

## Physics 360

### Electric fields in dielectrics

Atoms or molecules in a neutral insulator contain charges, of course, because every atomic nucleus has positive charge and is surrounded by a cloud of negative electrons. In some substances the fundamental molecules that make up the substance have charges that are distributed pretty symmetrically so that the dipole moment, as well as the monopole moment, of each molecule is zero. In others, like water, there is a non-zero dipole moment associated with each molecule. If either kind of substance is placed in a field-free region of space, it generates no electric field of its own: in the first case, because there are no non-zero multipoles; and in the second because the dipoles are aligned randomly, so that the net field due to all the dipoles is zero.

Now let's see what happens if we apply an electric field to the material. In the first case, the electric field pushes the positively charged nuclei in one direction and the negatively charged electron cloud in the other, causing each molecule to have a small dipole moment. The size of the dipole moment is directly proportional to the strength of the electric field.

$$\vec{p} = \alpha \vec{E}$$

where  $\alpha$  is the atomic polarizability. We can see how this happens by creating a crude model of an atom as a point nucleus with charge  $Ze$  and a spherical electron cloud with radius  $a$  and charge density  $\rho = -Ze/(4\pi a^3/3)$ . (This is pretty good for a single atom in its ground state.) We'll also assume that the electron cloud stays spherical and moves bodily to one side, so that the nucleus and the center of the cloud are offset by a distance  $d$ . Once equilibrium is reached there is no net charge on the nucleus. That means that the external field  $\vec{E}$  and the field inside the cloud at distance  $d$  from its center sum to zero. We can easily find from Gauss' law (LB Exercise 24.4) that the internal field is

$$\vec{E}_{\text{int}} = \frac{-Ze\vec{r}}{4\pi\epsilon_0 a^3}$$

Thus

$$\frac{Zed}{4\pi\epsilon_0 a^3} = E$$

The dipole moment of the configuration is

$$p = Zed = 4\pi\epsilon_0 a^3 E$$

Thus

$$\alpha = 4\pi\epsilon_0 a^3$$

Tabulated values usually give  $\alpha/4\pi\epsilon_0$  which is  $a^3$  in this simple model. Taking  $a =$  the Bohr radius,  $=6 \times 10^{-11}$  m, we have

$$\frac{\alpha}{4\pi\epsilon_0} = 2. \times 10^{-31} \text{ m}^3$$

Values $\frac{\alpha}{4\pi\epsilon_0}$ in units of $10^{-30} \text{ m}^3$	simple model	H	He	Ca	Na
	0.2	0.67	0.2	1.76	24.1

The simple model works pretty well for the inert gases He and Ne, and is not bad for Hydrogen.

Note here that if  $\vec{E}$  is very strong,  $d$  may be bigger than  $a$ ! This means that the nucleus is no longer in the electron cloud. The atom has been ionized! Once this happens all hell breaks loose – the free electrons make the insulator into a conductor. The electric field strength at which this happens is called the dielectric strength of the material.

Using our simple model again,

$$d = \frac{4\pi\epsilon_0 a^3}{Ze} E = a$$

for

$$E = \frac{Ze}{4\pi\epsilon_0 a^2}$$

and for hydrogen this would be

$$E = \frac{1.6 \times 10^{-19} \text{ C}}{4\pi (8.85 \times 10^{-12} \text{ F/m}) (6 \times 10^{-11} \text{ m})^2} = 4 \times 10^{11} \text{ V/m}$$

This is a very big field! In fact it is too big. Typical dielectric strengths are measured to be around a few MV/m.

More complicated molecules may be more easily polarized in one direction than another, so it is possible that the value of  $\alpha$  could depend on the relative orientation of the electric field and the molecule. Let's not get into that here.

If a molecule has a dipole moment, each molecule experiences a net torque  $\vec{\tau} = \vec{p} \times \vec{E}$  (LB 24.12) that tends to align it along the direction of  $\vec{E}$ .

In both types of material, then, we expect to find dipoles aligned with  $\vec{E}$  when we apply an electric field to the material. We say that the resulting material is *polarized*. The polarization  $\vec{P}$  of the material is the dipole moment per unit volume. If we have  $n$  molecules per unit volume, each with dipole moment  $\vec{p}$ , then

$$\vec{P} = n\vec{p}$$

Let's look at some consequences of this.

*Forces*

If a dipole is in a non-uniform electric field, there is a force on it. Using an ideal dipole as our model,

$$\vec{F} = \lim_{d \rightarrow 0} \left[ -q\vec{E}(\vec{r}) + q\vec{E}(\vec{r} + \vec{d}) \right]$$

We can do a Taylor series expansion on the function  $\vec{E}(x, y, z)$ . For example, if we put the  $x$ -axis along  $\vec{d}$ :

$$\begin{aligned} \vec{E}(x_0 + d, y, z) &= \vec{E}(x_0, y, z) + d \left. \frac{\partial}{\partial x} \vec{E}(x, y, z) \right|_{x=x_0} + \dots \\ \vec{E}(\vec{r} + \vec{d}) &= \vec{E}(\vec{r}) + (\vec{d} \cdot \vec{\nabla}) \vec{E} + \dots \end{aligned}$$

The second form of the result is coordinate-free, and so it is true for any orientation of  $\vec{d}$ . Thus

$$\begin{aligned}\vec{F} &= \lim_{d \rightarrow 0} -q \left[ \vec{E}(\vec{r}) - \left( \vec{E}(\vec{r}) + (\vec{d} \cdot \vec{\nabla}) \vec{E} + \dots \right) \right] \\ &= \lim_{d \rightarrow 0} -q \left[ -\vec{d} \cdot \vec{\nabla} \vec{E} \right] = (\vec{p} \cdot \vec{\nabla}) \vec{E}\end{aligned}$$

Thus a polarized material in a non-uniform field experiences a force per unit volume

$$\vec{f} = (\vec{P} \cdot \vec{\nabla}) \vec{E}$$

This force tends to draw the dielectric into higher field regions. More later....

#### *Energy*

A dipole in a uniform field  $\vec{E}$  possesses potential energy. We can calculate it in the usual way by imagining a way to put the system together. So let's bring in the two charges one at a time. To bring the charge  $-q$  in we exert a force  $q\vec{E}$  that is exactly equal and opposite to the force  $-q\vec{E}$  exerted by the field  $\vec{E}$ . The work we do is

$$W_1 = \int_{\infty}^0 q\vec{E} \cdot d\vec{s}$$

Now we bring in the second charge  $+q$  to its final position  $\vec{d}$ . We have to exert a force that is the exact opposite of the forces exerted by the charge  $-q$  and the field  $\vec{E}$

$$\vec{F} = -q\vec{E} + \frac{kq^2}{r^2} \hat{r}$$

and we do work

$$W_2 = \int_{\infty}^{\vec{d}} \left( -q\vec{E} + \frac{kq^2}{r^2} \hat{r} \right) \cdot d\vec{s}$$

The total work done is

$$\begin{aligned}W &= W_1 + W_2 = \int_{\infty}^0 q\vec{E} \cdot d\vec{s} + \int_{\infty}^{\vec{d}} \left( -q\vec{E} + \frac{kq^2}{r^2} \hat{r} \right) \cdot d\vec{s} \\ &= \int_{\infty}^{\vec{d}} q\vec{E} \cdot d\vec{s} + \int_{\vec{d}}^0 q\vec{E} \cdot d\vec{s} + \int_{\infty}^{\vec{d}} \left( -q\vec{E} + \frac{kq^2}{r^2} \hat{r} \right) \cdot d\vec{s} \\ &= \int_{\vec{d}}^0 q\vec{E} \cdot d\vec{s} + \int_{\infty}^{\vec{d}} \frac{kq^2}{r^2} \hat{r} \cdot d\vec{s}\end{aligned}$$

The first term is what we want. The second term is the self-energy of the dipole, and, as we did for point charges, we are going to neglect it. Finally as  $\vec{d} \rightarrow 0$ , we have

$$W = -q\vec{E} \cdot \vec{d} = -\vec{p} \cdot \vec{E}$$

Thus the potential energy is

$$U = -\vec{p} \cdot \vec{E}$$

and our polarized material has an energy density

$$u = -\vec{P} \cdot \vec{E} \quad (1)$$

due to the polarization. Here we can see that the energy becomes more negative the greater  $\vec{P} \cdot \vec{E}$  becomes. Left to their own devices, systems tend toward the minimum possible energy state, so again we see that our system will be drawn into greater field regions, and aligned with  $\vec{P}$  parallel to  $\vec{E}$ .

**The field produced by a polarized material**

A polarized material, whether the polarization is produced by an external field or something else, will generate its own field as a result of the dipoles within it. We already know the potential produced by each dipole:

$$V_{\text{one dipole at } \vec{r}'}(\vec{r}) = \frac{\vec{p} \cdot (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3}$$

So now we sum up the potentials due to all the dipoles in the material. This is relatively easy because the potential is a scalar. The potential produced by the dipoles in a volume  $d\tau'$  is

$$dV(\vec{r}) = n \frac{\vec{p} \cdot (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3} d\tau' = \frac{\vec{P} \cdot (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3} d\tau'$$

and so the potential produced by the whole material is

$$V(\vec{r}) = \int \frac{\vec{P} \cdot (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3} d\tau' \quad (2)$$

and the integral is over the whole region of space where  $\vec{P}$  is not zero.

Now here's a neat trick

$$\begin{aligned} \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|} &= \left( \hat{x} \frac{\partial}{\partial x'} + \hat{y} \frac{\partial}{\partial y'} + \hat{z} \frac{\partial}{\partial z'} \right) \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\ &= \frac{1}{2} \frac{-2(x-x')\hat{x} - 2(y-y')\hat{y} - 2(z-z')\hat{z}}{\left[ (x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{3/2}} \\ &= \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \end{aligned}$$

So we can write  $V$  as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \vec{P} \cdot \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|} d\tau'$$

Now we integrate by parts, by making use of the relation

$$\vec{\nabla} \cdot (\phi \vec{u}) = \phi \vec{\nabla} \cdot \vec{u} + \vec{u} \cdot \vec{\nabla} \phi$$

with  $\vec{u} = \vec{P}$  and  $\phi = 1/|\vec{r} - \vec{r}'|$ . Thus

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \left[ \vec{\nabla}' \cdot \left( \vec{P} \frac{1}{|\vec{r} - \vec{r}'|} \right) - \frac{\vec{\nabla}' \cdot \vec{P}}{|\vec{r} - \vec{r}'|} \right] d\tau'$$

Using the divergence theorem on the first part, we have:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\vec{P}}{|\vec{r} - \vec{r}'|} \cdot d\vec{A}' - \frac{1}{4\pi\epsilon_0} \int \frac{\vec{\nabla}' \cdot \vec{P}}{|\vec{r} - \vec{r}'|} d\tau'$$

The surface  $S$  is the boundary of the region where  $\vec{P}$  is not zero- usually the surface of our material. The first term looks like the field due to a surface charge density

$$\sigma_b = \vec{P} \cdot \hat{n}. \quad (3)$$

This is not surprising, because the ends of the dipole at the surface form a surface charge layer. We call this "bound" charge, because the charges are still attached to the atoms they came from. The second term looks like the potential due to a bound charge density

$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$

In many cases this bound volume charge density is zero, because  $\vec{P}$  is uniform within the material.

This whole discussion is based on the idea that we can average over a region that is microscopically large (contains very large numbers of molecules) but macroscopically differential (much less than all relevant dimensions of our material). The details are subtle. (see Griffiths section 4.2.3) If you want the truth, the whole truth etc etc look at Jackson section 4.5. The difficulty is in correctly relating the overall polarization  $\vec{P}$  to the individual molecular polarizability  $\alpha$ . As long as we are willing to believe that an external field creates a polarization  $\vec{P}$ , and we can measure how  $\vec{P}$  relates to  $\vec{E}$  as an experimental fact, we don't actually need the theoretical details of how  $\vec{P}$  is related to  $\alpha$ . Thus we have

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

where  $\chi_e$  is the electric susceptibility of the material and  $\vec{E}$  is the macroscopic electric field. Materials that obey this relation are described as *linear*. If the proportionality constant  $\chi_e$  is the same everywhere in the material, it is also described as *homogeneous*, and if  $\chi_e$  is independent of the direction of  $\vec{E}$ , then the material is also *isotropic*. Thus the simplest materials are the LIH (linear, isotropic, homogeneous) materials.

### A polarized sphere

Suppose a sphere of radius  $a$  has a uniform polarization  $\vec{P}$  throughout its interior. We choose a coordinate system with origin at the center of the sphere and with polar axis parallel to  $\vec{P}$ . Then

$$\sigma_b = \vec{P} \cdot \hat{n} = P \cos \theta$$

Then on the polar axis, with  $z > a$ ,

$$\begin{aligned} V(z) &= \frac{1}{4\pi\epsilon_0} \int_S \frac{\vec{P}}{|\vec{r} - \vec{r}'|} \cdot d\vec{A}' = V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{P \cos \theta'}{|\vec{r} - \vec{r}'|} dA' \\ &= \frac{1}{4\pi\epsilon_0 r} \sum_{l=0}^{\infty} \frac{1}{r^l} \int_S a^l P_l(\cos \theta) P \cos \theta' a^2 \sin \theta' d\theta' d\phi' \end{aligned}$$

where we used the expansion of  $1/|\vec{r} - \vec{r}'|$  in Legendre polynomials. The integral is zero unless  $l = 1$ , in which case

$$V(z) = \frac{2\pi}{4\pi\epsilon_0 r} \frac{a^3 P}{r} \frac{2}{3} = \frac{P}{3\epsilon_0} \frac{a^3}{r^2}$$

and thus at an arbitrary point outside the sphere

$$V(\vec{r}) = \frac{P}{3\epsilon_0} \frac{a^3}{r^2} P_1(\cos \theta) = \frac{P}{3\epsilon_0} \frac{a^3}{r^2} \cos \theta$$

This is a dipole potential with dipole moment  $\vec{p} = 4\pi\vec{P}a^3/3 = \vec{P} \times$  the volume of the sphere.

To get the potential inside, we use the boundary condition

$$E_{r,\text{out}} - E_{r,\text{in}} = \frac{\sigma}{\epsilon_0}$$

So

$$E_{r,\text{in}} = 2 \frac{P}{3\epsilon_0} \frac{a^3}{r^3} \cos \theta \Big|_{r=a} - \frac{P \cos \theta}{\epsilon_0} = -\frac{1}{3} \frac{P \cos \theta}{\epsilon_0}$$

Thus the potential inside is

$$V(\vec{r}) = V_0 + \frac{1}{3} \frac{P}{\epsilon_0} r \cos \theta$$

and since the potential is continuous at  $r = a$ ,

$$V_0 = 0$$

Thus

$$V(\vec{r}) = \frac{1}{3} \frac{P}{\epsilon_0} r \cos \theta = \frac{1}{3} \frac{P}{\epsilon_0} z$$

giving a uniform field inside

$$\vec{E} = -\vec{\nabla}V = -\frac{1}{3} \frac{P}{\epsilon_0} \hat{z} = -\frac{1}{3} \frac{\vec{P}}{\epsilon_0}$$

See also Griffiths example 4.3.

### A dielectric slab

Now let's see what happens when we apply a uniform external field to a dielectric slab. The slab becomes polarized, producing equal and opposite surface

charge layers on the two edges of the slab. These layers produce an internal field

$$E_{\text{int}} = \frac{\sigma_b}{\varepsilon_0}$$

opposite the original field, so the net field has magnitude

$$E_{\text{net}} = E_{\text{app}} - E_{\text{int}} < E_{\text{app}}$$

The electric field has been reduced, but, unlike a conductor, the field is not reduced to zero because of the limited mobility of the charges.

Now here's the interesting thing. The polarization is proportional to  $\vec{E}_{\text{net}}$ , which is due, in turn, to  $\vec{P}$ . Thus as the polarization forms, it continuously adjusts, along with  $\vec{E}$ , until equilibrium is achieved. As we saw with conductors, it takes a very short time for this equilibrium to be established.