Radiation damping, line profiles and Rayleigh scattering

S.M.Lea

1 Radiation damping.

An accelerating particle radiates energy at the rate

$$P = \frac{2}{3}\frac{e^2}{c^3}a^2$$

The energy radiated must come from the electron's kinetic energy, so

$$\frac{dE}{dt} = P = \frac{d}{dt}\frac{1}{2}mv^2 \sim \frac{mv^2}{t_{loss}}$$

And thus the time scale for the particle to lose an appreciable amount of energy is:

$$t_{loss} \sim \frac{3}{2}\frac{mc^3v^2}{e^2a^2} \sim \frac{3}{2}\frac{mc^3}{e^2}t_0^2 = t_0\left(\frac{t_0}{\tau}\right)$$

where $t_0 = v/a$ is a characteristic time assocuated with the particle's motion. (For circular motion or oscillatory motion, t_0 is approximately the period.) In this expression the timescale

$$\tau = \frac{2}{3}\frac{e^2}{mc^3} = \frac{2}{3}\frac{\left(4.0\times 10^{-10}\text{ esu}\right)^2}{\left(9\times 10^{-28}\text{ g}\right)\left(3\times 10^{10}\text{ cm/s}\right)^3} = 4\times 10^{-24}\text{ s} \tag{1}$$

where the numerical value is for an electron. Heavier particles have smaller values of τ. Thus the loss time is in general very long, and the radiation loss process is most important for systems with small values of $t_{0.}$ (This result is surprisng, perhaps.)

To find the force that acts on the electron to remove energy from it, look at the integrated energy loss:

$$\begin{aligned}\triangle E &= \int_{t_1}^{t_2} P(t)dt \\ &= \int_{t_1}^{t_2} -\tilde{\mathbf{F}}_{\text{rad}}\cdot\tilde{\mathbf{v}}\,dt = \int_{t_1}^{t_2}\frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\tilde{\mathbf{v}}\cdot\tilde{\mathbf{a}}\,dt\end{aligned}$$

Integrate by parts:

$$= \int_{t_1}^{t_2} -\tilde{\mathbf{F}}_{\text{rad}} \cdot \tilde{\mathbf{v}} dt = \frac{2}{3}\frac{e^2}{c^3}\tilde{\mathbf{v}} \cdot \tilde{\mathbf{a}}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\tilde{\mathbf{a}} \cdot \tilde{\mathbf{v}} dt$$

The integrated term is zero if the motion is periodic, and $t_2 - t_1$ is a whole number of periods, or for circular motion where $\tilde{\mathbf{v}}$ is perpendicular to $\tilde{\mathbf{a}}$. Then we have:

$$\int_{t_1}^{t_2} \left(\tilde{\mathbf{F}}_{\text{rad}} - \frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\tilde{\mathbf{a}}\right) \cdot \tilde{\mathbf{v}} dt = 0$$

Thus it is reasonable to suppose that in a time-averaged sense we may write:

$$\tilde{\mathbf{F}}_{\text{rad}} = \frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\tilde{\mathbf{a}} = m\tau\frac{d}{dt}\tilde{\mathbf{a}} \tag{2}$$

2 Application to atomic systems

Model an atom as a classical, damped oscillator with natural frequency ω_0. Then, including the radiation damping, the equation of motion for the system is:

$$m\frac{d^2\tilde{\mathbf{x}}}{dt^2} = -m\omega_0^2\tilde{\mathbf{x}} + m\tau\frac{d^3}{dt^3}\tilde{\mathbf{x}} \tag{3}$$

We expect a solution of the form:

$$\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_0(t)\cos(\omega_0 t)$$

where the amplitude $\tilde{\mathbf{x}}_0(t)$ is a slowly decreasing function of time due to the damping. Then from equation 3, we then have:

$$\begin{aligned} \frac{d^3}{dt^3}\tilde{\mathbf{x}} &= \frac{d}{dt}\left(\frac{d^2\tilde{\mathbf{x}}}{dt^2}\right) = \frac{d}{dt}\left(-\omega_0^2\tilde{\mathbf{x}} + \tau\frac{d^3}{dt^3}\tilde{\mathbf{x}}\right) \\ &\simeq -\omega_0^2\left(\frac{d\tilde{\mathbf{x}}}{dt}\right) \end{aligned}$$

to zeroth order in the small quantity $\omega_0\tau$. We only need zeroth order since this derivative is multiplied by another τ in equation 3. So thedifferential equation becomes:

$$\frac{d^2\tilde{\mathbf{x}}}{dt^2} = -\omega_0^2\tilde{\mathbf{x}} - \tau\omega_0^2\left(\frac{d\tilde{\mathbf{x}}}{dt}\right)$$

Now look for a solution of the form:

$$\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_0 e^{\alpha t}$$

We find

$$\alpha^2 + \omega_0^2 + \tau\alpha\omega_0^2 = 0$$

which has the solutions:

$$\alpha = \frac{-\omega_0^2\tau \pm \sqrt{\left(\omega_0^2\tau\right)^2 - 4\omega_0^2}}{2} = \pm i\omega_0\sqrt{1 - \Gamma^2/4} - \frac{\Gamma}{2}$$

where

$$\Gamma = \omega_0^2\tau \tag{4}$$

Thus the solution is a damped oscillation at a frequency very close to the natural oscillation frequency. The solution that satisfies the initial condition $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_0$ at $t = 0$ is:

$$\tilde{\mathbf{x}}(t) = \tilde{\mathbf{x}}_0 e^{-\Gamma t/2} \cos \omega_0 t$$

and its Fourier transform is:

$$\begin{aligned}
\tilde{\mathbf{x}}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \tilde{\mathbf{x}}_0 e^{-\Gamma t/2} \cos \omega_0 t e^{i\omega t} dt \\
&= \frac{1}{2\sqrt{2\pi}} \tilde{\mathbf{x}}_0 \left(\frac{\exp(-\Gamma t/2 + i\omega_0 t + i\omega t)}{-\Gamma/2 + i(\omega_0 + \omega)} - \frac{\exp(-\Gamma t/2 - i\omega_0 t + i\omega t)}{-\Gamma/2 + i(-\omega_0 + \omega)} \right) \Bigg|_0^\infty \\
&= \frac{1}{2\sqrt{2\pi}} \tilde{\mathbf{x}}_0 \left(-\frac{1}{-\Gamma/2 + i(\omega_0 + \omega)} + \frac{1}{-\Gamma/2 + i(-\omega_0 + \omega)} \right) \Bigg|
\end{aligned}$$

This function is strongly peaked around the two values of $\omega = \pm\omega_0$.

Now the energy radiated by the system per unit frequency is:

$$\begin{aligned}
\frac{dE}{d\omega} &= \frac{2}{3} \frac{e^2}{c^3} |\tilde{\mathbf{a}}(\omega)|^2 = \frac{2}{3} \frac{e^2}{c^3} \left| \omega^2 \tilde{\mathbf{x}}(\omega) \right|^2 \\
&= \frac{2}{3} \frac{e^2}{c^3} \frac{x_0^2}{8\pi} \frac{\omega^4}{(\Gamma/2)^2 + (\omega - \omega_0)^2}
\end{aligned}$$

for frequencies near ω_0. There is an equal contribution for frequncies near $-\omega_0$, which corresponds to the same real frequency. Thus, including both contributions, and using equation 1 we get:

$$\frac{dE}{d\omega} = \frac{x_0^2}{4\pi} m\tau\omega_0^2 \frac{\omega_0^2}{(\Gamma/2)^2 + (\omega - \omega_0)^2}$$

where we set $\omega \simeq \omega_0$ except where they are subtracted. Simplifying, and using equation 4:

$$\begin{aligned}
\frac{dE}{d\omega} &= \frac{x_0^2}{2\pi} m\omega_0^2 \frac{\Gamma/2}{(\Gamma/2)^2 + (\omega - \omega_0)^2} \\
&= \frac{1}{2} m\omega_0^2 x_0^2 \phi(\omega)
\end{aligned}$$

where $\frac{1}{2} m\omega_0^2 x_0^2$ is the oscillator's energy at $t = 0$ and $\phi(\omega)$ is the Lorentz line profile function that describes how energy is distributed in frequency. Note that:

$$\begin{aligned}
\int_0^\infty \phi(\omega)\, d\omega &= \frac{1}{\pi} \int_0^\infty \frac{\Gamma/2}{(\Gamma/2)^2 + (\omega - \omega_0)^2} d\omega \\
&= \frac{1}{\pi} \int_{\theta_{\min}}^{\pi/2} \frac{\sec^2\theta}{1 + \tan^2\theta} d\theta
\end{aligned}$$

where we used the substitution $\omega - \omega_0 = (\Gamma/2)\tan\theta$, and the lower limit is

$$\theta_{\min} = \tan^{-1}(-2\omega_0/\Gamma) \simeq -\pi/2$$

since $\Gamma \ll \omega_0$. Thus:

$$\int_0^\infty \phi(\omega)\, d\omega = \frac{1}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = 1$$

as required.

3 Scattering by atoms

If the oscillator is now driven by an externally applied force due to an incoming EM wave, The equation of motion becomes:

$$m\frac{d^2\tilde{\mathbf{x}}}{dt^2} = -m\omega_0^2\tilde{\mathbf{x}} + m\tau\frac{d^3}{dt^3}\tilde{\mathbf{x}} - e\tilde{\mathbf{E}}$$

or

$$\frac{d^2\tilde{\mathbf{x}}}{dt^2} + \omega_0^2\tilde{\mathbf{x}} - \boldsymbol{\tau}\frac{d^3}{dt^3}\tilde{\mathbf{x}} = -e\tilde{\mathbf{E}}_0\cos\omega t = -\frac{e}{m}\,\mathrm{Re}\,\tilde{\mathbf{E}}_0 e^{i\omega t}$$

The expected solution is an oscillation at the same frequency ω. So letting $\tilde{\mathbf{x}} = \mathrm{Re}\left(\tilde{\mathbf{x}}_0 e^{i\omega t}\right)$ we find:

$$\left(-\omega^2 + \omega_0^2 + i\tau\omega^3\right)\tilde{\mathbf{x}}_0 = -\frac{e}{m}\tilde{\mathbf{E}}_0$$

or

$$\tilde{\mathbf{x}}_0 = -\tilde{\mathbf{E}}_0\frac{e}{m}\frac{1}{\left(-\omega^2 + \omega_0^2 + i\tau\omega^3\right)}$$

and the radiated power at frequency ω is:

$$P(\omega) = \frac{2}{3}\frac{e^2}{c^3}a^2 = \frac{2}{3}\frac{e^2}{c^3}E_0^2\frac{e^2}{m^2}\frac{\omega^4}{\left(\left(\omega^2-\omega_0^2\right)^2 + \left(\tau\omega^3\right)^2\right)}\cos^2(\omega t + \phi)$$

where the phase shift ϕ is given by:

$$\tan\phi = \frac{\tau\omega^3}{\omega^2 - \omega_0^2}$$

Now we take the time average to get:

$$< P(\omega) > = \frac{1}{3}\frac{e^4}{m^2c^3}E_0^2\frac{\omega^4}{\left(\left(\omega^2-\omega_0^2\right)^2 + \left(\tau\omega^3\right)^2\right)}$$

and the scattering cross section is:

$$\begin{aligned}\sigma(\omega) &= \frac{< P(\omega) >}{cE_0^2/8\pi} = \frac{8\pi}{3}\frac{e^4}{m^2c^4}\frac{\omega^4}{\left(\left(\omega^2-\omega_0^2\right)^2 + \left(\tau\omega^3\right)^2\right)} \\ &= \sigma_T\frac{\omega^4}{\left(\left(\omega^2-\omega_0^2\right)^2 + \left(\tau\omega^3\right)^2\right)} \qquad (5)\end{aligned}$$

Now let's look at some limits:

For $\omega \gg \omega_0$

$$\sigma(\omega) = \sigma_T$$

At high frequencies the electron "looks like" a free electron, and the cross section is the Thomson scattering cross section

For $\omega \ll \omega_0$

$$\sigma(\omega) = \sigma_T\frac{\omega^4}{\omega_0^4}$$

which is Rayleigh scattering.

Finally, near the resonance, $\omega \approx \omega_0$, we have:

$$\begin{aligned}
\sigma(\omega) &= \sigma_T \frac{\omega^4}{\left((\omega-\omega_0)^2(\omega+\omega_0)^2 + (\tau\omega^3)^2\right)} \\
&\simeq \sigma_T \frac{\omega_0^4}{\left((\omega-\omega_0)^2(2\omega_0)^2 + (\tau\omega_0^3)^2\right)} \\
&= \sigma_T \frac{\omega_0^2}{\left((\omega-\omega_0)^2 4 + (\tau\omega_0^2)^2\right)} \\
&= \frac{\sigma_T}{2\tau} \frac{\omega_0^2\tau/2}{\left((\omega-\omega_0)^2 + (\Gamma/2)^2\right)} \\
&= \frac{8\pi}{3}\frac{e^4}{m^2c^4}\frac{3mc^3}{4e^2}\frac{\Gamma/2}{\left((\omega-\omega_0)^2 + (\Gamma/2)^2\right)} \\
&= 2\pi\frac{e^2}{mc}\frac{\Gamma/2}{\left((\omega-\omega_0)^2 + (\Gamma/2)^2\right)}
\end{aligned}$$

And we regain the Lorentz line profile. Using our previous result for the inetgral, we get:

$$\int \sigma(\omega)\, d\omega = 2\pi\left(\frac{\pi e^2}{mc}\right)$$

or, equivalently:

$$\int \sigma(\nu)\, d\nu = \frac{\pi e^2}{mc}$$

The classical calculation should be modified to allow for quantum mechanical effects by including the oscilltor strength f : Then

$$\sigma(\nu) = f\frac{\pi e^2}{mc}\phi(\nu)$$

Lorentz profile with $\Gamma = 0.1$

$\frac{1}{\pi}\frac{.1/2}{(.1/2)^2+(\omega-5)^2}$