Vlasov theory of Landau damping

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In Vlasov theory we use the full power of the Vlasov equation to discuss the evolution of a plasma, including the effects of details of the particle velocity distribution on the evolution of perturbations. We will study growth and damping of perturbations. The basic technique is the standard "perturb and linearize" approach that we have been using. Our first topic is the Landau damping of Langmuir waves.

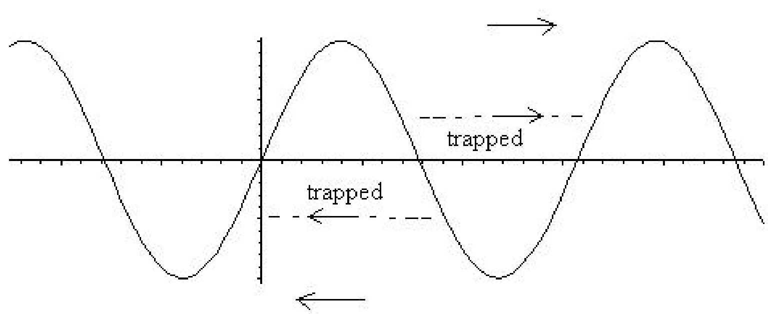
1 Landau damping

1.1 Qualitative discussion

First a qualitative analysis. The effect arises from an interaction of the individual particles with the wave potential, which translates at the wave phase speed $v_\phi = \omega/k$. Some particles travel faster than the wave and move to the right in the diagram below. The total energy of an electron is

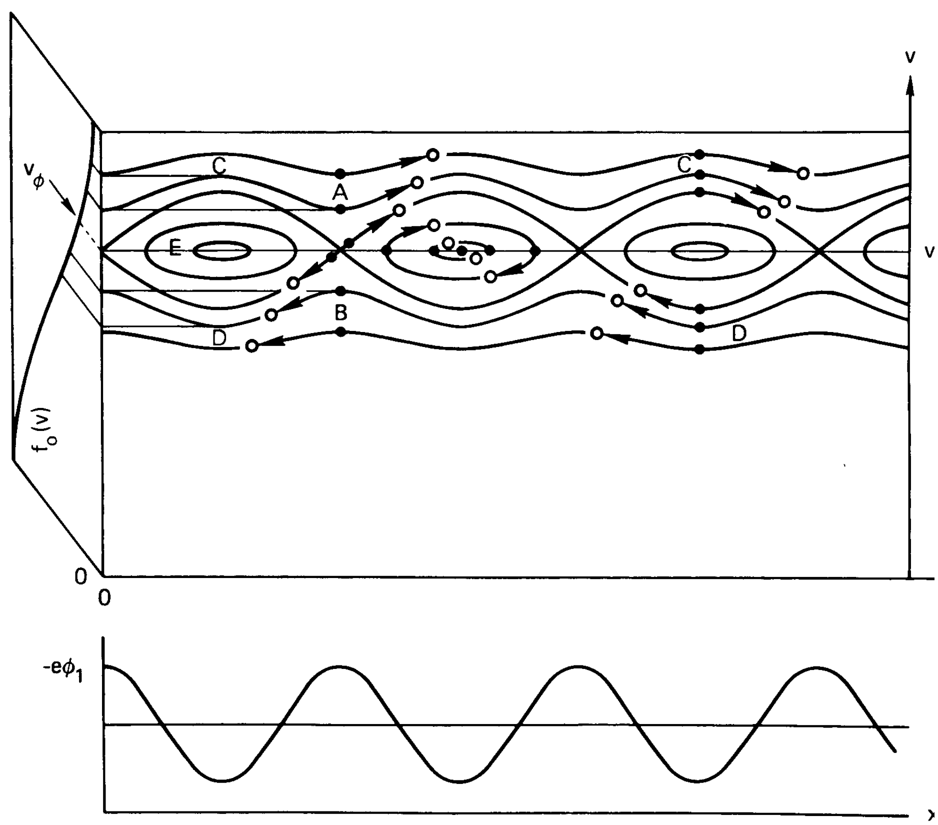
$$E = \frac{1}{2}mv^2 - e\phi = \frac{1}{2}mv^2 + U_E$$

One such particle that starts with $U_E = U_1 < e\phi_{\max}$ will ultimately be reflected by the wave potential: it will bounce back and forth in the potential well. But a particle with an initial $U_E > e\phi_{\max}$ will go over the hill (with some loss of speed). Similarly, a particle going slightly slower than the wave will move to the left in the wave frame and will be trapped. A particle going a lot slower is not trapped but continues to move to the left in the wave frame.



A particle that is initially moving with $v > v_\phi$ and becomes trapped ends up with an average speed equal to v_ϕ. Since its energy is decreased, the wave gains energy. But a particle moving with $v < v_\phi$ that gets trapped gains energy (in the lab frame) and so the wave loses energy. If v_ϕ is in the tail of the Maxwellian distribution, there are more particles that gain energy than particles that lose energy. and so the wave loses energy– it is damped. This is non-linear Landau damping.

To understand linear Landau damping we have to look at the initiation of this process. Let's see what happens to the particle velocities: (diagram from Chen p 255). The bottom graph shows the particle potential energy $-e\phi$. The upper panel shows the particle velocities in the wave frame. A particle initially at A gains energy during the first quarter cycle of the wave, while a particle intially at C loses energy. Similarly a particle initially at B loses energy during the first quarter cycle, while a particle at D gains energy. Since there are more particles at A than at C and more at D than at B, (see distribution to the left) the particles as a whole lose energy and so the wave damps. Thus linear Landau damping is a start-up effect. This is a clue– initial conditions are going to be important in our analysis.



1.2 Mathematical analysis.

Langmuir waves are high-frequency waves so the ions are unperturbed. The Vlasov equation (plasfluid notes eqn 9) with electrostatic fields only is:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{q}{m} \vec{E} \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

Now we perturb and linearize. A new feature we have not used before is the perturbation to the distribution function:

$$f(\vec{v}) = f_0(\vec{v}) + f_1(\vec{v})$$

Then, with $\vec{E}_0 = 0$, and keeping only first order terms,

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \vec{\nabla} f_1 + \frac{q}{m} \vec{E} \cdot \frac{\partial f_0}{\partial \vec{v}} = 0 \qquad (1)$$

Poisson's equation will allow us to relate the integral of f to $\vec{E}$. Remember (plasfluid notes eqn 6):

$$n = \int f(\vec{v}) d^3\vec{v}$$

Thus:

$$\nabla \cdot \vec{E} = -\frac{e}{\varepsilon_0} n_1 = -\frac{e}{\varepsilon_0} \int f_1(\vec{v})\ d^3\vec{v} \qquad (2)$$

Next Fourier transform these two equations (or, equivalently, assume that the perturbation is proportional to $\exp i\left(\vec{k} \cdot \vec{x} - \omega t\right)$):

$$-i\omega f_1 + i\vec{k} \cdot \vec{v} f_1 - \frac{e}{m} \vec{E} \cdot \frac{\partial f_0}{\partial \vec{v}} = 0 \qquad (3)$$

and

$$i\vec{k} \cdot \vec{E} = -\frac{e}{\varepsilon_0} \int f_1(\vec{v}) d^3\vec{v} \qquad (4)$$

Choosing coordinates with the $x-$axis along $\vec{E}$, we have:

$$E_x = i\frac{e}{\varepsilon_0 k_x} \int f_1(\vec{v}) d^3\vec{v}$$

and we find f_1 from (3).

$$f_1 = i\frac{e}{m} E_x \frac{\partial f_0}{\partial v_x} \frac{1}{\omega - \vec{k} \cdot \vec{v}} \qquad (5)$$

Substitute into the previous equation:

$$E_x = i\frac{e}{\varepsilon_0 k_x} \int i\frac{e}{m} E_x \frac{\partial f_0}{\partial v_x} \frac{1}{\left(\omega - \vec{k} \cdot \vec{v}\right)} d^3\vec{v} = -\frac{n_0 e^2}{\varepsilon_0 m k} E_x \int \frac{\partial \hat{f}_0}{\partial v_x} \frac{1}{(\omega - k v_x)} d^3\vec{v}$$

where in the last line we assumed longitudinal waves with $\vec{E}$ parallel to $\vec{k}$. Then for E_x not zero, we have the dispersion relation:

$$1 = \frac{\omega_p^2}{k^2} \int \frac{\partial \hat{f}_0}{\partial v_x} \frac{1}{(v_x - \omega/k)} dv_y dv_z dv_x \qquad (6)$$

Now in principle the problem is solved. But how do we do that integral? There is no problem with the integral over v_y and v_z. But since ω is, in general, a complex number, we are actually doing an integral along the real axis in the complex v_x plane, and such integrals are path dependent unless the integrand is analytic everywhere. Our integrand has a pole at $v_x = \omega/k$. As the imaginary part of ω gets smaller, the pole approaches the real axis, and we shall have to deform our path to go around the pole. So how do we know what path to use?

Well, remember that Landau damping is a result of initial conditions. So we should really be solving an initial value problem, for which the Laplace transform usually works better than the Fourier transform. So let's go back to 1 and Laplace transform it in time while retaining the Fourier transform in space:

$$sF_1 - f_1(0) + i\vec{k}\cdot\vec{v}F_1 - \frac{e}{m}\mathcal{L}(E_x)\frac{\partial f_0}{\partial v_x} = 0$$

where F_1 is the Laplace transform of f_1 and $f_1(0)$ is the intitial value of f_1. Now we solve for F_1 :

$$F_1 = \frac{f_1(0) + \frac{e}{m}\mathcal{L}(E_x)\frac{\partial f_0}{\partial v_x}}{s + ikv_x}$$

From Poisson's equation (2), we get:

$$ik\mathcal{L}(E_x) = -\frac{e}{\varepsilon_0}\int F_1(\vec{v})d^3\vec{v} = -\frac{e}{\varepsilon_0}\int \frac{f_1(0) + \frac{e}{m}\mathcal{L}(E_x)\frac{\partial f_0}{\partial v_x}}{s + ikv_x}d^3\vec{v}$$

Factor out the density and write $f_0 = n_0\hat{f}_0$ to get:

$$\mathcal{L}(E_x)\left(ik + \frac{n_0 e^2}{\varepsilon_0 m}\int \frac{\partial \hat{f}_0}{\partial v_x}\frac{1}{s + ikv_x}d^3\vec{v}\right) = -\frac{e}{\varepsilon_0}\int \frac{f_1(0)}{s + ikv_x}d^3\vec{v}$$

or

$$\begin{aligned}
\mathcal{L}(E_x) &= \frac{i\frac{e}{\varepsilon_0 k}\int \frac{f_1(0)}{s+ikv_x}d^3\vec{v}}{1 + \frac{n_0 e^2}{i\varepsilon_0 mk}\int \frac{\partial \hat{f}_0}{\partial v_x}\frac{1}{s+ikv_x}d^3\vec{v}} = \frac{i\frac{e}{\varepsilon_0 k}\int \frac{f_1(0)}{s+ikv_x}d^3\vec{v}}{1 + .\frac{\omega_p^2}{k^2}\int \frac{\partial \hat{f}_0}{\partial v_x}\frac{1}{is/k - v_x}d^3\vec{v}} \\
&= \frac{i\frac{e}{\varepsilon_0 k}\int \frac{f_1(0)}{s+ikv_x}d^3\vec{v}}{1 - .\frac{\omega_p^2}{k^2}\int \frac{\partial \hat{f}_0}{\partial v_x}\frac{1}{v_x - is/k}d^3\vec{v}} \\
&\equiv \frac{i\frac{e}{\varepsilon_0 k}\int \frac{f_1(0)}{s+ikv_x}d^3\vec{v}}{\varepsilon(is, k)} \qquad (7)
\end{aligned}$$

where I have used the definition

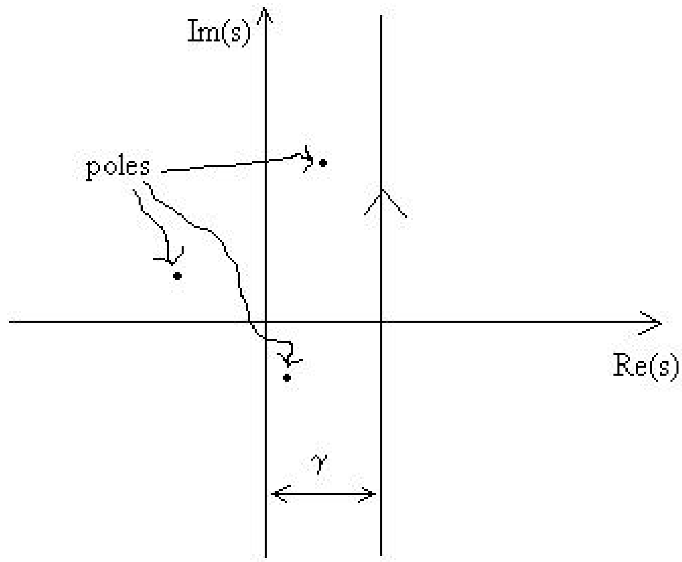
$$\varepsilon(is, k) \equiv 1 - .\frac{\omega_p^2}{k^2}\int \frac{\partial \hat{f}_0}{\partial v_x}\frac{1}{v_x - is/k}d^3\vec{v}$$

and ε is the dielectric constant (see below). Here we see that our previous dispersion relation 6 is just $\varepsilon(\omega, k) = 0$, where ε appears in the denominator of $\mathcal{L}(E_x)$, with is replaced with ω.

To solve for E_x, we use the Mellin inversion integral.

$$E_x(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty}\mathcal{L}(E_x)e^{st}ds$$

The prescription says that we must choose a path of integration that passes to the right of all the poles of $\mathcal{L}(E_x)$. Note that these poles are just the zeroes of the denominator, i.e. the roots of the dispersion relation.



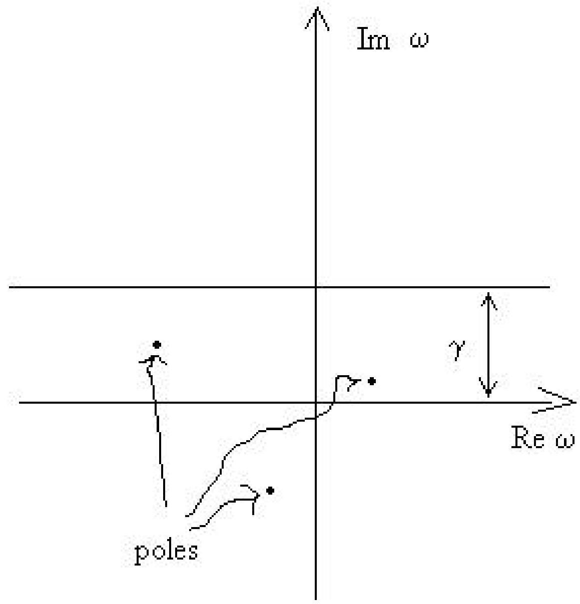
Thus on the integration path we have $\mathrm{Re}(s) = \gamma$. Noting that this s is related to the previous ω by $is = \omega$, we have $\mathrm{Re}(s) = \mathrm{Re}(-i\omega) = \mathrm{Im}(\omega)$. Thus on the integration path in the $\omega-$plane, $\mathrm{Im}(\omega) = \gamma$.

Now we can take this information and use it in our previous problem. The poles of $\mathcal{L}(E_x)$ determine the behavior of the integral that gives E_x, that is they determine the behavior of the plasma. In fact, if we denote the roots of $\varepsilon = 0$ as ω_n, (and assume these are simple poles) then we can evaluate the integral to get:

$$\begin{aligned} E_x(t) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\mathcal{L}(E_x)\,e^{st}ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{i\frac{e}{\varepsilon_0 k}\int\frac{f_1(0)}{s+ikv_x}d^3\vec{v}}{\varepsilon(is,k)}e^{st}ds \\ &= \sum_n \lim_{s\to -i\omega_n}\frac{s+i\omega_n}{\varepsilon(\omega,k)}\frac{ie}{\varepsilon_0 k}\int\frac{f_1(0)}{s+ikv_x}\,d^3\vec{v}\,e^{-i\omega_m t} \end{aligned}$$

where s is greater than $\mathrm{Re}(-i\omega_n)$ until the limit is taken.

Now let's look at the $\omega-$plane. Remember that $\omega = is$ means the $\omega-$plane is rotated 90^o relative to the $s-$plane: $s = +\infty$ corresponds to $\omega = +i\infty$, $s = +i\infty$ corresponds to $\omega = -\infty$, and $s = \gamma$ corresponds to $\omega = i\gamma$.



In this plane all the poles (normal mode frequencies) are below the path of integration and $\omega > \omega_n$.

Now let's go to the v plane. We integrate along (or close to) the real axis, but the imaginary part of ω is positive and equals $i\gamma$ (as can be seen from the graph above). Thus the path of integration passes *below* the poles of this integrand at $v = \omega/k = i\gamma/k$. This is the Landau prescription.

Now of course we (almost) never actually do the Laplace transform, but in order to get correct results we must always have the integration path in the $v-$plane pass *beneath* the poles at $v = \omega/k$.

Now let's evaluate the frequencies using equation 6. We can split the integral over v_x up into 2 pieces

1. The principal value:

$$P(I) = \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{\omega_r - \varepsilon} + \int_{\omega_r + \varepsilon}^{+\infty} \right) \frac{\partial \hat{f}_0}{\partial v_x} \frac{1}{v_x - \omega/k} dv_x$$

2. The contribution from the pole.

To get the principal value, first integrate by parts:

$$P(I) = \left. \frac{\hat{f}_0}{v_x - \omega/k} \right|_{-\infty}^{+\infty} + \int \hat{f}_0 \frac{1}{(v_x - \omega/k)^2} dv_x$$

The integrated term is zero, since $f_0 \to 0$ as $v_x \to \pm\infty$, and the denominator helps. Now let's assume that the wave phase speed is large compared with the electron thermal speed. Both f_0 and $\partial f_0/\partial v$ get small as v gets large, if f_0 is a Maxwellian. Thus the integral is dominated by the range where $v \ll \omega/k$. Thus we may expand the $1/\left(v_x - \omega/k\right)^2$ factor:

$$\left(\frac{1}{v_x - \omega/k}\right)^2 = \frac{k^2}{\omega^2}\frac{1}{\left(1 - kv_x/\omega\right)^2} = \frac{k^2}{\omega^2}\left(1 + 2\frac{kv_x}{\omega} + 3\left(\frac{kv_x}{\omega}\right)^2 + \cdots\right)$$

so we have:

$$\begin{aligned}\int P\left(I\right) dv_y dv_z &= \int \hat{f}_0 \left(1 + 2\frac{kv_x}{\omega} + 3\left(\frac{kv_x}{\omega}\right)^2 + ...\right) dv_x dv_y dv_z \\ &= \frac{k^2}{\omega^2}\left(1 + 0 + \frac{3k^2}{\omega^2}\int \hat{f}_0 v_x^2 dv_x dv_y dv_z + ...\right) \\ &= \frac{k^2}{\omega^2}\left(1 + \frac{3k^2}{\omega^2}\frac{k_B T_e}{m} + ...\right)\end{aligned}$$

where the average of one component of $\vec{v}$, squared, over the distribution function, is $k_B T_e/m$, and the average of one component by itself is zero because of the symmetry of the Maxwellian.

For the contribution from the pole, we have to deform the contour so that it passes *under* the pole. It is a simple pole, and if the imaginary part of ω is small, we are getting $\frac{1}{2}$ of a circle around the pole. We are going around the pole counter-clockwise, and so we get

$$+\pi i \left(\text{residue at the pole}\right) = \pi i \left.\frac{\partial \hat{f}}{\partial v_x}\right|_{v_x = \omega/k}$$

Putting it all together, we have:

$$\begin{aligned} 1 &= \frac{\omega_p^2}{k^2}\left[\frac{k^2}{\omega^2}\left(1 + \frac{3k^2}{\omega^2}\frac{k_B T_e}{m} + ...\right) + \pi i \left.\frac{\partial \hat{f}}{\partial v_x}\right|_{v_x = \omega/k}\right] \\ &= \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2}{\omega^2}\frac{k_B T_e}{m} + ...\right) + i\pi\frac{\omega_p^2}{k^2}\left.\frac{\partial \hat{f}\left(v_x\right)}{\partial v_x}\right|_{v_x = \omega/k} \end{aligned} \quad (8)$$

where $\hat{f}\left(v_x\right)$ denotes the distribution function integrated over v_y and v_z

The real part of (8) gives the usual dispersion relation for Langmuir waves:

$$\omega^2 = \omega_p^2 + 3k^2\frac{k_B T_e}{m}$$

where we took $\omega \simeq \omega_p$ in the second term. Now let's assume that the imaginary part is small, $\omega = \omega_r + i\gamma$, $\gamma \ll \omega_r$, and $\omega^2 \simeq \omega_r^2 + 2i\omega_r\gamma$, and let's neglect the small correction term $3v_{th}^2/v_\phi^2$ to the real part. Then the imaginary part of (8) gives:

$$2i\omega_r\gamma = i\pi\omega_r^2\frac{\omega_p^2}{k^2}\left.\frac{\partial \hat{f}\left(v_x\right)}{\partial v_x}\right|_{v_x = \omega/k} \Rightarrow \gamma = \frac{\pi}{2}\frac{\omega_p^3}{k^2}\left.\frac{\partial \hat{f}\left(v_x\right)}{\partial v_x}\right|_{v_x = \omega/k}$$

If $v_\phi = \omega/k$ is in the tail of the distribution, the derivative $\partial \hat{f}\left(v_x\right)/\partial v_x$ is small and negative. Since γ is negative, the wave damps. Chen evaluates γ and gets:

$$\gamma = -\frac{0.22\sqrt{\pi}}{(k\lambda_D)^3} \exp\left(\frac{-1}{2k^2\lambda_D^2}\right)$$

for a Maxwellian. Thus the damping is very small for wavelengths that are much greater than the Debye length ($k\lambda_D \ll 1$). Damping is greatest when λ λ_D. This theoretical result is amply confirmed by experiment.

2 The plasma dielectric constant

This theory gives us a lot more than Landau damping. Why did I call the denominator of equation 7 ε? Let's imagine imposing an external charge density ρ_{ext} into the plasma. Then the Fourier-transformed Poisson's equation (4), expressed in terms of the potential, becomes:

$$k^2\Phi = \frac{1}{\varepsilon_0}\left(\sum_s q_s n_{1s} + \rho_{ext}\right)$$

As before we use the distribution function to get n_1, and use the Vlasov equation to relate f_1 to Φ (eqn 5):

$$f_{1s} = -\frac{q_s}{m_s}k\Phi\frac{\partial f_{0s}}{\partial v_x}\frac{1}{\omega - kv_x}$$

Thus:

$$k^2\Phi = \frac{1}{\varepsilon_0}\left(\sum_s q_s \int -\frac{q_s}{m_s}k\Phi\frac{\partial f_{0s}(v_x)}{\partial v_x}\frac{1}{\omega - kv_x}dv_x + \rho_{ext}\right)$$

So

$$\Phi\left(k^2 + \sum_s \omega_s^2 \int k\frac{\partial \hat{f}_{0s}(v_x)}{\partial v_x}\frac{1}{\omega - kv_x}dv_x\right) = \frac{\rho_{ext}}{\varepsilon_0} \tag{9}$$

where ω_s is the plasma frequency for species s, or, integrating by parts:

$$k^2\Phi\varepsilon_0\left(1 - \sum_s \int \hat{f}_{0s}(v_x)\frac{\omega_s^2}{(\omega - kv_x)^2}dv_x\right) = \rho_{ext}$$

Thus we can identify

$$\varepsilon_0\left(1 - \sum_s \int \hat{f}_{0s}(v_x)\frac{\omega_s^2}{(\omega - kv_x)^2}dv_x\right) = \varepsilon$$

as the dielectric constant for the plasma, and our previous dispersion relation, or any dispersion relation for electrostatic waves, is found by setting $\varepsilon = 0$.

We can also write $\varepsilon = \varepsilon_0(1 + \chi)$, where χ is the susceptibility, and

$$\chi = -\sum_s \int \hat{f}_{0s}(v_x)\frac{\omega_s^2}{(\omega - kv_x)^2}dv_x \tag{10}$$

The sum is over all the particle species in the plasma.

It is also possible to write the susceptibility as:

$$\chi = \sum_s \omega_s^2 \frac{\partial}{\partial \omega_r} \int \hat{f}_{0s}(v_x) \frac{1}{(\omega_r + i\gamma - kv_x)} dv_x \tag{11}$$

Then χ will have a real and an imaginary part, $\chi = \chi_r + i\chi_i$. When $\gamma \ll \omega_r$

$$\chi_r = \sum_s \omega_s^2 \frac{\partial}{\partial \omega_r} P \int \hat{f}_{0s}(v_x) \frac{1}{(\omega_r - kv_x)} dv_x$$

and if we do a Taylor series expansion, we get:

$$\chi(\omega_r + i\gamma) = \chi(\omega_r) + i\gamma \frac{\partial \chi}{\partial \omega_r} + ... = \chi_r + i\chi_i \tag{12}$$

In most cases the imaginary part of χ is due to the contribution of the pole:

$$\chi_i = -i\frac{\pi}{k} \sum_s \omega_s^2 \frac{\partial}{\partial \omega_r} \hat{f}_{0s}\left(\frac{\omega}{k}\right)$$

And thus from equation 12 we have:

$$\gamma = \frac{-\frac{\pi}{k} \sum_s \omega_s^2 \frac{\partial}{\partial \omega_r} \hat{f}_{0s}\left(\frac{\omega}{k}\right)}{\partial \chi / \partial \omega_r} = -\frac{\pi}{k^2} \frac{\sum_s \omega_s^2 \left. \frac{\partial}{\partial v} \hat{f}_{0s}(v) \right|_{v=\frac{\omega}{k}}}{\partial \chi / \partial \omega_r}$$

However, we must be alert for situations where χ has an additional imaginary part.

3 The two-stream distribution revisited

If the electrons are moving at a non-zero velocity with respect to the ions, then a disturbance with $\omega/k < v_0$ in the lab (ion) frame falls on the electron distribution function where the slope $\partial f_0/\partial v$ is positive. These waves will be unstable. We previously found $kv_0/\omega \sim (M/m)^{2/3} > 1$, which satisfies this condition.