Scattering

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1 Polarization

Let's describe the electric field $\vec{E}$ in terms of two orthogonal polarization vectors $\hat{e}_1$ and $\hat{e}_2$. Then:

$$\vec{E} = \vec{E} \cdot \hat{e}_1 + \vec{E} \cdot \hat{e}_2$$

and

$$\left|\vec{E}\right|^2 = \left(\vec{E} \cdot \hat{e}_1\right)^2 + \left(\vec{E} \cdot \hat{e}_2\right)^2$$

Thus the power radiated, which is proportional to $\left|\vec{E}\right|^2$, is the sum of the powers radiated into the two orthogonal polarizations.

Now in the non-relativistic case, or with $\vec{a}$ parallel to $\vec{\beta}$, $\vec{E} \propto \hat{n} \times (\hat{\mathbf{n}} \times \tilde{\mathbf{a}})$ and so

$$\begin{aligned} \vec{E} \cdot \hat{e}_1 &\propto (\hat{n} \times (\hat{n} \times \vec{a})) \cdot \hat{e}_1 \\ &= (\hat{n}\,(\hat{n} \cdot \vec{a}) - \vec{a}) \cdot \hat{e}_1 \\ &= (\hat{n} \cdot \vec{a})\,\hat{n} \cdot \hat{e}_1 - \vec{a} \cdot \hat{e}_1 \end{aligned}$$

But since $\hat{n}$ is perpendicular to $\hat{e}_1$, then

$$\vec{E} \cdot \hat{e}_1 \propto \vec{a} \cdot \hat{e}_1$$

Generally this is the easiest way to compute the power.

2 Thomson scattering

Let the incident wave be described by:

$$\vec{E}\,(\vec{x}, t) = \hat{\varepsilon}_0 E_0 \exp\left(i\left[\vec{k} \cdot \vec{x} - \omega t\right]\right)$$

where $\hat{\varepsilon}_0$ is a vector describing the polarization. The electric field is incident on an electron and gives it an acceleration:

$$\vec{a} = \frac{-e}{m}\hat{\varepsilon}_0 E_0 \exp\left(i\left[\vec{k} \cdot \vec{x} - \omega t\right]\right)$$

and then the power radiated into polarization i is:

$$\begin{aligned} \frac{dP_i}{d} &= \frac{e^2}{4\pi c^3} \left|\vec{a}\cdot\hat{e}_1\right|^2 \\ &= \frac{e^2}{4\pi c^3}\left(\frac{e}{m}E_0\right)^2 \left(\hat{\varepsilon}_0 \exp\left(i\left[\vec{k}\cdot\vec{x}-\omega t\right]\right)\cdot\hat{e}_1^*\right)^2 \end{aligned}$$

and the time average is

$$< \frac{dP_i}{d} >= \frac{e^4}{4\pi m^2 c^3}\frac{E_0^2}{2}\left(\hat{\varepsilon}_0\cdot\hat{e}_i^*\right)^2$$

Note: by using the complex conjugate of the polarization vector to compute the absolute value, we can include circular polarizations in our results.

The differential scattering cross section is defined as:

$$\frac{d\sigma}{d} \equiv \frac{\text{energy radiated/unit time/unit solid angle}}{\text{incident flux}}$$

and thus

$$\begin{aligned} \frac{d\sigma_i}{d} &= \frac{\frac{e^4}{8\pi m^2 c^3}E_0^2\left(\hat{\varepsilon}_0\cdot\hat{e}_i^*\right)^2}{cE_0^2/8\pi} \\ &= \frac{e^4}{m^2c^4}\left(\hat{\varepsilon}_0\cdot\hat{e}_i^*\right)^2 \\ &= r_0^2\left(\hat{\boldsymbol{\varepsilon}}_0\cdot\hat{\mathbf{e}}_i^*\right)^2 \end{aligned}$$

where

$$r_0 = \frac{e^2}{mc^2}$$

is the classical electron radius.

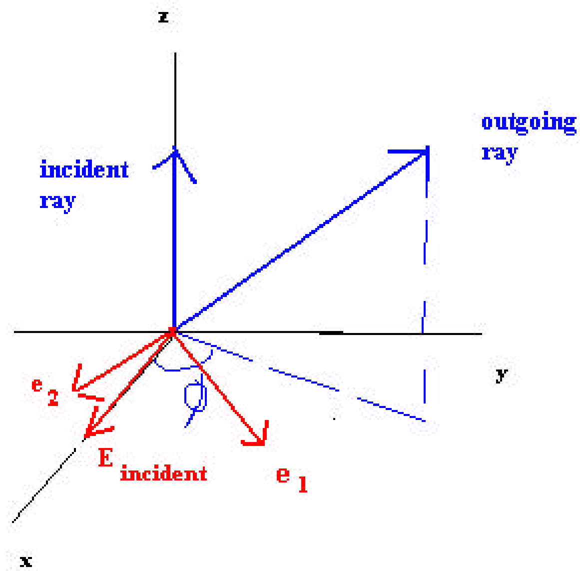
Let's assume the incident wave is incident along the $z-$axis and is linearly polarized along the $\hat{x}-$axis: $\hat{\varepsilon}_0 = \hat{x}$. An outgoing wave scattered at angle θ may be described in terms of the polarization vectors

$$\hat{\mathbf{e}}_1 = \cos\theta\left(\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi\right) - \hat{\mathbf{z}}\sin\theta \tag{1}$$

and

$$\hat{\mathbf{e}}_2 = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

as shown in the diagram:



Then

$$\hat{\boldsymbol{\varepsilon}}_0 \cdot \hat{\mathbf{e}}_1 = \cos\theta \cos\phi$$

and

$$\hat{\boldsymbol{\varepsilon}}_0 \cdot \hat{\mathbf{e}}_2 = -\sin\phi$$

Thus:

$$\begin{aligned} \frac{d\sigma}{d} &= \frac{d\sigma_1}{d} + \frac{d\sigma_2}{d} \\ &= r_0^2 \left(\cos^2\theta \cos^2\phi + \sin^2\phi\right) \end{aligned}$$

For the perpendicular incident polarization $\hat{\boldsymbol{\varepsilon}}_0 = \hat{\mathbf{y}}$, we have:

$$\frac{d\sigma}{d} = r_0^2 \left(\cos^2\theta \sin^2\phi + \cos^2\phi\right)$$

and thus the differential scattering cross section for unpolarized incident radiation is:

$$\frac{d\sigma}{d} = \frac{r_0^2}{2} \left(1 + \cos^2\theta\right)$$

and thus the total scattering cross section is:

$$\begin{aligned} \sigma &= \int \frac{d\sigma}{d} d \quad = 2\pi \frac{r_0^2}{2} \int_{-1}^{+1} \left(1 + \mu^2\right) d\mu = \pi r_0^2 \left(\mu + \frac{\mu^3}{3}\right)\Big|_{-1}^{+1} \\ &= \frac{8\pi}{3} r_0^2 = 0.665 \times 10^{-24} \ \text{cm}^2 \end{aligned}$$

This result is valid for $h\nu \ll mc^2 = 511$ keV, and an electron at rest or moving with $v \ll c$.

3 Compton scattering

For high-energy photons it is important to include the photon momentum and regard the scattering process as a particle collision. Then we have:

Energy conservation:

$$mc^2 + h\nu = \gamma mc^2 + h\nu' \tag{2}$$

and momentum conservation: $x-$component

$$\frac{h\nu}{c} = \gamma mv\cos\phi + \frac{h\nu'}{c}\cos\theta \tag{3}$$

and $y-$component

$$0 = \gamma mv\sin\phi - \frac{h\nu'}{c}\sin\theta \tag{4}$$

Now square equations (3) and (4) and add:

$$\begin{aligned}(\gamma mv\cos\phi)^2 + (\gamma mv\sin\phi)^2 &= (\gamma mv)^2 = \left(\frac{h\nu'}{c}\sin\theta\right)^2 + \left(\frac{h\nu'}{c}\cos\theta - \frac{h\nu}{c}\right)^2 \\ &= \left(\frac{h\nu'}{c}\right)^2 + \left(\frac{h\nu}{c}\right)^2 - \frac{2h^2\nu\nu'}{c^2}\cos\theta\end{aligned}$$

But

$$\beta^2 = 1 - \frac{1}{\gamma^2} \Rightarrow \gamma^2\beta^2 = \gamma^2 - 1$$

So

$$\left(\gamma^2 - 1\right)\left(\frac{mc^2}{h\nu}\right)^2 = 1 + \left(\frac{\nu'}{\nu}\right)^2 - 2\frac{\nu'}{\nu}\cos\theta \tag{5}$$

Then squaring equation (2) gives:

$$(\gamma - 1)^2\left(\frac{mc^2}{h\nu}\right)^2 = \left(1 - \frac{\nu'}{\nu}\right)^2 = 1 + \left(\frac{\nu'}{\nu}\right)^2 - 2\frac{\nu'}{\nu} \tag{6}$$

Subtracting equations (6) and (5), we get:

$$2(\gamma - 1)\left(\frac{mc^2}{h\nu}\right)^2 = 2\frac{\nu'}{\nu}(1 - \cos\theta)$$

Then using equation (2) again gives:

$$\begin{aligned}\frac{\nu'}{\nu}(1 - \cos\theta) &= \left(\frac{mc^2}{h\nu}\right)^2\frac{h\nu}{mc^2}\left(1 - \frac{\nu'}{\nu}\right) \\ \frac{\nu'}{\nu}\left(1 - \cos\theta + \frac{mc^2}{h\nu}\right) &= \frac{mc^2}{h\nu}\end{aligned}$$

and so

$$\nu' = \nu \left(\frac{mc^2}{h\nu}\right) \frac{1}{1 - \cos\theta + \left(\frac{mc^2}{h\nu}\right)}$$
$$= \frac{\nu}{1 + \frac{h\nu}{mc^2}(1 - \cos\theta)}$$

The differential cross section is modified by the frequency shift:

$$\left.\frac{d\sigma}{d}\right|_{QM} = r_0^2 \left(\hat{\boldsymbol{\varepsilon}}_0 \cdot \hat{\mathbf{e}}_i^*\right)^2 \left(\frac{\nu'}{\nu}\right)^2$$