^{ikx-i\omega t}$$

Thus can be Fourier-transformed to yield a Gaussian in $k-$space.

A general disturbance of the system may be written as a superposition of plane waves. With $x-$axis chosen along the direction of propagation for simplicity:

$$u\left(x,t\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}A\left(k\right)e^{ikx-i\omega t}dk$$

But ω is related to k through the dispersion relation:

$$\omega=\omega\left(k\right)=\omega_{0}+\left(k-k_{0}\right)\left.\frac{d\omega}{dk}\right|_{k_{0}}+\cdots$$

where k_0 is the wave number at which $A\left(k\right)$ peaks. Then the integral becomes:

$$\begin{aligned}
u\left(x,t\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}A\left(k\right)e^{ikx-i\omega_{0}t}\exp\left[-i\left(k-k_{0}\right)\left.\frac{d\omega}{dk}\right|_{k_{0}}t\right]dk \\
&= \frac{1}{\sqrt{2\pi}}\exp\left[i\left(k_{0}\left.\frac{d\omega}{dk}\right|_{k_{0}}-\omega_{0}\right)t\right]\int_{-\infty}^{+\infty}A\left(k\right)\exp\left[ik\left(x-\left.\frac{d\omega}{dk}\right|_{k_{0}}t\right)\right]dk \\
&= \exp\left[i\left(k_{0}\left.\frac{d\omega}{dk}\right|_{k_{0}}-\omega_{0}\right)t\right]\ u\left(x-\left.\frac{d\omega}{dk}\right|_{k_{0}}t,0\right)
\end{aligned}$$

Thus, apart from an overall phase factor, the disturbance propagates at the *group speed*

$$v_{g}=\left.\frac{d\omega}{dk}\right|_{k_{0}} \tag{3}$$

Information travels at this *group speed.*

Additional terms we have neglected in the exponent (involving the higher derivatives $d^{n}\omega/dk^{n}$) give rise to pulse spreading and other factors generally referred to as dispersion. (More on this in Phys 704!) Hence the name "dispersion relation".

Our goal will be to identify the different wave modes that occur in the plasma, and to find the dispersion relation $\omega\left(\vec{k}\right)$ for each. If the frequency has an imaginary part, that indicates damping or growth of the wave.

2 Plasma oscillations

In this wave we assume that the initial condition is a cold, uniform, unmagnetized plasma with $n=n_{0}$, $T\equiv 0$ and $\vec{v}_{0}=0$. Then the equation of motion is:

$$mn_{0}\left(\frac{\partial}{\partial t}\vec{v}_{1}+\left(\vec{v}_{1}\cdot\vec{\nabla}\right)\vec{v}_{1}\right)=-en_{0}\vec{E}$$

We ignore the second term on the left, because it involves a small quantity squared. Then we use the assumed wave form to obtain

$$-i\omega m\vec{v}_{1}=-e\vec{E} \tag{4}$$

Next the continuity equation gives

$$\begin{aligned}\frac{\partial}{\partial t}(n_0+n_1)+\vec{\nabla}\cdot(n_0+n_1)\vec{v}_1 &= 0\\ \frac{\partial}{\partial t}n_1+n_0\vec{\nabla}\cdot\vec{v}_1+\vec{\nabla}\cdot(n_1\vec{v}_1) &= 0\end{aligned}$$

We used the facts that $\partial n_0/\partial t$ and $\vec{\nabla}n_0$ are zero. Again we ignore the last term because it involves the product of two small quantities. Then:

$$-i\omega n_1+n_0 i\vec{k}\cdot\vec{v}_1=0 \tag{5}$$

Or, solving for n_1,

$$n_1=n_0\frac{\vec{k}\cdot\vec{v}_1}{\omega} \tag{6}$$

Thus the density is not perturbed unless $\vec{v}_1$ has a component along $\vec{k}$. This means that transverse waves with $\vec{v}_1$ perpendicular to $\vec{k}$ do not perturb the plasma density.

Finally we use Poisson's equation, since this is a high-frequency wave:

$$\vec{\nabla}\cdot\vec{E}=-\frac{e}{\varepsilon_0}\left[n_0-(n_0+n_1)\right]=-\frac{e}{\varepsilon_0}n_1$$

This equation is already linear, and we get:

$$i\vec{k}\cdot\vec{E}=-\frac{e}{\varepsilon_0}n_1 \tag{7}$$

Now we dot equation (4) with $\vec{k}$ and substitute in from equations (5) and (7).

$$\begin{aligned}-i\omega m\vec{k}\cdot\vec{v}_1 &= -e\vec{k}\cdot\vec{E}\\ -i\omega m\frac{\omega n_1}{n_0} &= -e\left(-\frac{e}{i\varepsilon_0}n_1\right)\\ \omega m\frac{\omega n_1}{n_0} &= \frac{e^2}{\varepsilon_0}n_1\end{aligned}$$

Now either $n_1=0$ (not the result we want) or

$$\omega_p^2=\frac{n_0e^2}{\varepsilon_0 m} \tag{8}$$

The frequency of the disturbance is the *plasma frequency* ω_p. Notice that the result is independent of k. There is no phase speed or group speed: the wave is an oscillation that does not propagate. As we shall see, introduction of a non-zero temperature causes the wave to propagate.

We can also see from equation (7) that the electric field is out of phase with the density by $\pi/2$, while the velocity is in phase (equation 6).

This oscillation is a fundamental mode of the plasma and has many ramifications. Our next step is to begin to investigate its importance.

3 Electromagnetic waves in an unmagnetized plasma

To understand the importance of the plasma frequency a bit more, let's look at the propagation of electromagnetic waves in the plasma. Maxwell's equations are already linear. We have Poissons' equation for $\vec{E}$ (7) and $\vec{B}$,

$$i\vec{k}\cdot\vec{B}_1 = 0 \tag{9}$$

Faraday's Law

$$i\vec{k}\times\vec{E} = i\omega\vec{B}_1 \tag{10}$$

and Ampere's law

$$i\vec{k}\times\vec{B} = \mu_0\left(\vec{j} + i\omega\varepsilon_0\vec{E}\right) \tag{11}$$

The current in the plasma is due to the electron and ion motions:

$$\vec{j} = n_0 e\vec{v}_i - (n_0 + n_1)\, e\vec{v}_e$$

For high frequency waves we may neglect the ion motion $(\vec{v}_i \simeq 0)$ since the massive ions do not respond rapidly enough as the driving force due to $\vec{E}$ changes. We also ignore the product of the two small quantities n_1 and v_e to get:

$$\vec{j} = -n_0 e\vec{v}_e \tag{12}$$

so that Ampere's law becomes:

$$i\vec{k}\times\vec{B}_1 = \mu_0\left(-n_0 e\vec{v}_e + i\omega\varepsilon_0\vec{E}\right) \tag{13}$$

Finally we need the equation of motion for the electrons:

$$-i\omega m\vec{v}_1 = -e\left(\vec{E} + \vec{v}_e\times\vec{B}_1\right) \tag{14}$$

which reduces to equation (4) when we ignore the non-linear term.

Equation (6) shows that the density variations are zero for a transverse wave $\left(\vec{v} \text{ perpendicular to } \vec{k}\right)$, and Poisson's equation becomes $\vec{k}\cdot\vec{E} = 0$, which is consistent with the assumption of a transverse wave with $\vec{v}$ parallel to $\vec{E}$ (equation 14). So let us look for transverse waves.

Cross equation (13) with $\vec{k}$ to get:

$$\begin{aligned} i\vec{k}\times\left(\vec{k}\times\vec{B}_1\right) &= \mu_0\left(-n_0 e\vec{k}\times\vec{v}_e + i\omega\varepsilon_0\vec{k}\times\vec{E}\right) \\ \vec{k}\left(\vec{k}\cdot\vec{B}_1\right) - k^2\vec{B}_1 &= -\mu_0\frac{n_0 e}{i}\vec{k}\times\vec{v}_e - \mu_0\omega\varepsilon_0\omega\vec{B}_1 \end{aligned}$$

where we used Faraday's law. Further substitutions from equations (9) and (4) give:

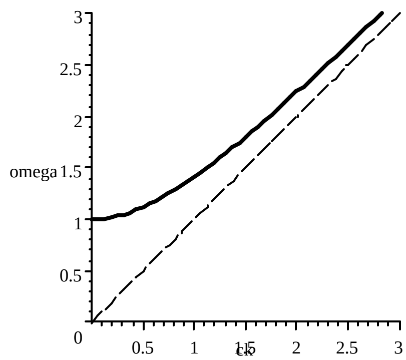
$$\begin{aligned} -k^2\vec{B}_1 &= -\mu_0\frac{n_0 e}{i}\vec{k}\times\left(\frac{e\vec{E}}{i\omega m}\right) - \mu_0\varepsilon_0\omega^2\vec{B}_1 \\ &= \mu_0 n_0\frac{e^2}{\omega m}\omega\vec{B} - \mu_0\varepsilon_0\omega^2\vec{B}_1 \\ &= \mu_0\varepsilon_0\left(\omega_p^2 - \omega^2\right)\vec{B}_1 \end{aligned} \tag{15}$$

Again we argue that the solution $\vec{B}_1 = 0$ is uninteresting. The solution with non-zero $\vec{B}$ has

$$\omega^2 = k^2c^2 + \omega_p^2 \tag{16}$$

$$\omega = \sqrt{k^2c^2 + \omega_p^2} \tag{17}$$

Thus at high frquencies ($\omega \gg \omega_p$) we have the usual vacuum relation $\omega = ck$.



However, the wave number

$$k = \frac{\sqrt{\omega^2 - \omega_p^2}}{c}$$

becomes imaginary ($k = i\gamma$) when $\omega < \omega_p$. The wave form becomes

$$\exp\left[i\left(i\gamma\right)x - i\omega t\right] = e^{-\gamma x}e^{i\omega t}$$

The wave does not propagate, and the disturbance damps as $\exp\left(-\gamma x\right)$. The waves have a *cut-off* at $\omega = \omega_p$.

An electromagnetic wave propagating at $\omega \leq \omega_p$ is able to excite plasma oscillations in the plasma, thus draining energy out of the wave and into the motion of plasma particles. Hence the wave damps.

In the ionosphere with $n \sim 10^{12}\ \text{m}^{-3}$, the plasma frequency is

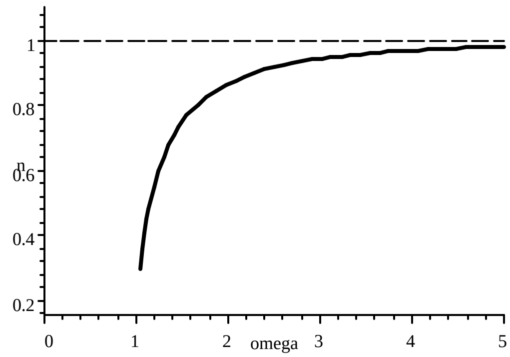
$$f_p = \frac{1}{2\pi}\sqrt{\frac{\left(10^{12}\ \text{m}^{-3}\right)\left(1.6\times 10^{-19}\ \text{C}\right)^2}{\left(8.85\times 10^{-12}\ \text{F/m}\right)\left(9\times 10^{-31}\ \text{kg}\right)}} = 9.\,\text{MHz}$$

and waves at lower frequencies do not propagate through the ionosphere. This is an intrinsic limit on our ability to do radio astronomy from the Earth's surface. If we do build a base on the moon, a radio observatory will quickly follow!

The refractive index of the plasma is

$$n = \frac{c}{v_\phi} = \frac{ck}{\omega} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{18}$$

and is less than one for all $\omega < \infty$.



Snells' law thus predicts the phenomenon of total *external* reflection when an EM wave of frequency $\omega < \omega_p$ is incident on a plasma. Waves incident on the ionosphere from Earth at $<$ 9 MHz are reflected back to earth. X-rays incident on glass from air are also reflected– that is how x-ray mirrors (such as those in the orbiting x-ray observatory AXAF) work. At the high frequencies of x-rays, the electrons in the glass behave like free electrons in a plasma.

The group speed is found by differentiating equation (16):

$$2\omega \frac{d\omega}{dk} = 2kc^2$$

Thus

$$v_g = \frac{d\omega}{dk} = \frac{c^2}{v_\phi} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{19}$$

and is less than c at all frequencies $\omega > \omega_p$.

Equation (15) may also be written

$$k^2 \vec{B}_1 = \mu_0 \varepsilon \omega^2 \vec{B}_1$$

and thus we obtain another expression for the plasma dielectric constant, this one valid for high-frequency transverse oscillations in a field-free plasma:

$$\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{20}$$

Notice that for $\omega < \omega_p$ the dielectric constant is negative. The refractive index is imaginary, corresponding to absorption of the wave.

4 Langmuir waves

Let's go back and reconsider the plasma oscillations (longitudinal waves) but now with a non-zero T. The momentum equation (4) has an extra term due to the pressure gradient:

$$-i\omega m n_0 \vec{v}_1 = -en_0 \vec{E} - i\vec{k} P_1$$

We must add the equation of state

$$P = K\rho^{\gamma}$$

which becomes, to first order,

$$\begin{aligned} P_0 + P_1 &= K\left(\rho_0 + \rho_1\right)^{\gamma} \simeq K\rho_0^{\gamma}\left(1 + \gamma\frac{\rho_1}{\rho_0}\right) = P_0\left(1 + \gamma\frac{\rho_1}{\rho_0}\right) \\ P_1 &= \gamma\frac{P_0}{\rho_0}\rho_1 \end{aligned} \tag{21}$$

Poisson's equation and the continuity equation are unchanged. Dotting the momentum equation with $\vec{k}$, we get:

$$-i\omega m n_0 \vec{k}\cdot\vec{v}_1 = -en_0\vec{k}\cdot\vec{E} - ik^2 P_1$$

and then using equations (21), (5) and (7), we have

$$-i\omega m n_0 \frac{\omega n_1}{n_0} = -en_0\left(-\frac{e}{i\varepsilon_0}n_1\right) - ik^2\gamma\frac{P_0}{\rho_0}n_1 m$$

leading to the dispersion relation

$$\omega^2 = \omega_p^2 + k^2\gamma\frac{k_B T_e}{m} = \omega_p^2 + k^2 c_s^2 \tag{22}$$

where

$$c_s = \sqrt{\gamma\frac{k_B T_e}{m}}$$

is the electron sound speed. Equation (22) is the Bohm-Gross dispersion relation for Langmuir waves– longitudinal waves in a plasma. Notice its similarity to equation (16) – the sound speed replaces the speed of light. The phase speed of these waves is

$$v_\phi = \frac{\omega}{k} = \sqrt{\frac{\omega_p^2}{k^2} + c_s^2}$$

and the group speed is

$$v_g = \frac{d\omega}{dk} = \frac{c_s^2 k}{\omega} = \frac{c_s^2}{v_\phi}$$

5 Sound waves

5.1 Fluid sound waves

If we perturb and linearize the momentum equation and the continuity equation for a field-free plasma, we get:

$$-i\omega\rho_0\vec{v}_1 = -i\vec{k}\gamma\frac{P_0}{\rho_0}\rho_1 \tag{23}$$

and

$$-i\omega\rho_1 + i\vec{k}\cdot\left(\rho_0\vec{v}_1\right) = 0 \tag{24}$$

Dotting the first equation (23) with $\vec{k}$ and substituting into the second (24), we get:

$$-\omega\rho_1 + \rho_0 \frac{k^2}{\omega\rho_0}\gamma\frac{P_0}{\rho_0}\rho_1 = 0$$

and the dispersion relation is

$$\omega^2 = k^2\gamma\frac{P_0}{\rho_0} = k^2 c_s^2$$

waves with $v_\phi = v_g = c_s$.

These waves rely on collisions in the fluid to provide the restoring force.

5.2 Ion sound waves

Even if collisions are unimportant, sound waves, being longitudinal waves, generate density fluctuations which in turn generate electric fields that can provide the necessary restoring force. When ion motion is involved, we know that the waves must be low frequency, so we can use the *plasma approximation,* $n_e \simeq n_i \simeq n_0$. We are still assuming that there is no magnetic field.

The momentum equation for the ions is:

$$Mn_0\left[\frac{\partial}{\partial t}\vec{v}_1 + \left(\vec{v}_1\cdot\vec{\nabla}\right)\vec{v}_1\right] = en_0\vec{E} - \vec{\nabla}\left(P_i + P_{i,1}\right)$$

and for the electrons

$$mn_0\left[\frac{\partial}{\partial t}\vec{v}_1 + \left(\vec{v}_1\cdot\vec{\nabla}\right)\vec{v}_1\right] = -en_0\vec{E} - \vec{\nabla}\left(P_e + Pe_{,1}\right)$$

The continuity equation is

$$\frac{\partial}{\partial t}n_1 + n_0\vec{\nabla}\cdot\vec{v}_1 + \vec{\nabla}\cdot n_1\vec{v}_1 = 0$$

and linearizing, we get:

$$\frac{\partial}{\partial t}n_1 = -n_0\vec{\nabla}\cdot\vec{v}_1$$

Thus

$$\frac{\partial}{\partial t}\left(n_e - n_i\right) = -n_0\vec{\nabla}\cdot\left(\vec{v}_{e1} - \vec{v}_{i1}\right)$$

Thus if the ion and electron velocities differ, the densities will become different too. Thus the plasma approximation also requires $\vec{v}_{e1} = \vec{v}_{i1}$ (at least to first order).

Using this result, we add the two linearized momentum equations. The electric field terms cancel, leaving:

$$\begin{aligned}(m+M)\,n_0\frac{\partial}{\partial t}\vec{v}_1 &= -\vec{\nabla}\left(P_{i,1} + Pe_{,1}\right)\\ -i\omega\,(m+M)\,n_0\vec{v}_1 &= -i\vec{k}n_1\left(\gamma_i k_B T_i + \gamma_e k_B T_e\right)\end{aligned}$$

(Compare this equation with (23)). Dot with $\vec{k}$, and combine with equation (6) to get:

$$\begin{aligned} -i\omega\,(m+M)\,n_0\omega\frac{n_1}{n_0} &= -ik^2 n_1\,(\gamma_i k_B T_i + \gamma_e k_B T_e) \\ \omega^2 &= k^2\left(\frac{\gamma_i k_B T_i + \gamma_e k_B T_e}{m+M}\right) \end{aligned} \tag{25}$$

There are several things to note about this result.

1. It is essentially identical to the result for fluid sound waves even though at a microscopic level there are profound differences. The coupling here is electrostatic not collisional.
2. The electrons move very rapidly, and the distribution may be assumed to be isothermal, $\gamma_e = 1$.
3. The electron mass is negligible compared with the ion mass in the denominator. However, Vlasov theory (a detailed study of the effect of the particle velocity distributions) shows that the wave is strongly damped unless the electron temperature greatly exceeds the ion temperature. Thus the ion sound speed is determined by the electron temperature and the ion mass.

$$v_{\text{is}} = \sqrt{\frac{k_B T_e}{M}} \tag{26}$$

We may now consider the electric field necessary to effect the coupling. Poisson's equation is:

$$i\vec{k}\cdot\vec{E} = k^2\phi = \frac{e}{\varepsilon_0}\,(n_i - n_e)$$

Now we allow for small differences between the electron and ion densities. The ion density is given by the continuity equation:

$$n_i = n_0 + \frac{\vec{k}\cdot\vec{v}}{\omega} n_0$$

while the electrons respond rapidly to the electric field, and so follow the Boltzman relation:

$$n_e = n_0 \exp\left(\frac{e\phi}{k_B T_e}\right) \simeq n_0\left(1 + \frac{e\phi}{k_B T_e}\right)$$

Thus

$$k^2\phi = \frac{en_0}{\varepsilon_0}\left(\frac{\vec{k}\cdot\vec{v}}{\omega} - \frac{e\phi}{k_B T_e}\right)$$

Rearranging, we get:

$$\phi\left(k^2 + \frac{en_0}{\varepsilon_0}\frac{e\phi}{k_B T_e}\right) = \frac{en_0}{\varepsilon_0}\frac{\vec{k}\cdot\vec{v}}{\omega}$$

We should recognize the second term in the parentheses as $1/\lambda_D^2$. Now we rewrite the momentum equation for the ions, substituting this expression for ϕ in the electric field term

$(\vec{E} = -\vec{\nabla}\phi = -i\vec{k}\phi)$

$$
\begin{aligned}
-i\omega M n_0 \vec{v}_1 &= -en_0 i\vec{k}\phi - i\vec{k}n_1\gamma_i k_B T_i \\
-\omega M n_0 \vec{k}\cdot\vec{v}_1 &= -en_0 \frac{en_0}{\varepsilon_0}\frac{\vec{k}\cdot\vec{v}_1}{\omega}\left(\frac{1}{1+1/k^2\lambda_D^2}\right) - k^2 n_1 \gamma_i k_B T_i \\
\omega M n_0 \omega \frac{n_1}{n_0} &= en_0 \frac{en_0}{\varepsilon_0}\frac{n_1}{n_0}\left(\frac{1}{1+1/k^2\lambda_D^2}\right) + k^2 n_1 \gamma_i k_B T_i \\
\omega^2 &= e^2 \frac{n_0}{M\varepsilon_0}\left(\frac{1}{1+1/k^2\lambda_D^2}\right) + k^2 \frac{\gamma_i k_B T_i}{M} \\
&= k^2 \left(\frac{\gamma_i k_B T_i}{M} + \frac{\lambda_D^2 \ {}_p^2}{1+k^2\lambda_D^2}\right)
\end{aligned}
$$

where ${}_p$ is the ion plasma frequency.

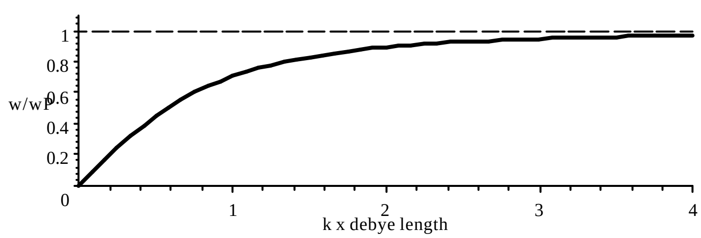
The numerator in the second term is:

$$\lambda_D^2 \ {}_p^2 = \frac{\varepsilon_0}{en_0}\frac{k_B T_e}{e} e^2 \frac{n_0}{M\varepsilon_0} = k_B \frac{T_e}{M}$$

Thus the new result is identical to the previous one except for the denominator $1 + k^2\lambda_D^2$. Thus the correction is necessary only when $k\lambda_D$ is not small, that is when the wavelength is less than or equal to the Debye length. The full wavelength is within the region where we would expect the plasma approximation to fail. When $k\lambda_D \gg 1$ we find

$$\omega \simeq \ {}_p$$

and we have oscillations at the ion plasma frequency. The wave reduces to plasma oscillations of the ions.



Dispersion relation for ion sound waves

6 Electrostatic waves in magnetized plasmas

Now we look at a plasma that has a uniform magnetic field $\vec{B}_0$ in the initial, unperturbed state. The motion of the plasma particles is affected by the magnetic field when they try to move across $\vec{B}$. Thus we expect to find that waves travelling along $\vec{B}$ and waves travelling across $\vec{B}$ will behave differently.

Electrostatic waves are longitudinal waves, and thus when electrostatic waves propagate along the magnetic field, the particle motions are also along $\vec{B}$, and these motions are not affected by the magnetic force. Thus the previous dispersion relations for Langmuir waves (eqn 22) and ion sound waves (eqn 25) are unaffected. We obtain interesting new effects when the waves propagate perpendicular to the magnetic field.

Note that for electrostatic waves, there is no perturbation to the magnetic field (equation 10) because $\vec{E}$ is parallel to $\vec{k}$.

6.1 High frequency electrostatic waves propagating perpendicular to $\vec{B}_0$

We begin by taking the electron temperature to be zero, as we did when studying Langmuir waves. With a non-zero magnetic field, we have an additional term in the equation of motion (4):

$$-i\omega m\vec{v} = -e\vec{E} - e\vec{v}\times\vec{B}_0 \tag{27}$$

The additional equations we need are the continuity equation in the form (6) and Poisson's equation (7). As usual, we dot the equation of motion with $\vec{k}$:

$$\begin{aligned} -i\omega m\vec{k}\cdot\vec{v} &= -e\vec{k}\cdot\vec{E} - e\vec{k}\cdot\left(\vec{v}\times\vec{B}_0\right) \\ &= -e\vec{k}\cdot\vec{E} - e\vec{B}_0\cdot\left(\vec{k}\times\vec{v}\right) \end{aligned}$$

The next step is to get $\vec{k}\times\vec{v}$ from equation (27):

$$\begin{aligned} -i\omega m\vec{k}\times\vec{v} &= -e\vec{k}\times\vec{E} - e\vec{k}\times\left(\vec{v}\times\vec{B}_0\right) \\ &= 0 - e\vec{v}\left(\vec{k}\cdot\vec{B}_0\right) + e\vec{B}_0\left(\vec{k}\cdot\vec{v}\right) \end{aligned}$$

where we used the result that $\vec{k}\times\vec{E} = 0$ for these waves. Also, for propagation across $\vec{B}_{0,}$ $\vec{k}\cdot\vec{B}_0 = 0$. Thus

$$\begin{aligned} -i\omega m\vec{k}\cdot\vec{v} &= -e\vec{k}\cdot\vec{E} - e\vec{B}_0\cdot\frac{e\vec{B}_0\left(\vec{k}\cdot\vec{v}\right)}{-i\omega m} \\ \vec{k}\cdot\vec{v}\left(1-\frac{\omega_c^2}{\omega^2}\right) &= \frac{e\vec{k}\cdot\vec{E}}{i\omega m} \end{aligned} \tag{28}$$

Finally we substitute in from the continuity and Poisson equations, to get

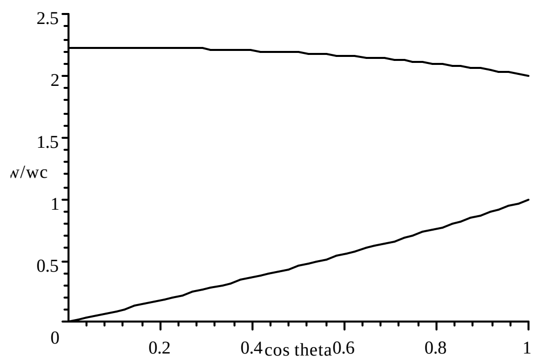
$$\frac{n_1}{n_0}\omega\left(1-\frac{\omega_c^2}{\omega^2}\right) = \frac{e}{i\omega m}\left(\frac{-en_1}{i\varepsilon_0}\right)$$
$$\omega^2-\omega_c^2 = \omega_p^2$$

Thus we obtain a new oscillation frequency, called the *upper hybrid frequency:*

$$\omega_H^2 = \omega_c^2 + \omega_p^2 \tag{29}$$

It is "hybrid" because it is a mixture of the plasma and cyclotron frequencies. It is a higher frequency than either because the additional restoring force leads to a higher oscillation frequency. As the electric field in the wave sets the particles moving across $\vec{B}$, the magnetic force causes them to gyrate. The resulting particle orbits are elliptical.

We might ask what happens when the wave propagates at a finite angle $\theta < \pi/2$ to $\vec{B}$. We would expect the magnetic force to act, but less strongly, and thus the frequency will have an intermediate value. We find that as $\theta \to 0$, $\omega \to$ the larger of ω_p and ω_c. When $\omega_c > \omega_p$ there is a second wave at $\theta = 0$ (the plasma oscillation at $\omega = \omega_p$). In fact there are two oscillations at all intermediate angles, as shown in the figure below which shows ω/ω_c versus $\cos\theta$. (See Problem 4-8). (We should be suspicious of the region near $\omega = 0$ because ion motion will become important here.)



$\omega_p = 2\omega_c$

The waves do not propagate with $T = 0$. As for the Langmuir waves, they do propagate when $T > 0$, but the analysis is more complicated and we leave it for another day.

6.2 Electrostatic ion waves propagating perpendicular to $\vec{B}_0$

Here we must include the ion motion, but we expect to use the plasma approximation. We have the two equations of motion, for the electrons

$$-i\omega m\vec{v} = -e\vec{E} - e\vec{v}\times\vec{B}_0$$

and for the ions

$$-i\omega M\vec{v} = e\vec{E} + e\vec{v}\times\vec{B}_0$$

(same $\vec{v}$ because of the plasma approximation) and

$$n_1 = \frac{\vec{k}\cdot\vec{v}}{\omega}n_0$$

The method is the same as in the previous section. Dot and cross the equations of motion with $\vec{k}$ to get both components of $\vec{v}$. We obtain two versions of equation (28), one for the ions and one for the electrons:

$$\begin{aligned}\vec{k}\cdot\vec{v}\left(1-\frac{\omega_c^2}{\omega^2}\right) &= \frac{e\vec{k}\cdot\vec{E}}{i\omega m}\\ \vec{k}\cdot\vec{v}\left(1-\frac{{}_c^2}{\omega^2}\right) &= -\frac{e\vec{k}\cdot\vec{E}}{i\omega M}\end{aligned} \tag{30}$$

Now divide the two equations, to obtain:

$$\frac{\left(\omega^2-\omega_c^2\right)}{\left(\omega^2-\ {}_c^2\right)} = -\frac{M}{m}$$

or

$$\begin{aligned}\omega^2\left(1+\frac{M}{m}\right) &= \frac{M}{m}\ {}_c^2+\omega_c^2 = e^2B_0^2\left(\frac{1}{Mm}+\frac{1}{m^2}\right) = \frac{e^2B_0^2}{Mm}\left(1+\frac{M}{m}\right)\\ \omega_{LH}^2 &= \omega_c\ {}_c\end{aligned} \tag{31}$$

This is the *lower hybrid* frequency. In the previous graph, the lower branch goes to ω_{LH} not zero.

These waves are almost never observed experimentally because it is necessary that they propagate *exactly* across $\vec{B}$, and experimentally that is impossible to achieve. Suppose that the waves propagate at an angle $\pi/2-\theta$ to $\vec{B}$, where $\theta \ll 1$. Electrons are free to migrate along $\vec{B}$, and do so very rapidly. Because the wave is not exactly perpendicular to $\vec{B}$, the electrostatic potential is not constant along $\vec{B}$ and the electrons will establish a Boltzmann distribution by travelling long distances along the field lines. Thus we should replace the electron density with $n_{1,e} = n_0 e\phi/k_BT$, while retaining equation (6) for the ions.

(Look at the electron density we used before. It is

$$\begin{aligned}n_1 &= \frac{\vec{k}\cdot\vec{v}}{\omega}n_0 = \frac{n_0}{\omega}\frac{e\vec{k}\cdot\vec{E}}{i\omega m\left(1-\frac{\omega_c^2}{\omega^2}\right)} = \frac{n_0k^2\phi}{im\left(\omega^2-\omega_c^2\right)}\\ &= n_0\frac{e\phi}{k_BT_e}\frac{k^2\left(k_BT_e/m\right)}{i\left(\omega^2-\omega_c^2\right)}\\ &= n_{\text{Boltzmann}}\left(\frac{\text{distance travelled per period at } v_{\text{e,th}}}{\text{wavelength}}\right)^2\end{aligned}$$

The factor in parentheses must be much greater than one if the argument we have given is correct, and so $n_{\text{Boltzmann}}$ is much less than our previous n_1 for the electrons.)

Now we use the plasma approximation:

$$n_i = \frac{\vec{k} \cdot \vec{v}}{\omega} n_0 = n_0 \frac{e\phi}{k_B T_e} = n_e$$

and combine with the equation of motion (30) for the ions:

$$\vec{k} \cdot \vec{v} \left(1 - \frac{{}^2_c}{\omega^2}\right) = -\frac{e\vec{k} \cdot \vec{E}}{i\omega M} = \frac{ek^2\phi}{\omega M} = \frac{ek^2}{\omega M} \frac{\vec{k} \cdot \vec{v}}{\omega} \frac{k_B T_e}{e}$$

Thus

$$\omega^2 - \ {}^2_c = k^2 \frac{k_B T_e}{M} = k^2 v_{\text{is}}^2 \tag{32}$$

where the speed on the right is the ion sound speed (26). These waves are called *electrostatic ion-cyclotron waves.* Unlike lower hybrid oscillations, they are easily observed experimentally. They propagate with phase speed

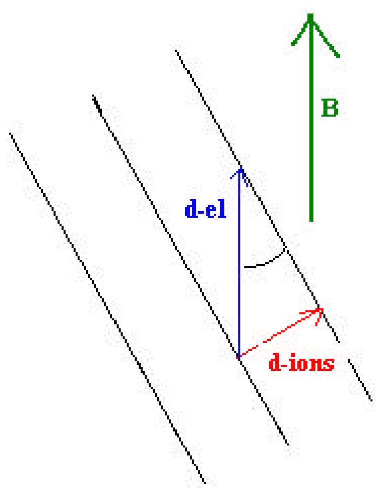
$$v_\phi = \frac{\omega}{k} = \frac{\omega}{\sqrt{\omega^2 + \ {}^2_c}} v_{\text{is}} \to v_{\text{is}} \text{ as } \omega \to \infty$$

and group speed

$$\frac{d\omega}{dk} = \frac{v_{\text{is}}^2}{v_\phi}$$

Dispersion relation (32) has the familiar form we have seen before.

Exactly how close to perpendicular to $\vec{B}$ does the wave vector need to be to get the lower hybrid oscillations?



From the diagram, we need

$$\sin\theta < \frac{d_{\text{ions}}}{d_{\text{electrons}}} = \frac{v_{\text{th, ions}}}{v_{\text{th, elec}}} = \frac{m}{M} \frac{T_i}{T_e}$$

With equal temperatures, we find $\theta < \sqrt{1/1837} = 2 \times 10^{-2}$ radians = 1.1°.

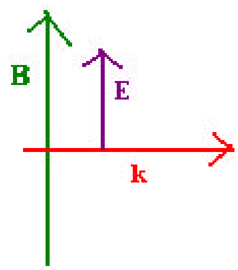
7 Electromagnetic waves in magnetized plasmas

7.1 Electromagnetic waves propagating across $\vec{B}$

Electromagnetic waves in an unmagnetized plasma are transverse and thus do not generate density fluctuations. Thus the plasma particles affect the waves through the currents that they generate. The same is true for magnetized plasmas, but we will have to be alert for the possibility that a longitudinal component of the waves may be generated as well.

An additional effect that we did not have to consider with electrostatic waves is the polarization.

7.1.1 The ordinary wave.



When the wave propagates perpendicular to $\vec{B}_0$, one polarization has its electric field vector parallel to $\vec{B}_0$. This is called the ordinary (or "O" mode) because the plasma particles can move freely along $\vec{B}_0$ as they are accelerated by the electric field. Thus the magnetic field has no effect, and we regain the dispersion relation (16). We can see this explicitly from the electron equation of motion:

$$-i\omega m\vec{v} = -e\left(\vec{E} + \vec{v} \times \vec{B}_0\right) \tag{33}$$

which gives rise the the current density

$$\vec{j} = -n_0 e\vec{v}$$

and then Ampere's law gives

$$\begin{aligned} \vec{k} \times \left(i\vec{k} \times \vec{B}_1\right) &= \vec{k} \times \left(\mu_0 \vec{j} - i\frac{\omega}{c^2}\vec{E}\right) \\ \vec{k}\left(\vec{k} \cdot \vec{B}_1\right) - k^2\vec{B}_1 &= \mu_0 \vec{k} \times \left(\frac{-n_0 e\vec{v}}{i}\right) - \frac{\omega^2}{c^2}\vec{B}_1 \end{aligned}$$

And since $\vec{k} \cdot \vec{B}_1 = 0$ (remember that $\vec{\nabla} \cdot \vec{B}_0$ must be zero too), we obtain the wave equation:

$$\left(\frac{\omega^2}{c^2} - k^2\right)\vec{B}_1 = i\mu_0 n_0 e\vec{k} \times \vec{v} \tag{34}$$

From the equation of motion:

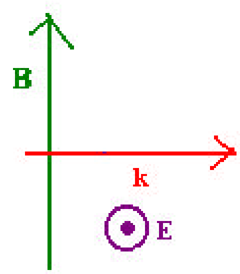
$$\begin{aligned}
-i\omega m\vec{k}\times\vec{v} &= -e\left[\vec{k}\times\vec{E}+\vec{k}\times\left(\vec{v}\times\vec{B}_0\right)\right] \\
&= -e\omega\vec{B}_1 - e\left[\vec{v}\left(\vec{k}\cdot\vec{B}_0\right)-\vec{B}_0\left(\vec{k}\cdot\vec{v}\right)\right] \\
&= -e\omega\vec{B}_1
\end{aligned}$$

since $\vec{k}$ is perpendicular to $\vec{B}_0$ by assumption, and $\vec{k}\cdot\vec{v}=0$ follows immediately from equation (33). Substituting into Ampere's law, we find:

$$\begin{aligned}
\left(\frac{\omega^2}{c^2}-k^2\right)\vec{B}_1 &= i\mu_0 n_0 e\frac{e\omega\vec{B}_1}{i\omega m} \\
\omega^2 &= \omega_p^2 + k^2c^2 \qquad (35)
\end{aligned}$$

7.1.2 The extraordinary wave

Now we look at the case of polarization perpendicular to $\vec{B}_0$. Here the electric field drives electron motion perpendicular to $\vec{B}_0$, and thus the magnetic force comes into play. Notice also that as the electron gyrates, it will have a component of velocity along $\vec{k}$, and so the wave cannot be purely transverse– it is a mixture.



Let's begin by solving the equation of motion, because the results will be useful for other waves too. Choose a coordinate system with $z-$axis parallel to $\vec{B}_0$. Then the $z-$component is the easiest:

$$-i\omega m v_z = -eE_z \Rightarrow v_z = -i\frac{e}{\omega m}E_z \qquad (36)$$

The x-and y-component equations are coupled:

$$\begin{aligned}
-i\omega m v_x &= -eE_x - e\left(\vec{v}\times\vec{B}_0\right)_x = -eE_x - ev_yB_0 \\
-i\omega m v_y &= -eE_y - e\left(\vec{v}\times\vec{B}_0\right)_y = -eE_y + ev_xB_0
\end{aligned}$$

Thus, using the second relation in the first,

$$\begin{aligned}
-i\omega m v_x &= -eE_x - eB_0 \frac{-eE_y + ev_x B_0}{-i\omega m} \\
v_x &= \frac{-e}{-i\omega m} E_x + e^2 \frac{B_0}{-\omega^2 m^2} E_y - e^2 \frac{B_0^2}{-\omega^2 m^2} v_x \\
&= -\frac{ie}{\omega m} E_x - e \frac{\omega_c}{\omega^2 m} E_y + \frac{\omega_c^2}{\omega^2} v_x \\
v_x &= -\frac{e}{\omega m} \frac{iE_x + \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} \qquad (37)
\end{aligned}$$

and then using the v_y equation

$$\begin{aligned}
v_y &= \frac{e}{i\omega m} E_y + \frac{\omega_c}{i\omega} \frac{e}{\omega m} \frac{iE_x + \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} \\
&= \frac{e}{\omega m} \frac{\frac{\omega_c}{\omega} E_x - iE_y}{1 - \frac{\omega_c^2}{\omega^2}} \qquad (38)
\end{aligned}$$

From these relation we obtain the components of the current:

$$j_z = -nev_z = ine \frac{e}{\omega m} E_z = i\varepsilon_0 \frac{\omega_p^2}{\omega} E_z \qquad (39)$$

and

$$\begin{aligned}
j_x &= -nev_x = \frac{ne^2}{\omega m} \frac{iE_x + \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} = \varepsilon_0 \frac{\omega_p^2}{\omega} \left(\frac{iE_x + \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} \right) \qquad (40) \\
j_y &= \varepsilon_0 \frac{\omega_p^2}{\omega} \left(\frac{iE_y - \frac{\omega_c}{\omega} E_x}{1 - \frac{\omega_c^2}{\omega^2}} \right) \qquad (41)
\end{aligned}$$

Note the resonance at $\omega = \omega_c$.

Now we are ready to find the dispersion relation for the extraordinary (X-) wave. First we choose the $x-$axis along $\vec{k}$, with $\vec{B}_0$ in the $z-$direction, and $\vec{E}$ in the $x - y$ plane. Then Ampere's law becomes:

$$\left(\frac{\omega^2}{c^2} - k^2 \right) \vec{B}_1 = -i\mu_0 \vec{k} \times \vec{j}$$

Here the wave number and current lie in the $x - y$ plane, so $\vec{B}_1$ is in the $z-$ direction. (We already know $\vec{B}_1$ has no $x-$component, because its divergence is zero.) Thus

$$\left(\frac{\omega^2}{c^2} - k^2 \right) B_z = -i\mu_0 k j_y = -i\mu_0 k \varepsilon_0 \frac{\omega_p^2}{\omega} \left(\frac{iE_y - \frac{\omega_c}{\omega} E_x}{1 - \frac{\omega_c^2}{\omega^2}} \right) \qquad (42)$$

To find the components of $\vec{E}$, we use Faraday's law:

$$\begin{aligned}
\vec{k} \times \vec{E} &= \omega \vec{B}_1 \\
kE_y &= \omega B_z \qquad (43)
\end{aligned}$$

We are short one equation, and that must be Poisson's equation, which determines the longitudinal component of $\vec{E}$:

$$\begin{aligned} ikE_x &= -\frac{n_1 e}{\varepsilon_0} = -\frac{e}{\varepsilon_0} n_0 \frac{\vec{k}\cdot\vec{v}}{\omega} = -\frac{e}{\varepsilon_0} n_0 \frac{k}{\omega} v_x \\ &= \frac{e}{\varepsilon_0} n_0 \frac{k}{\omega} \frac{e}{\omega m} \frac{iE_x + \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} \\ iE_x &= \frac{\omega_p^2}{\omega^2} \frac{iE_x + \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} \end{aligned}$$

Thus, rearranging and using Faraday's law (43):

$$\begin{aligned} E_x \left(1 - \frac{\omega_c^2}{\omega^2} - \frac{\omega_p^2}{\omega^2}\right) &= \frac{\omega_c}{i\omega} \frac{\omega_p^2}{\omega^2} E_y \\ E_x \left(1 - \frac{\omega_H^2}{\omega^2}\right) &= -i \frac{\omega_c}{\omega} \frac{\omega_p^2}{\omega^2} E_y = -i \frac{\omega_c}{k} \frac{\omega_p^2}{\omega^2} B_z \end{aligned}$$

Substituting into Ampere's law (42), we have:

$$\begin{aligned} \left(\frac{\omega^2}{c^2} - k^2\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right) B_z &= -\frac{i}{c^2} \frac{\omega_p^2}{\omega} \left(i\omega + i \frac{\omega_c^2}{\omega} \frac{\frac{\omega_p^2}{\omega^2}}{\left(1 - \frac{\omega_H^2}{\omega^2}\right)}\right) B_z \\ \left(\omega^2 - c^2 k^2\right) &= \omega_p^2 \frac{1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_c^2}{\omega^2} + \frac{\omega_p^2}{\omega^2}\frac{\omega_c^2}{\omega^2}}{\left(1 - \frac{\omega_c^2}{\omega^2}\right)\left(1 - \frac{\omega_H^2}{\omega^2}\right)} = \omega_p^2 \frac{\left(\omega^2 - \omega_p^2\right)}{\left(\omega^2 - \omega_H^2\right)} \end{aligned}$$

So the dispersion relation for the X-wave is

$$\omega^2 = c^2 k^2 + \omega_p^2 \frac{\left(\omega^2 - \omega_p^2\right)}{\left(\omega^2 - \omega_H^2\right)} \tag{44}$$

OK, let's investigate this dispersion relation. First note that for $\omega \gg \omega_{p,}\ \omega_H$ we retrieve the zero-field result (16), and as $\omega \to \infty$, we get back the vacuum relation $\omega = ck$. Now let's see whether there are any stop bands – bands of frequencies at which the wave cannot propagate.

Cut-offs occur where $k = 0$. The frequencies are then given by:

$$\omega^2 = \omega_p^2 \frac{\left(\omega^2 - \omega_p^2\right)}{\left(\omega^2 - \omega_H^2\right)} = \omega_p^2 \frac{1 - \frac{\omega_p^2}{\omega^2}}{1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_c^2}{\omega^2}} \tag{45}$$

Rearranging and dividing, we find:

$$\frac{\omega_p^2}{\omega^2} = 1 - \frac{\omega_c^2/\omega^2}{1 - \frac{\omega_p^2}{\omega^2}}$$

or

$$\left(1-\frac{\omega_p^2}{\omega^2}\right)^2=\frac{\omega_c^2}{\omega^2}$$

and taking the square root:

$$1-\frac{\omega_p^2}{\omega^2}=\pm\frac{\omega_c}{\omega}$$

There are two roots, so we get two quadratics for ω :

$$\omega^2\pm\omega\omega_c-\omega_p^2=0 \tag{46}$$

with the four solutions:

$$\omega=\frac{\mp\omega_c\pm\sqrt{\omega_c^2+4\omega_p^2}}{2}$$

Negative frequencies do not have physical meaning, so the two meaningful solutions are

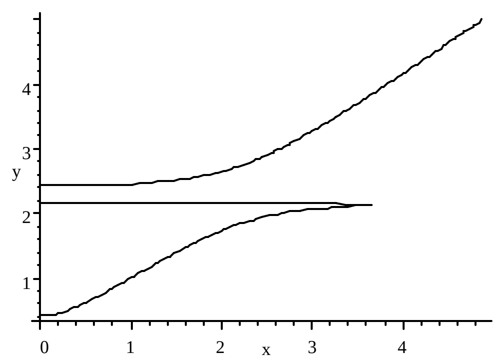
$$\omega_R=\frac{1}{2}\left(\sqrt{\omega_c^2+4\omega_p^2}+\omega_c\right) \tag{47}$$

and

$$\omega_L=\frac{1}{2}\left(\sqrt{\omega_c^2+4\omega_p^2}-\omega_c\right) \tag{48}$$

The subscripts refer to R for right and L for left. We'll see why later.

Let $y=\omega/\omega_p$, $\omega_c/\omega_p=2$, $\omega_H/\omega_p=\sqrt{5}$. Then we have $\frac{\omega_R}{\omega_p}=\frac{1}{2}\left(\sqrt{8}+2\right)=2.4142$ and $\frac{\omega_L}{\omega_p}=\frac{1}{2}\left(\sqrt{8}-2\right)=.41421$. Then the plot of ω versus k looks like:



ω versus k for the X-mode

There is a stop band between ω_R and ω_H and again below ω_L.

We can gain more insight by looking at the refractive index:

$$\begin{aligned} n^2 &= \frac{c^2k^2}{\omega^2}=1-\frac{\omega_p^2}{\omega^2}\frac{\left(\omega^2-\omega_p^2\right)}{\left(\omega^2-\omega_H^2\right)} \\ &= \frac{\omega^2\left(\omega^2-\omega_H^2\right)-\omega_p^2\left(\omega^2-\omega_p^2\right)}{\omega^2\left(\omega^2-\omega_H^2\right)} \end{aligned}$$

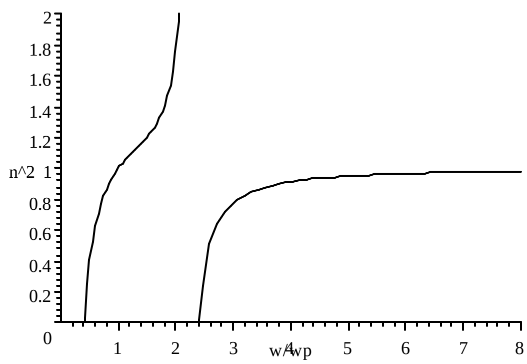
Comparing with equation (45), we see that we can factor the numerator:

$$n^2 = \frac{\left(\omega^2 - \omega_R^2\right)\left(\omega^2 - \omega_L^2\right)}{\omega^2\left(\omega^2 - \omega_H^2\right)}$$

This confirms our previous result that there are cutoffs at ω_R and ω_L, and shows that the stop bands ($n^2 < 0$) occur for $\omega_R > \omega > \omega_H$ and $\omega < \omega_L$. (To obtain these results, note that $\omega_R > \omega_H > \omega_L$)

There is also a *resonance* ($n \to \infty$) at $\omega = \omega_H$. (Also $n^2 \to -\infty$ at $\omega = 0$, but this one arises because we have neglected ion motion.) Note also that $n \to 1$ as $\omega \to \infty$.

The plot looks like:



n^2 versus ω for the X-wave

7.2 Electromagnetic waves propagating along $\vec{B}_0$.

Transverse waves propagating along $\vec{B}_0$ have the electric field vector in the plane perpendicular to $\vec{B}_0$, and thus the electron motions, while affected by the magnetic force, remain transverse. Thus this wave is purely electromagnetic. We again choose our $z-$axis along the magnetic field direction. We can use our previous results (37) and (38) (or equivalently 40 and 41) for the particle motions. Now Faraday's law gives both components of $\vec{E}$, but there are also two components of $\vec{B}_1$:

$$kE_y = -\omega B_x \text{ and } kE_x = \omega B_y$$

Ampere's law (34) has two components. Using relation (38) for the electron velocity, the $x-$component is

$$\left(\frac{\omega^2}{c^2} - k^2\right) B_x = i\mu_0 k j_y \tag{49}$$

$$-\left(\frac{\omega^2}{c^2} - k^2\right)\frac{k}{\omega} E_y = i\mu_0 k \varepsilon_0 \frac{\omega_p^2}{\omega}\left(\frac{iE_y - \frac{\omega_c}{\omega}E_x}{1 - \frac{\omega_c^2}{\omega^2}}\right) \tag{50}$$

and similarly for the $y-$component

$$\begin{aligned}\left(\frac{\omega^2}{c^2}-k^2\right)B_y &= -i\mu_0 k j_x &(51)\\ \left(\frac{\omega^2}{c^2}-k^2\right)\frac{k}{\omega}E_x &= -i\mu_0 k\varepsilon_0\frac{\omega_p^2}{\omega}\left(\frac{iE_x+\frac{\omega_c}{\omega}E_y}{1-\frac{\omega_c^2}{\omega^2}}\right) &(52)\end{aligned}$$

Rearrange (50) to get:

$$i\omega_p^2\left(iE_y-\frac{\omega_c}{\omega}E_x\right)=-\left(\omega^2-k^2c^2\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)E_y$$

$$i\omega_p^2\frac{\omega_c}{\omega}E_x+E_y\left(\omega_p^2-\left(\omega^2-k^2c^2\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)\right)=0 \tag{53}$$

and (52) becomes

$$-i\omega_p^2\left(iE_x+\frac{\omega_c}{\omega}E_y\right)=\left(\omega^2-k^2c^2\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)E_x$$

$$E_x\left(\omega_p^2-\left(\omega^2-k^2c^2\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)\right)-i\omega_p^2\frac{\omega_c}{\omega}E_y=0 \tag{54}$$

For a non-zero solution for E_x and E_y, the determinant of the coefficients in equations (53) and (54) must be zero, so:

$$\left(\omega_p^2\frac{\omega_c}{\omega}\right)^2-\left[\omega_p^2-\left(\omega^2-k^2c^2\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)\right]^2=0$$

or, taking the square root,

$$\begin{aligned}\omega_p^2\frac{\omega_c}{\omega} &= \pm\left[\omega_p^2-\left(\omega^2-k^2c^2\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)\right]\\ \omega_p^2\left(1\pm\frac{\omega_c}{\omega}\right) &= \left(\omega^2-k^2c^2\right)\left(1-\frac{\omega_c^2}{\omega^2}\right)\\ \omega_p^2 &= \left(\omega^2-k^2c^2\right)\left(1\pm\frac{\omega_c}{\omega}\right)\end{aligned}$$

and so we obtain the dispersion relation

$$\omega^2=k^2c^2+\frac{\omega_p^2}{\left(1\pm\frac{\omega_c}{\omega}\right)} \tag{55}$$

There is a resonance at $\omega=\omega_c$ for one of the two possible frequencies. Note again that for $\omega\gg\omega_c$ we get back the result for an unmagnetized plasma, and we retrieve $\omega=ck$ as $\omega\to\infty$. The cutoffs ($k=0$) are at

$$\begin{aligned}\omega^2 &= \frac{\omega_p^2}{\left(1\pm\frac{\omega_c}{\omega}\right)}\\ \omega^2\left(1\pm\frac{\omega_c}{\omega}\right) &= \omega_p^2\end{aligned}$$

which is equation (46), with the same two solutions ω_R (bottom sign) and ω_L (top sign).

Now let's solve for the electric field components, using equation (53) with our solution for ω inserted:

$$\begin{aligned} i\omega_p^2\frac{\omega_c}{\omega}E_x + E_y\left[\omega_p^2 - \frac{\omega_p^2}{\left(1\pm\frac{\omega_c}{\omega}\right)}\left(1-\frac{\omega_c^2}{\omega^2}\right)\right] &= 0 \\ i\frac{\omega_c}{\omega}E_x + E_y\left[1-\left(1\mp\frac{\omega_c}{\omega}\right)\right] &= 0 \\ i\frac{\omega_c}{\omega}E_x + E_y\left(\pm\frac{\omega_c}{\omega}\right) &= 0 \\ E_y &= \mp iE_x \end{aligned}$$

This solution corresponds to circularly polarized waves. If

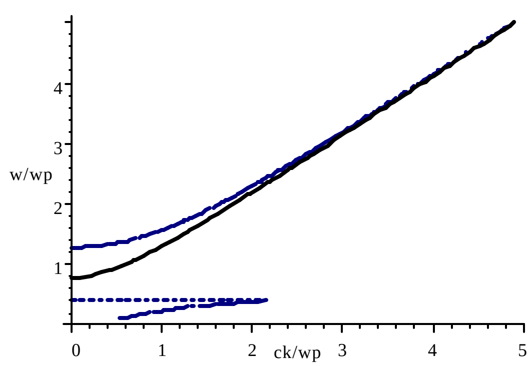
$$E_x = E_0\cos\omega t = \mathrm{Re}\left(E_0e^{-i\omega t}\right)$$

then

$$\begin{aligned} E_y &= \mathrm{Re}\left(\mp iE_0e^{-i\omega t}\right) = \mp E_0\,\mathrm{Re}\left(i\cos\omega t + \sin\omega t\right) \\ &= \mp E_0\sin\omega t \end{aligned}$$

Thus with the plus (lower) sign, the $\vec{E}$ vector rotates counter-clockwise, and with the minus (upper) sign the vector rotates clockwise. The waves have circular polarization. Putting your thumb along the direction of propagation (the $z-$direction), your right hand gives the direction of rotation with the plus sign- these are right-hand circular waves. Conversely, with the minus sign you need your left hand– these are left hand circular waves. Thus the RH circular wave has a cutoff at ω_R and the left hand circular wave has a cutoff at ω_L.

For the plot, let $\omega_p/\omega_c = 2$.

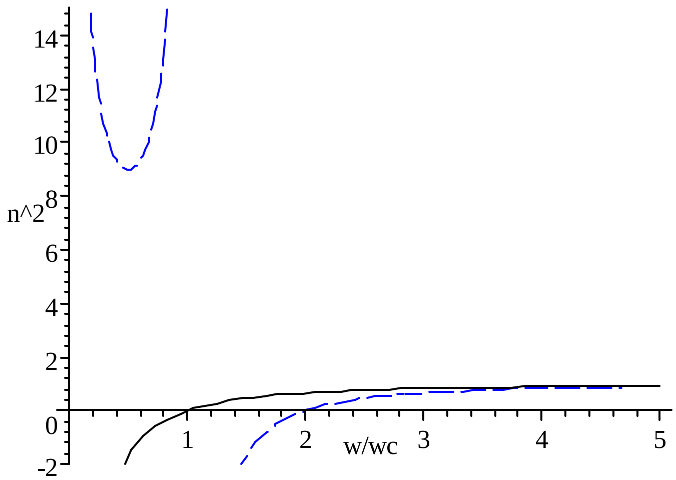


ω versus k for R and L circular waves

The refractive index is:

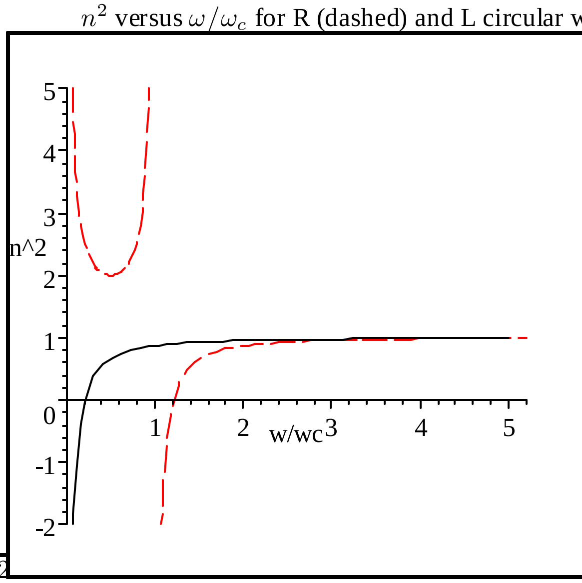
$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{\left(1\pm\frac{\omega_c}{\omega}\right)} \tag{56}$$

and there is a resonance at $\omega = \omega_c$ for the right circular wave. In this wave the electric field circulates in the same direction as the electron, and when $\omega = \omega_c$ the field is able to transfer energy to the electron continuously, producing the resonance. Below we shall show that when ion motion is included the left circular wave exhibits a resonance at $_c$.



$\omega_p/\omega_c =$

n^2 versus ω/ω_c for R (dashed) and L circular waves



$\sqrt{2}$ $\omega_p/\omega_c = 1/2$

n^2 versus ω/ω_c

7.3 Observational effects

The dispersion relations that we have derived for EM waves propagating in plasmas have interesting observational consequences.

7.3.1 Plasma dispersion

When a signal has a well-defined time-structure, the dispersion relation can affect how we observe it. For example, radio pulsars produce a well-defined pulse over a range of radio frequencies ranging from MHz to GHz. The different frequencies travel at different group speeds, so the pulse does not arrive at Earth at the same time at each frequency.

In an unmagnetized plasma (or for $\omega \gg \omega_c$) we have the dispersion relation

$$\omega^2 = k^2c^2 + \omega_p^2$$

and thus

$$\begin{aligned} 2\omega\frac{d\omega}{dk} &= 2kc^2 \\ \frac{d\omega}{dk} &= v_g = \frac{c}{\omega}\sqrt{\omega^2 - \omega_p^2} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \\ &\simeq c\left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right) \quad \text{for } \omega \gg \omega_p \end{aligned}$$

Thus the pulses at higher frequencies travel faster, and arrive sooner, than those at lower frequencies.

In the interstellar medium, the electron density is about 0.05 electrons/cm^3, so the plasma frequency is

$$\begin{aligned} f_p &= \frac{1}{2\pi}\sqrt{\frac{ne^2}{\varepsilon_0 m}} = \frac{1}{2\pi}\sqrt{\frac{(5 \times 10^4 \text{ m}^{-3})\,(1.6 \times 10^{-19} \text{ C})^2}{(8.85 \times 10^{-12} \text{ F/m})\,(9 \times 10^{-31} \text{ kg})}} \\ &= 2. \times 10^3 \text{ Hz}, \end{aligned}$$

much less than the observed frequencies in the radio band.

The magnetic field is about 10^{-6} Gauss, so $\omega_c \simeq \frac{1.6\times10^{-19}\text{ C}}{9\times10^{-31}\text{ kg}}10^{-6}\text{ G} = 1.\,8 \times 10^5$ Hz is also much less than the observed frequencies.

The time taken for a pulse to reach earth is

$$T = \frac{L}{v_g}$$

where L is the distance to the pulsar. The difference in arrival times between two pulses at frequencies f_1 and f_2 is

$$\begin{aligned}
\Delta T &= \frac{L}{v_{g1}} - \frac{L}{v_{g2}} = \frac{L}{c}\left[\frac{1}{1-\frac{1}{2}\frac{f_p^2}{f_1^2}} - \frac{1}{1-\frac{1}{2}\frac{f_p^2}{f_2^2}}\right] \\
&= \frac{L}{2c}\left(\frac{f_p^2}{f_1^2} - \frac{f_p^2}{f_2^2}\right) \\
&= \frac{L}{2c} f_p^2 \left(\frac{1}{f_1^2} - \frac{1}{f_2^2}\right) \qquad (57)
\end{aligned}$$

For a typical pulsar at a distance of approximately 1 kpc= 3×10^{19} m, the time delay between two pulses at 1 and 5 Ghz is

$$\begin{aligned}
\Delta T &= \frac{3 \times 10^{19}\ \text{m}}{6 \times 10^{8}\ \text{m/s}} \left(2.\ \times 10^3\ \text{Hz}\right)^2 \left(\frac{1}{10^9\ \text{Hz}}\right)^2 \left(1 - \frac{1}{25}\right) \\
&= 0.192\ \text{s}
\end{aligned}$$

This time delay is easily detectable.

Since $f_p^2 \propto n_0$, measurement of the time delay provides a measurement of the plasma density along the line of sight to the pulsar. See. eg, The Astrophysical Journal, Volume 645, Issue 1, pp. 303-313, Astrophysical Journal, Part 1 (ISSN 0004-637X), vol. 385, Jan. 20, 1992, p. 273-281

7.3.2 Faraday Rotation

When a wave travels along the magnetic field, we have shown that the normal modes are the left-and right circularly polarized waves. A wave that is emitted as a linearly polarized wave will travel as the sum of two circularly polarized waves. We write the electric field vector for a wave travelling in the $z-$direction as:

$$\vec{E} = E_0 \hat{\mathbf{x}} \exp\left(ikz - i\omega t\right)$$

where as usual the real part is assumed, and we have chosen the $x-$axis along the electric field direction. We decompose the wave at the source into two circularly polarized waves:

$$\begin{aligned}
\vec{E}\left(0\right) &= E_0 \left(\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{2} + \frac{\hat{\mathbf{x}}-i\hat{\mathbf{y}}}{2}\right) \\
&= \vec{E}_R + i\vec{E}_L
\end{aligned}$$

The right and left circularly polarized waves have different phase speeds, so at some distance z from the source, we have:

$$\begin{aligned}
\vec{E}\left(z\right) &= E_0 \left(\frac{\hat{\mathbf{x}}+i\hat{\mathbf{y}}}{2} \exp\left(ik_R z\right) + \frac{\hat{\mathbf{x}}-i\hat{\mathbf{y}}}{2} \exp\left(ik_L z\right)\right) e^{-i\omega t} \\
&= E_0 \left(\frac{\hat{\mathbf{x}}}{2}\left(\exp\left(ik_R z\right) + \exp\left(ik_L z\right)\right) + i\frac{\hat{\mathbf{y}}}{2}\left(\exp\left(ik_R z\right) - \exp\left(ik_L z\right)\right)\right) e^{-i\omega t}
\end{aligned}$$

The angle between the electric field vector and the $x-$axis is given by:

$$\tan\theta = \frac{E_y}{E_x} = i\frac{\exp(ik_R z) - \exp(ik_L z)}{\exp(ik_R z) + \exp(ik_L z)}$$

which is not zero for $z > 0$ if $k_R \neq k_L$. Thus the wave vector rotates as the wave propagates. This phenomenon is called *Faraday rotation*.

We use the dispersion relation (55) (or equivalently (56)) to get

$$\frac{c^2 k_{R,L}^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{\left(1 \pm \frac{\omega_c}{\omega}\right)}$$

First let's assume $\omega \gg \omega_p,\, \omega_c$ which is always true in astronomical applications (but may *not* be in lab situations- so beware!). Then we can expand to get

$$\begin{aligned} \frac{ck_{R,L}}{\omega} &= \sqrt{1 - \frac{\omega_p^2/\omega^2}{\left(1 \pm \frac{\omega_c}{\omega}\right)}} = 1 - \frac{1}{2}\frac{\omega_p^2/\omega^2}{\left(1 \pm \frac{\omega_c}{\omega}\right)} \\ &= 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\left(1 \mp \frac{\omega_c}{\omega}\right) \qquad (58) \end{aligned}$$

where the top sign refers to the left circular wave and the bottom to the right circular wave. The two results differ only in the small term in $\omega_p^2\omega_c/\omega^3$.

Let's rewrite our result for the angle to exhibit its dependence on the difference of the two wave numbers:

$$\begin{aligned} \tan\theta &= \frac{E_y}{E_x} = i\frac{\exp(ik_R z) - \exp(ik_L z)}{\exp(ik_R z) + \exp(ik_L z)} \frac{\exp\left[-i\frac{z}{2}(k_L + k_R)\right]}{\exp\left[-i\frac{z}{2}(k_L + k_R)\right]} \\ &= -\frac{1}{i}\frac{\exp\left[i(k_R - k_L)\frac{z}{2}\right] - \exp\left[-i(k_R - k_L)\frac{z}{2}\right]}{\exp\left[i(k_R - k_L)\frac{z}{2}\right] + \exp\left[-i(k_R - k_L)\frac{z}{2}\right]} \\ &= \frac{\sin\left[\left(k_L - k_R\frac{z}{2}\right)\right]}{\cos\left[(k_L - k_R)\frac{z}{2}\right]} = \tan\left[(k_L - k_R)\frac{z}{2}\right] \end{aligned}$$

Thus

$$\theta = \left[(k_L - k_R)\frac{z}{2}\right] \pm 2n\pi$$

Using our result (58), we find

$$\begin{aligned} k_L - k_R &= \frac{\omega}{c}\left[-\frac{1}{2}\frac{\omega_p^2}{\omega^2}\left(1 - \frac{\omega_c}{\omega}\right) + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\left(1 + \frac{\omega_c}{\omega}\right)\right] \\ &= \frac{\omega}{c}\frac{\omega_p^2}{\omega^2}\left(\frac{\omega_c}{\omega}\right) = \frac{\omega_p^2\omega_c}{\omega^2 c} \\ &= \frac{n_0 e^2}{\varepsilon_0 m}\frac{eB_0}{m}\frac{1}{\omega^2 c} = \frac{e^3}{\varepsilon_0 m^2 c}\frac{n_0 B_0}{\omega^2} \end{aligned}$$

and thus the rotation angle determines the product $n_0 B_0$ of the plasma density and magnetic field strength.

$$\theta = \frac{e^3}{\varepsilon_0 m^2 c}\frac{n_0 B_0}{\omega^2}\frac{z}{2} \pm 2n\pi$$

where z is the distance between source and observer. Measurements at several frequencies are necessary to remove the ambiguity due to the unknown factor $2n\pi$.

For a pulsar at a distance of 1 kpc with $n_0 = 0.05\ \text{cm}^{-3}$ and $B_0 = 3\ \mu\text{G}$, the expected rotation is:

$$\begin{aligned}
\theta &= \frac{e^3}{2\varepsilon_0 m^2 c} n_0 B_0 \frac{(2\pi)^2}{f^2} z \\
&= \frac{2\pi^2 e^3}{\varepsilon_0 m^2 c} n_0 B_0 \frac{z}{f^2} \\
&= \frac{2\pi^2 \left(1.6 \times 10^{-19}\ \text{C}\right)^3}{\left(8.85 \times 10^{-12}\ \text{F/m}\right)\left(9 \times 10^{-31}\ \text{kg}\right)^2 \left(3 \times 10^8\ \text{m/s}\right)} \frac{n_0 B_0 z}{f^2} \\
&= 3.76 \times 10^7 \frac{\text{C}^3}{\text{F} \cdot \text{kg}^2} \frac{n_0 B_0 z}{f^2}
\end{aligned}$$

Now a Farad is a C/V and a V/m is a N/C, so a F = C^2/J, thus the units are C·J·s/kg^2 =C·kg·m^2/kg^2.s= C·m^2/kg·s. Now put in the numbers for the physical parameters n_0, B_0 and z :

$$\begin{aligned}
\theta &= 3.76 \times 10^7 \frac{\text{C} \cdot \text{m}^2}{\text{kg} \cdot \text{s}} \frac{\left(5 \times 10^4\ \text{m}^{-3}\right)\left(3 \times 10^{-10}\ \text{T}\right)\left(3 \times 10^{19}\ \text{m}\right)}{f^2} \\
&= \frac{1.692 \times 10^{22}}{f^2} \frac{\text{C} \cdot \text{T}}{\text{kg} \cdot \text{s}}
\end{aligned}$$

Now a N equals a C·T·m/s (from the force law), so the units now are

$$\frac{\text{C} \cdot \text{T}}{\text{kg} \cdot \text{s}} = \frac{\text{N}}{\text{kg} \cdot \text{m}} = \frac{1}{\text{s}^2}$$

which is exactly what we need!

At $f = 1$ GHz we get

$$\theta = \frac{1.7 \times 10^{22}}{10^{18}} = 1.7 \times 10^4 \text{ radians}$$

which shows that we do indeed expect the wave to rotate around more than once under some circumstances.

See, eg, The Astrophysical Journal, Volume 642, Issue 2, pp. 868-881. (2006)

7.3.3 Whistlers

Let's look at waves on the lower branch of the R-wave dispersion relation, i.e. at frequencies below ω_c. (This is the dashed "bucket" in the upper left corner of the n^2 versus ω plots.) Here we have the dispersion relation

$$\frac{c^2 k_R^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{\left(1 - \frac{\omega_c}{\omega}\right)} = 1 + \frac{\omega_p^2/\omega^2}{\left(\frac{\omega_c}{\omega} - 1\right)}$$

At frequencies well below the cyclotron frequency, we can approximate the denominator to get:

$$c^2 k_R^2 \simeq \omega^2 + \omega_p^2 \left(\frac{\omega}{\omega_c}\right)$$

and if the frequency ω is also well below ω_p, we will have the approximate relation

$$ck_R \simeq \omega_p \sqrt{\frac{\omega}{\omega_c}}$$

or equivalently

$$\omega = \frac{(ck)^2\, \omega_c}{\omega_p^2}$$

These waves have a phase speed

$$v_\phi = \frac{\omega}{k} = c\frac{\sqrt{\omega\omega_c}}{\omega_p}$$

and a group speed

$$v_g = \frac{d\omega}{dk} = 2\frac{c^2 k\omega_c}{\omega_p^2} = 2\frac{c\omega_c}{\omega_p^2}\omega_p\sqrt{\frac{\omega}{\omega_c}} = 2c\frac{\sqrt{\omega\omega_c}}{\omega_p} = 2v_\phi$$

In the ionosphere at 5 R_E the plasma frequency is about 10^4 Hz and the cyclotron frequency is about 2.9 MHz. Thus waves in the kHz band satisfy the constraints we have imposed, and would have this group speed. The group speed decreases with frequency, thus a pulsed signal would arrive at the observer high frequency first, and lower frequencies later. Put through an audio amplifier, the signal would sound like a whistle, hence the name: *whistler.*

During WWI radio operators used frequencies around 10 kHz and heard whistling signals that they interpreted as enemy shells. They turned out to be lighning pulses from the southern hemisphere that had travelled along the field lines to the north.

See also Journal of Geophysical Research, vol. 86, June 1, 1981, p. 4471-4492. for example.

Or The Astrophysical Journal, Volume 610, Issue 1, pp. 550-571 (2004)

7.4 Low frequency waves: propagation along the magnetic field.

For low-frequency waves we need to include the ion motion in the current.

$$\vec{j} = n_0 e\left(\vec{v}_i - \vec{v}_e\right)$$

For waves propagating along $\vec{B}$ (purely transverse waves) there is no density perturbation, and no electrostatic field. We may use all the results from our previous derivation, provided that we adjust the current accordingly. We may also use equations (37), and (38) for the ion motion, if we make the appropriate changes to the charge and mass of the particle. Thus:

$$
\begin{aligned}
j_x &= n_0 e \left[\frac{e}{\omega M} \frac{iE_x - \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} - \left(-\frac{e}{\omega m} \frac{iE_x + \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} \right) \right] \\
&= \left(\frac{n_0 e^2}{\omega M} \frac{iE_x - \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} + \frac{n_0 e^2}{\omega m} \frac{iE_x + \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} \right) \\
&= \frac{\omega_p^2}{\omega} \frac{iE_x - \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} + \frac{\omega_p^2}{\omega} \frac{iE_x + \frac{\omega_c}{\omega} E_y}{1 - \frac{\omega_c^2}{\omega^2}} \\
&= \frac{iE_x}{\omega} \left(\frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} + \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} \right) + \frac{E_y}{\omega^2} \left(\frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} - \frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} \right)
\end{aligned}
$$

and similarly for the $y-$component:

$$
\begin{aligned}
j_y &= n_0 e \left[\frac{-e}{\omega M} \frac{-\frac{\omega_c}{\omega} E_x - iE_y}{1 - \frac{\omega_c^2}{\omega^2}} - \frac{e}{\omega m} \frac{\frac{\omega_c}{\omega} E_x - iE_y}{1 - \frac{\omega_c^2}{\omega^2}} \right] \\
&= \frac{1}{\omega} \left[\omega_p^2 \frac{\frac{\omega_c}{\omega} E_x + iE_y}{1 - \frac{\omega_c^2}{\omega^2}} - \omega_p^2 \frac{\frac{\omega_c}{\omega} E_x - iE_y}{1 - \frac{\omega_c^2}{\omega^2}} \right] \\
&= \frac{1}{\omega} \left[\frac{E_x}{\omega} \left(\frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} - \frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} \right) + iE_y \left(\frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} + \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} \right) \right]
\end{aligned}
$$

and hence Ampere's law takes the form (cf 52):

$$
E_x \left(\frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} + \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} \right) - i\frac{E_y}{\omega} \left(\frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} - \frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} \right) = \left(\omega^2 - k^2 c^2 \right) E_x
$$

and (50)

$$
E_y \left(\frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} + \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} \right) + i\frac{E_x}{\omega} \left(\frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} - \frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} \right) = \left(\omega^2 - k^2 c^2 \right) E_y
$$

giving the dispersion relation

$$
\begin{aligned}
\left(\frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} + \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} - \omega^2 + k^2 c^2 \right)^2 &= \frac{1}{\omega^2} \left(\frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} - \frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} \right)^2 \\
\frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} + \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}} - \omega^2 + k^2 c^2 &= \pm \frac{1}{\omega} \left(\frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} - \frac{\omega_p^2 \omega_c}{1 - \frac{\omega_c^2}{\omega^2}} \right)
\end{aligned}
$$

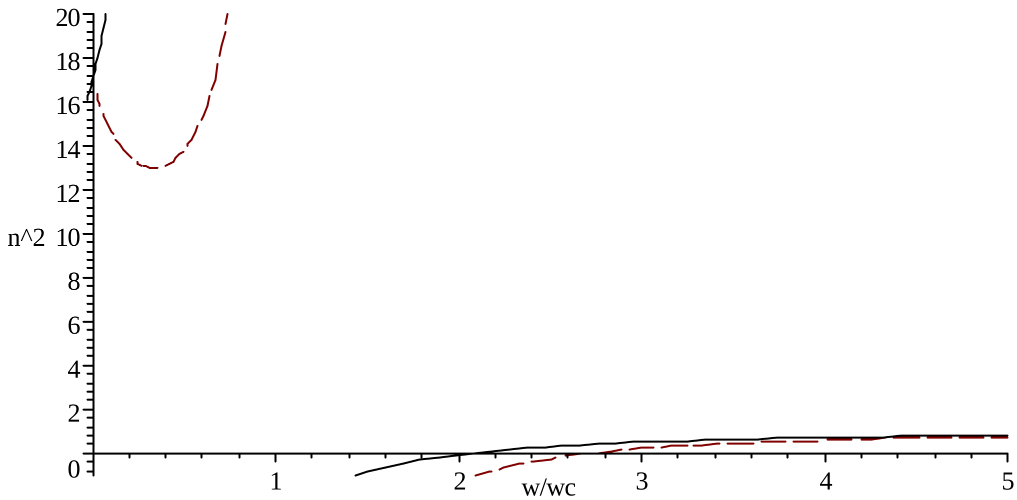
Rearranging:

$$\begin{aligned}\omega^2 &= k^2c^2 + \frac{{}_p^2}{1-\frac{{}_c^2}{\omega^2}}\left(1 \pm \frac{{}_c}{\omega}\right) + \frac{\omega_p^2}{1-\frac{\omega_c^2}{\omega^2}}\left(1 \mp \frac{\omega_c}{\omega}\right) \\ &= k^2c^2 + \frac{{}_p^2}{1 \mp \frac{{}_c}{\omega}} + \frac{\omega_p^2}{1 \pm \frac{\omega_c}{\omega}} \end{aligned} \tag{59}$$

We obtain a second resonance at the ion cyclotron frequency, but this time it is the L wave that has the resonance. The refractive index is:

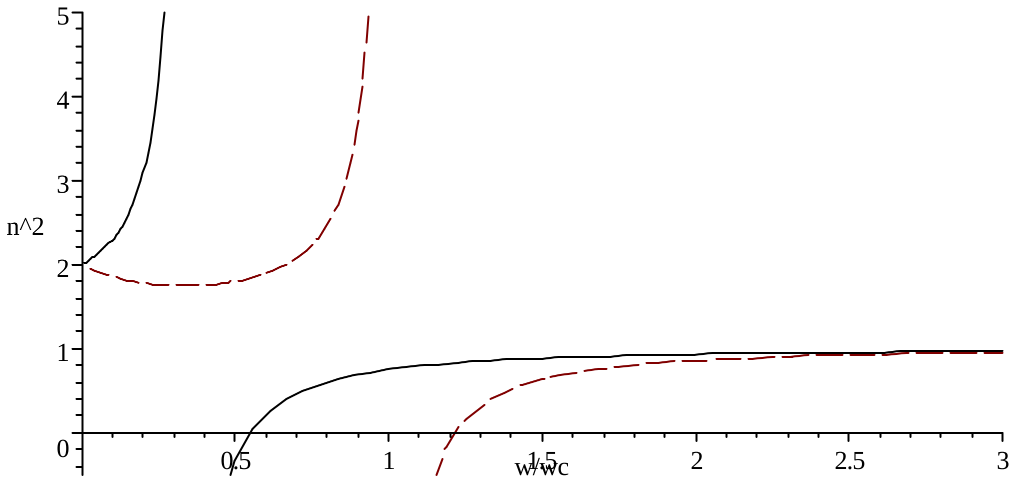
$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{{}_p^2}{\omega\left(\omega \pm {}_c\right)} - \frac{\omega_p^2}{\omega\left(\omega \mp \omega_c\right)} \tag{60}$$

Taking the mass ratio to be 3 rather than the real value of about 2000, and $\omega_p/\omega_c = 2$, the plot now looks like this:



n^2 versus ω/ω_c : solid line- L wave; dashed, R wave

Below: same plot with ω_p/ω_c = 1/2 rather than 2:



n^2 versus ω/ω_c : solid line- L wave; dashed, R wave

Notice that the resonance at $\omega = 0$ has disappeared, and both R and L circular waves have the same n as $\omega \to 0$.

7.5 Low frequency propagation across $\vec{B}_0$.

When the wave is polarized with $\vec{E}$ along $\vec{B}_0$ (O-wave), the ions as well as the electrons can move freely along the field lines, so we expect to get only a minor change to the previous dispersion relation. The current is

$$\begin{aligned}\vec{j} &= n_0 e\left(\vec{v}_i - \vec{v}_e\right) = n_0 e\left(\frac{e}{M} - \frac{-e}{m}\right)\vec{E} \\ &= n_0 e^2\left(\frac{1}{M} + \frac{1}{m}\right)\vec{E}\end{aligned}$$

Combining with Ampere's law, we get the dispersion relation:

$$\omega^2 = c^2 k^2 + \omega_p^2 + \ _p^2$$

Since there is no propagation at frequencies less than ω_p (now strictly $\sqrt{\omega_p^2 + \ _p^2}$) we never have to worry about low frequency waves.

The other polarization (X-wave) is more interesting. The $\vec{E} \times \vec{B}$ drift is parallel to $\vec{k}$, suggesting that we will have a longitudinal component to this wave. We already saw this for the high frequency waves. So let's include the pressure term. The equation of motion is:

$$-i\omega M\vec{v}_i = e\left(\vec{E} + \vec{v}\times\vec{B}_0\right) - i\gamma k_B T_i \vec{k}\frac{n_1}{n_0}$$

and the continuity equation gives the by-now familiar expression for n_1 :

$$\frac{n_1}{n_0} = \frac{\vec{k}\cdot\vec{v}}{\omega}$$

Notice that the density perturbation is more important at low frequencies. Then

$$-i\omega M\vec{v}_i = e\left(\vec{E} + \vec{v}\times\vec{B}_0\right) - i\gamma k_B T_i \vec{k}\frac{\vec{k}\cdot\vec{v}}{\omega}$$

Since these are low frequency waves, we'll also use the plasma approximation: $n_i \simeq n_e$ and consequently $\vec{k}\cdot\vec{v}_i = \vec{k}\cdot\vec{v}_e$. At low frequencies, then, the electrostatic field is shielded and $\vec{k}\cdot\vec{E} = 0$. Taking the dot product of the equation of motion with $\vec{k}$, we get:

$$\begin{aligned}\vec{k}\cdot\vec{v}_i &= \frac{e}{-i\omega M}\vec{k}\cdot\left(\vec{v}\times\vec{B}_0\right) - \frac{\gamma k_B T_i k^2}{-\omega^2 M}\vec{k}\cdot\vec{v} \\ \vec{k}\cdot\vec{v}_i\left(1 - \frac{\gamma k_B T_i k^2}{\omega^2 M}\right) &= \frac{e}{-i\omega M}\left(\vec{k}\times\vec{v}\right)\cdot\vec{B}_0\end{aligned}$$

or, in components, with x along $\vec{k}$ and $\vec{B}$ along z. :

$$v_{ix}\left(1 - \frac{\gamma k_B T_i k^2}{\omega^2 M}\right) = \frac{eB_0}{-i\omega M}v_{iy} = \frac{c}{-i\omega}v_{iy}$$

The $y-$component is the same as before:

$$-i\omega M v_{iy} = eE_y - ev_{ix}B_0 = eE_y - \frac{eB_0}{\left(1 - \frac{\gamma k_B T_i k^2}{\omega^2 M}\right)}\frac{c}{-i\omega}v_{iy}$$

The solution for v_y is modified:

$$v_{iy}\left(1 - \frac{eB_0}{\left(\omega^2 - \frac{\gamma k_B T_i k^2}{M}\right)}\frac{c}{M}\right) = i\frac{e}{\omega M}E_y$$

The parenthesis in the denominator is:

$$\omega^2 - k^2 v_{i,th}^2$$

and thus

$$v_{iy} = i\frac{e}{\omega M}E_y\left(\frac{\omega^2 - k^2 v_{i,th}^2}{\omega^2 - {}^2_c - k^2 v_{i,th}^2}\right)$$

The result for electrons is the same with the usual switch of $e \to -e$ and $m \to M$.

Then substituting into Ampere's law, we have:

$$\left(\frac{\omega^2}{c^2} - k^2\right)\vec{B}_1 = -i\mu_0\vec{k}\times\vec{j} = -i\mu_0\vec{k}\times n_0 e\left(\vec{v}_i - \vec{v}_e\right)$$

The interesting component is the $z-$component:

$$\begin{aligned}\left(\frac{\omega^2}{c^2}-k^2\right)B_z &= -i\mu_0 n_0 ek\left(v_{iy}-v_{ey}\right)\\ &= -i\mu_0 n_0 ek\left(i\frac{e}{\omega M}E_y\left(\frac{\omega^2-k^2v_{i,th}^2}{\omega^2-{}_c^2-k^2v_{i,th}^2}\right)+i\frac{e}{\omega m}E_y\left(\frac{\omega^2-k^2v_{e,th}^2}{\omega^2-\omega_c^2-k^2v_{e,th}^2}\right)\right)\end{aligned}$$

and finally Faraday's law gives us

$$E_y = \frac{\omega}{k}B_z$$

So the dispersion relation is

$$\omega^2 - c^2k^2 = {}_p^2\left(\frac{\omega^2-k^2v_{i,th}^2}{\omega^2-{}_c^2-k^2v_{i,th}^2}\right)+\omega_p^2\left(\frac{\omega^2-k^2v_{e,th}^2}{\omega^2-\omega_c^2-k^2v_{e,th}^2}\right) \tag{61}$$

Check that we have the right limit as $T\to 0$.

Now we recall that these are low frequency waves ($\omega \ll \omega_p$). We are going to make the additional assumptions:

$$\omega_c \gg kv_{th,e}$$

(the electrons travel a distance much less than a wavelength in one gyro-period) and also

$$\omega \ll {}_c \ll \omega_c$$

and

$$kv_{th,i} \ll {}_c$$

With these approximations, the dispersion relation simplifies:

$$\begin{aligned}\omega^2 - c^2k^2 &= {}_p^2\left(\frac{\omega^2-k^2v_{i,th}^2}{-{}_c^2}\right)+\omega_p^2\left(\frac{\omega^2-k^2v_{e,th}^2}{-\omega_c^2}\right)\\ \omega^2\left(1+\frac{{}_p^2}{{}_c^2}+\frac{\omega_p^2}{\omega_c^2}\right) &= k^2\left(c^2+\frac{{}_p^2}{{}_c^2}v_{i,th}^2+\frac{\omega_p^2}{\omega_c^2}v_{e,th}^2\right)\end{aligned}$$

The ratio

$$\frac{{}_p^2}{{}_c^2} = \frac{n_0e^2}{\varepsilon_0 M}\frac{M^2}{e^2B_0^2} = \frac{n_0}{\varepsilon_0}\frac{M}{B_0^2} \tag{62}$$

and similarly for $(\omega_p/\omega_c)^2$. Thus the dispersion relation is:

$$\begin{aligned}\omega^2\left(1+\frac{n_0}{\varepsilon_0}\frac{(M+m)}{B_0^2}\right) &= k^2\left(c^2+\frac{n_0}{\varepsilon_0}\frac{M}{B_0^2}\frac{k_BT_i}{M}+\frac{n_0}{\varepsilon_0}\frac{m}{B_0^2}\frac{k_BT_e}{m}\right)\\ \omega^2\left(1+\frac{\rho}{\varepsilon_0 B_0^2}\right) &= k^2\left(c^2+\frac{n_0}{\varepsilon_0 B_0^2}k_B\left(T_i+T_e\right)\right)\end{aligned}$$

Define a new speed called the Alfven speed:

$$v_A^2 = \frac{B_0^2}{\mu_0\rho} \tag{63}$$

Then

$$\omega^2 = k^2c^2\frac{(v_A^2 + v_s^2)}{(v_A^2 + c^2)} \tag{64}$$

It is often the case that $v_A \ll c$, in which case the relation simplifies again, to give:

$$\omega^2 = k^2 \left(v_A^2 + v_s^2\right) \tag{65}$$

The dispersion relation (64) or (65) decribes *magnetosonic waves.*

7.6 Very Low frequency propagation along $\vec{B}_0$.

We start with our previous result (59)

$$\omega^2 = k^2c^2 + \frac{\omega_p^2}{1 \mp \frac{\omega_c}{\omega}} + \frac{{}_p^2}{1 \pm \frac{{}_c}{\omega}}$$

Now we approximate for $\omega \ll {}_c$, (and hence also $\omega \ll \omega_c$). Then

$$\begin{aligned}
\omega^2 &= k^2c^2 \mp \frac{\omega\omega_p^2}{\omega_c\left(1 \mp \frac{\omega}{\omega_c}\right)} \pm \frac{\omega \; {}_p^2}{{}_c\left(1 \pm \frac{\omega}{{}_c}\right)} \\
&= k^2c^2 \mp \omega\frac{\omega_p^2}{\omega_c}\left(1 \pm \frac{\omega}{\omega_c} + \frac{\omega^2}{\omega_c^2}\right) \pm \omega\frac{{}_p^2}{{}_c}\left(1 \mp \frac{\omega}{{}_c} + \frac{\omega^2}{{}_c^2}\right) \\
&= k^2c^2 \mp \omega\frac{\omega_p^2}{\omega_c} - \frac{\omega_p^2\omega^2}{\omega_c^2} \mp \frac{\omega_p^2\omega^3}{\omega_c^3} \pm \omega\frac{{}_p^2}{{}_c} - \frac{{}_p^2\omega^2}{{}_c^2} + \frac{{}_p^2\omega^3}{{}_c^3}
\end{aligned}$$

Now

$$\frac{\omega_p^2}{\omega_c} = \frac{n_0e^2}{\varepsilon_0 m}\frac{m}{eB_0} = \frac{n_0e}{\varepsilon_0 B_0} = \frac{{}_p^2}{{}_c}$$

and thus the first order terms in ω cancel. Dropping the third order terms, we get:

$$\omega^2 = k^2c^2 - \frac{\omega_p^2\omega^2}{\omega_c^2} - \frac{{}_p^2\omega^2}{{}_c^2}$$

Now

$$\frac{\omega_p^2}{\omega_c^2} = \frac{n_0e^2}{\varepsilon_0 m}\left(\frac{m}{eB_0}\right)^2 = \frac{n_0}{\varepsilon_0}\frac{m}{B_0^2}$$

(compare 62). Thus

$$\omega^2\left(1 + \frac{n_0}{\varepsilon_0}\frac{(m+M)}{B_0^2}\right) = k^2c^2$$

or

$$\omega = kv_A$$

where

$$v_A = \frac{c}{\sqrt{1 + \rho/\varepsilon_0 B^2}} \tag{66}$$

is the Alfven speed. Let's compare with equation (63). From (66),

$$\begin{aligned} v_{A,2}^2 &= \frac{1}{\mu_0 \varepsilon_o \left(1+\rho/\varepsilon_0 B^2\right)} = \frac{1}{\mu_0 \varepsilon_o + \mu_0 \frac{\rho}{B^2}} = \frac{1}{1/c^2 + 1/v_{A,1}^2} \\ &= \frac{B^2}{\rho\mu_0 \left(1 + B^2/\rho\mu_0 c^2\right)} \end{aligned}$$

Thus the two expressions are the same when

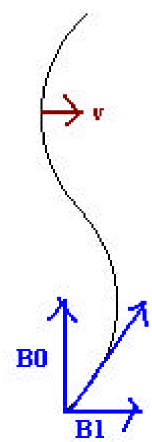
$$\frac{B^2}{\rho\mu_0 c^2} \ll 1$$

This is almost always the case. However, there are some extreme astrophysical situations (near pulsars, for example) where B is very large and ρ is very small and this ratio approaches 1. In these cases we have to use the more exact result (66).

Now let's take a closer look at what is going on in this wave. The wave vector is along $\vec{B}_0$, and both the electric and magnetic fields of the wave lie in the $x-y$ plane. In the low-frequency limit, both the L and R circular waves travel at the same speed. This means that linearly polarized waves are also normal modes. With $\vec{E}$ in the $x-$direction, $\vec{B}_1 = \frac{1}{\omega}\vec{k} \times \vec{E} = \frac{k}{\omega} E\hat{y}$ is in the $y-$direction. The plasma particles undergo an $\vec{E} \times \vec{B}$ drift which is (to first order)

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}_0}{B_0^2} = \frac{E}{B_0}\left(-\hat{\mathbf{y}}\right)$$

The perturbation $\vec{B}_1$ causes the magnetic field to change direction, and since $\vec{B}_1$ has a wave form, the field line has a ripple:



As $\vec{B}_1$ oscillates, the field line ripple moves back and forth. The field line behaves like a vibrating string. The speed of the sideways motion is

$$v = v_\phi \frac{B_1}{B_0} = \frac{\omega B_1}{k B_0} = \frac{E}{B_0}$$

which is the same as the particle drift speed. Thus the particles and the field lines move *together*. This phenomenon is sometimes called *flux freezing* as the magnetic flux is frozen into the plasma.

The Alfven speed

$$v = \sqrt{\frac{B^2}{\rho\mu_0}}$$

may be viewed as the speed of a wave on a string with tension

$$T_{\rm mag} = \frac{B^2}{\mu_0}$$

and density ρ. We'll return to these ideas later when we study magnetohydrodynamics (MHD).

The Alfven speed plays an important role in plasmas because flows that are super-Alfvenic often behave like supersonic flows in ordinary fluids. (Notice that as $T \to 0$ the magnetosonic wave also travels at the Alfven speed.) For example, shock waves may form.

The solar wind parameters are: speed about 400 km/s near the Earth, density $n \sim 5$ cm^{-3}, proton cyclotron frequency (from ACE news release) of about 0.17 Hz.

$$\frac{eB}{m} = 0.17 \times 2\pi \text{ rad/s} \Rightarrow B = \frac{.17 \times 2\pi \times 1.7 \times 10^{-27} \text{ kg}}{\text{s}\,(1.6 \times 10^{-19} \text{ C})} = .11349 \times 10^{-7} \frac{\text{kg}}{\text{s} \cdot \text{C}}$$

Now the units: 1 C·T·m/s=1 N=1 kg·m/s^2 and so 1 T = 1 kg/s·C. Thus we have B of about 10^{-8} T. The Alfven speed is then

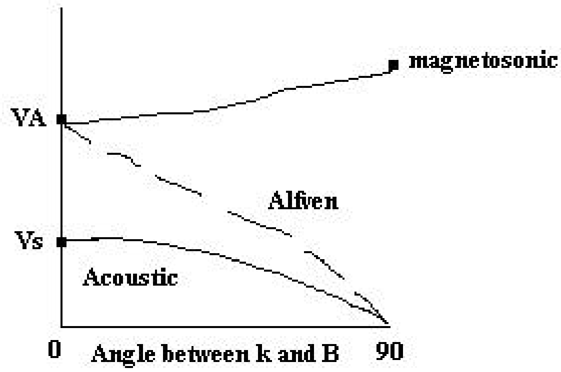
$$v_A \sim \frac{10^{-8}\text{ T}}{\sqrt{\left(5 \times 10^6 \times 1.7 \times 10^{-27} \text{ kg/m}^3\right) 4\pi \times 10^{-7} \text{ N/A}^2}} = 9.6759 \times 10^4 \frac{\text{T}}{\sqrt{\left(\frac{\text{kg}}{\text{m}^3}\frac{\text{kg}\cdot\text{m}}{\text{A}^2\text{s}^2}\right)}}$$

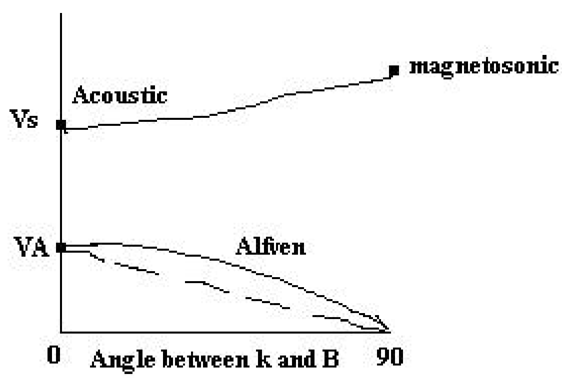
$$= 9.676 \times 10^4 \frac{\text{kg/s} \cdot \text{C}}{\frac{\text{kg}}{\text{m}\cdot\text{C}}} = 9.7 \times 10^4 \text{ m/s}$$

Thus the solar wind speed of about 4×10^5 m/s exceeds the Alfven speed.

7.7 Low frequency waves at intermediate angles.

When waves propagate along the magnetic field we get electrostatic waves (ion-acoustic waves) propagating at phase speed v_s, and electromagnetic Alfven waves, travelling at phase speed v_A. Propagating across B we get the hybrid magnetosonic waves: (the X-mode is a mixture of electromagnetic and electrostatic components). (The O-wave does not propagate at low-frequency.) The speed of the magnetosonic waves is $\sqrt{v_s^2 + v_A^2}$. . At intermediate angles we expect the waves to transition from one mode to another. The Alfven wave does propagate at all angles to B, and the phase speed is $B_\parallel/\sqrt{\rho\mu_0} = v_A \cos\theta$ where $B_\parallel$ is the component of B parallel to the wave vector $\vec{k}$. When $v_A > v_s$, the magnetosonic wave transitions to the Alfven wave as $\theta \to 0$, but when $v_s > v_A$, the magnetosonic wave transitions to the acoustic wave. We'll be able to investigate these low frequency waves more easily when we study MHD (magnetohydrodynamics).





There are still 3 waves here. The lowest branch is called the modified Alfven wave.