

1 Radiative transfer etc

Last time we derived the transfer equation

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

where I_ν is the intensity, $S_\nu = j_\nu/\alpha_\nu$ is the source function and $\tau_\nu = \int \alpha_\nu d\ell$ is the optical depth.

The formal solution is

$$I_\nu(\tau) = I_\nu(0) e^{-\tau} + \int_0^\tau S_\nu(\tau') e^{-(\tau-\tau')} d\tau' \quad (1)$$

1.1 Special cases: No emission

If $S_\nu \equiv 0$, then the intensity is given by the first term in equation 1. The intensity decreases exponentially. The Earth's atmosphere provides an example. The optical depth varies with angle: $\tau = \tau_0 \sec \theta$ where τ_0 is the optical depth along a vertical path. Scattering is very important in this situation, so we'll defer further discussion for the moment.

1.2 Optically thin source

When $\tau \ll 1$, the solution reduces to:

$$I_\nu(\tau) = I_\nu(0) + \int_0^\tau S_\nu(\tau') d\tau' \approx I_\nu(0) + S_\nu \tau = I_\nu(0) + j_\nu L$$

where L is the total path length through the source.

Example: HII region in the radio. The emission is due to thermal bremsstrahlung (more later)

$$j_\nu = 6 \times 10^{-39} \frac{n^2}{T^{1/2}} e^{-h\nu/kT} \text{ erg}/(\text{cm}^3 \cdot \text{s} \cdot \text{ster} \cdot \text{Hz}) \quad (2)$$

and in the radio ($h\nu \ll kT$, $T \approx 10^4 \text{ K}$)

$$\alpha_\nu \simeq 0.01 \frac{n^2}{T^{3/2} \nu^2} \text{ cm}^{-1} \quad (3)$$

Thus for typical values of $n \sim 7000 \text{ cm}^{-3}$, $T \sim 10^4 \text{ K}$, and $\nu \sim 2 \text{ GHz}$, we have

$$\alpha_\nu \sim 0.01 \frac{7000^2}{10^6 \times 4 \times 10^{18}} \text{ cm}^{-1} = 1.2 \times 10^{-19} \text{ cm}^{-1}$$

and for a path length of 1/2 pc,

$$\tau = 1.2 \times 10^{-19} \times 1.5 \times 10^{18} = 0.18$$

Thus we can compute the emitted intensity by a straightforward multiplication of emission coefficient times path length.

1.3 Optically thick source

If the source function is a constant, and $\tau \gg 1$, then

$$I_\nu(\tau) = I_\nu(0) e^{-\tau} + \int_0^\tau S_\nu(\tau') e^{-(\tau-\tau')} d\tau' = I_\nu(0) e^{-\tau} + S_\nu (1 - e^{-\tau})$$

$$\simeq S_\nu$$

and so the intensity equals the source function in this case.

1.4 Thermal emission

In thermal equilibrium, we reach a situation in which the intensity does not change within the source, and thus

$$I_\nu = S_\nu$$

Thus very large optical depth corresponds to thermal equilibrium. We know what the intensity is in this case: it is just the black body (Planck) function:

$$I_\nu = B_\nu(T) = S_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

and thus when we have thermal radiation, the source function equals the planck function. This is a fundamental relation, since j_ν and α_ν involve only the microphysics of the emission and absorption processes. It follows that

$$j_\nu = \alpha_\nu B_\nu(T) \quad (4)$$

Thus once we find j , we can find α immediately (or vice versa). That is how I obtained equation (3) from equation (2).

Now the transfer equation for a medium in thermal equilibrium has become

$$\frac{dI_\nu}{d\tau_\nu} = B_\nu(T) - I_\nu(\tau)$$

Now R&L equation 1.13 for a uniformly bright sphere becomes:

$$F_\nu = \pi B_\nu \left(\frac{R}{r} \right)^2$$

and integrating over frequency

$$F = \sigma T^4 \left(\frac{R}{r} \right)^2$$

where $\sigma = 5.67 \times 10^{-5} \text{erg}/(\text{cm}^2 \cdot \text{deg}^4 \cdot \text{s})$

Sometimes we use the relation $I_\nu \equiv B_\nu(T_b)$ to define a temperature called the brightness temperature T_b . Radio astronomers like to do this. The brightness temperature equals the thermodynamic temperature only in the optically thick case. The general solution 1 may be written as:

$$T_b = T_{\text{bo}} e^{-\tau} + T (1 - e^{-\tau})$$

or, if there is no background source,

$$T_b = T (1 - e^{-\tau})$$

from which it is clear that the brightness temperature can never exceed the thermodynamic temperature, and T_b approaches T as $\tau \rightarrow \infty$.

(Never is a strong word. Perhaps we should be more careful. Exceptions may occur if the source is not at least locally in thermodynamic equilibrium. More later)

low frequency limit:

$h\nu/kT \ll 1$. We obtain the Rayleigh-Jeans law:

$$B_\nu(T) \simeq \frac{2\nu^2}{c^2}kT$$

high frequency limit:

$h\nu/kT \gg 1$. Here we obtain the Wien law

$$B_\nu(T) \simeq \frac{2h\nu^3}{c^2}e^{-h\nu/kT}$$

which shows that the spectrum decreases exponentially at high frequencies.

Wien displacement law

The peak of the spectrum is found by setting the derivative to zero:

$$\begin{aligned} \frac{c^2}{2h} \frac{dB_\nu}{d\nu} &= \frac{3\nu^2}{e^{h\nu/kT} - 1} - \frac{h\nu^3 e^{h\nu/kT}}{kT (e^{h\nu/kT} - 1)^2} \\ &= \frac{\nu^2}{(e^{h\nu/kT} - 1)^2} \left(3(e^{h\nu/kT} - 1) - \frac{h\nu}{kT} e^{h\nu/kT} \right) = 0 \end{aligned}$$

We can solve this equation numerically:

$$1 - e^{-x} = \frac{x}{3}$$

where $x = h\nu/kT$. Since e^{-x} is small for $x > 1$, we expect that $x \simeq 3$. So use this as a first guess, and then evaluate

$$x_{\text{next}} = 3(1 - e^{-x})$$

The results are:

try	x	x_{next}
1	3	$3(1 - e^{-3}) = 2.8506$
2	2.8506	$3(1 - e^{-2.8506}) = 2.8266$
3	2.8266	$3(1 - e^{-2.8266}) = 2.8224$
4	2.8224	$3(1 - e^{-2.8224}) = 2.8216$
5	2.8216	$3(1 - e^{-2.8216}) = 2.82147$
6	2.82147	$3(1 - e^{-2.82147}) = 2.82144$
7	2.8214	$3(1 - e^{-2.82144}) = 2.82144$

so the result is $h\nu = 2.82144kT$. The algorithm converges quickly.

Now try the same thing to find wavelength of the maximum. Write the spectrum as

$$B_\lambda = \frac{x^5}{e^x - 1}$$

where $x = hc/\lambda kT$. Use a similar numerical algorithm to show that

$$x_{\max} = 4.965$$

and thus

$$\lambda_{\max}T = 0.29 \text{ cm} \cdot \text{K}$$

This is the Wien displacement law.

Suppose we calculate the wavelength corresponding to the frequency maximum. We'd get

$$\lambda_{\max}T = \frac{hc}{2.82144k} = 0.51 \text{ cm} \cdot \text{K}$$

Why do we not obtain the same value for λ_{\max} in the two cases? Is one preferable to the other?

2 The Einstein coefficients

In applying our results to atomic absorption and emission, it is convenient to use the Einstein coefficients. They are defined as follows:

The Einstein B -coefficient B_{01} describes absorption:

$$\sigma_{\nu} = \frac{h\nu}{4\pi} B_{01} \phi_{\nu}$$

the function ϕ_{ν} is the line profile function. It describes how the absorption is distributed around the line center frequency. Broadening is due to "natural" broadening (radiation damping), Doppler effect due to motion of the atoms, etc. We must also include stimulated emission, described by B_{10} . In terms of the B s, the absorption coefficient is:

$$\alpha_{\nu} = \frac{h\nu}{4\pi} \phi_{\nu} (n_0 B_{01} - n_1 B_{10})$$

where n_0 and n_1 are the number of atoms in the lower (upper) level respectively. The Boltzmann relation relates the two populations:

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-h\nu/kT}$$

where g_1 and g_0 are the statistical weights of the two levels. Thus

$$\alpha_{\nu} = \frac{h\nu}{4\pi} n_0 \phi_{\nu} \left(B_{01} - B_{10} \frac{g_1}{g_0} e^{-h\nu/kT} \right)$$

Emission is described by the Einstein A coefficient:

$$j_{\nu} = \frac{h\nu}{4\pi} n_1 A_{10} \phi_{\nu}$$

where A_{10} is the transition probability per unit time for spontaneous emission.

Now we know that

$$\frac{j_{\nu}}{\alpha_{\nu}} = B_{\nu}(T) = \frac{n_1 A_{10}}{n_0 \left(B_{01} - B_{10} \frac{g_1}{g_0} e^{-h\nu/kT} \right)} = \frac{A_{10}}{\frac{g_0}{g_1} e^{h\nu/kT} B_{01} - B_{10}} = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$$

Thus we obtain the relations

$$B_{01} = \frac{g_1}{g_0} B_{10}$$

and

$$A_{10} = \frac{2hv^3}{c^2} B_{10}$$

or

$$B_{10} = \frac{\lambda^2}{2hv} A_{10}$$

Interesting things can happen when the populations are not governed by the Boltzmann relation, i.e. when some non-thermal process operates. For example, if the population is *inverted* ($n_1 > n_0$), the stimulated emission dominates and we can obtain maser emission.

3 Scattering

As we have mentioned, scattering can increase the intensity (due to scattering from other directions) and can decrease the intensity (due to scattering out). We can define an emission coefficient for scattering as

$$j_{\nu, \text{scatt}} = n_s \sigma_{\nu} J_{\nu}$$

where n_s is the number density of scatterers and σ_{ν} is the scattering cross section. Notice that the mean intensity appears here, since photons going in any direction can be scattered into the beam. For the moment I shall assume that the scattering cross section has no dependence on direction. This is close to the truth for Thomson (electron) scattering. Then the transfer equation for scattering only becomes:

$$\frac{dI_{\nu}}{ds} = n_s \sigma_{\nu} (J_{\nu} - I_{\nu})$$

Including both absorption and scattering, we have:

$$\begin{aligned} \frac{dI_{\nu}}{ds} &= n_s \sigma_{\nu} (J_{\nu} - I_{\nu}) + j_{\nu} - \alpha_{\nu} I_{\nu} \\ &= n_s \sigma_{\nu} J_{\nu} + j_{\nu} - (\alpha_{\nu} + n_s \sigma_{\nu}) I_{\nu} \end{aligned}$$

We can write

$$n_s \sigma_{\nu} = \alpha_s$$

Then

$$\frac{dI_{\nu}}{ds} = \alpha_s J_{\nu} + \alpha_{\nu} B_{\nu} - (\alpha_{\nu} + \alpha_s) I_{\nu}$$

Now let's define a source function

$$S_{\nu} \equiv \frac{\alpha_s J_{\nu} + \alpha_{\nu} B_{\nu}}{\alpha_{\nu} + \alpha_s}$$

so that the equation takes the form

$$\frac{dI_{\nu}}{ds} = (\alpha_{\nu} + \alpha_s) (S_{\nu} - I_{\nu})$$

Defining $d\tau = (\alpha_{\nu} + \alpha_s) ds$, we retrieve the previous transfer equation in terms of τ and S . The solution is more complicated, however, because S_{ν} depends on J_{ν} , and hence on I_{ν} .

Note here that $1/(\alpha_\nu + \alpha_s)$ is the average distance a photon will travel before being removed from the beam by either an absorption or a scattering event.

$$l = \frac{1}{\alpha_\nu + \alpha_s}$$

On the other hand, the photon will travel a longer distance

$$l_a = \frac{1}{\alpha_\nu}$$

before being absorbed. The path will be a random walk. Each step of the walk will end by a scattering, with probability

$$p_s = \frac{\alpha_s}{\alpha_\nu + \alpha_s}$$

or with an absorption, with probability

$$p_a = \frac{\alpha_\nu}{\alpha_\nu + \alpha_s}$$

Thus the source function is

$$S_\nu = (1 - p_a)J_\nu + p_a B_\nu$$

3.1 Theory of random walks

A particle has a mean step length ℓ . The total displacement is

$$\vec{D} = \sum_{i=1}^N \vec{D}_i$$

and the mean distance travelled is

$$\begin{aligned} \langle |\vec{D}| \rangle &= \langle \sqrt{\vec{D} \cdot \vec{D}} \rangle = \langle \sqrt{\left(\sum_{i=1}^N \vec{D}_i \right) \cdot \left(\sum_{i=1}^N \vec{D}_i \right)} \rangle \\ &= \langle \sqrt{\sum_{i=1}^N \vec{D}_i \cdot \vec{D}_i + \sum_{i=1}^N \sum_{j=1, j \neq i}^N \vec{D}_i \cdot \vec{D}_j} \rangle \\ &= \sqrt{N\ell^2} = \sqrt{N}\ell = L \end{aligned}$$

since the directions of \vec{D}_i and \vec{D}_j are not correlated, and thus the average value of $\vec{D}_i \cdot \vec{D}_j$ is zero.

In our problem, the walk starts when the photon is created and ends when it is absorbed. The step length is $l = 1/(\alpha_\nu + \alpha_s)$. The probability that the path ends, at any one event, is p_a . Thus the total number of steps expected is $1/p_a$. The mean distance travelled is:

$$L = \sqrt{\frac{1}{p_a} \frac{1}{(\alpha_\nu + \alpha_s)}} = \sqrt{\frac{\alpha_\nu + \alpha_s}{\alpha_\nu} \frac{1}{(\alpha_\nu + \alpha_s)}} = \frac{1}{\sqrt{\alpha_\nu (\alpha_\nu + \alpha_s)}}$$

Thus the *effective optical depth due to absorption and scattering* is

$$\tau_{\text{eff}} = \frac{1}{L} = \sqrt{\alpha_\nu (\alpha_\nu + \alpha_s)}$$