**Gauss' Law**

**Field lines and Flux**

Field lines are drawn so that $\vec{E}$ is tangent to the field line at every point. Field lines give us information about the direction of $\vec{E}$, but also about its magnitude, since the relative density of field lines is a measure of the relative strength of the electric field. To measure the density, we construct a patch of area perpendicular to the field lines, and measure the number of fields lines passing through the patch. Then

$$\text{density} = \frac{N}{A}$$

If we draw a total number $N$ of field lines to represent the field due to a point charge, then $N$ field lines pass through every sphere surrounding the charge. Thus

$$\text{density} = \frac{N}{4\pi r^2}$$

and is proportional to $1/r^2$, as given by Coulomb's law. Since we can model any charge distribution as a collection of point charges, it is true in general that field strength is proportional to number of lines per unit area.

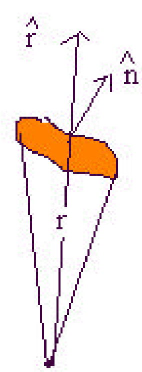
Since field lines begin on positive charge and end on negative charge, the number of field lines emerging from or entering any closed box is a measure of the net charge inside the box. This is Gauss' law.

To get a more quantitative measure of "number of field lines" we use the mathematical quantity *flux*. The flux of any vector field $\vec{v}$ through a surface $S$ is given by the surface integral

$$\Phi_{\vec{v}} = \int_s \vec{v} \cdot \vec{n} \, dA = \int_S \vec{v} \cdot d\vec{A}$$

where the vector differential area $d\vec{A} = \hat{n} \, dA$, and $\hat{n}$ is the normal to that patch of surface.

**Solid angle**



The solid angle subtended at $P$ by a patch of surface $dA$ is defined to be

$$d\quad = \frac{dA_\perp}{r^2} = \frac{d\vec{A}\cdot\hat{r}}{r^2} \tag{1}$$

where $dA_\perp$ is the projection of $dA$ perpendicular to $\hat{r}$. Imagine an ice-cream cone with the area as the surface of the ice-cream. For any closed surface surrounding $P$, the total solid angle subtended at $P$ is $4\pi$ steradians. You can see this by replacing the area patches by pieces of a spherical surface. If we look at pieces of spheres along the same cone, (and thus with the same solid angle) $dA/r^2 =$ constant. so we can shift the surface pieces inward or outward as needed until we get a complete sphere. For a sphere,

$$= \int \frac{dA}{r^2} = \frac{1}{r^2}\int dA = \frac{4\pi r^2}{r^2} = 4\pi.$$

We can get an expression for $d\quad$ in spherical coordinates, again using a piece of a sphere.

$$d\quad = \frac{dA}{r^2} = \frac{r^2\sin\theta d\theta d\phi}{r^2} = \sin\theta\ d\theta\ d\phi \tag{2}$$

or equivalently, the area element on a sphere may be usefully expressed as

$$dA = r^2 d$$

**Gauss' Law**

Let's see how flux of electric field relates to charge. For a closed surface $S$ that contains a point charge $q$ somewhere inside it,

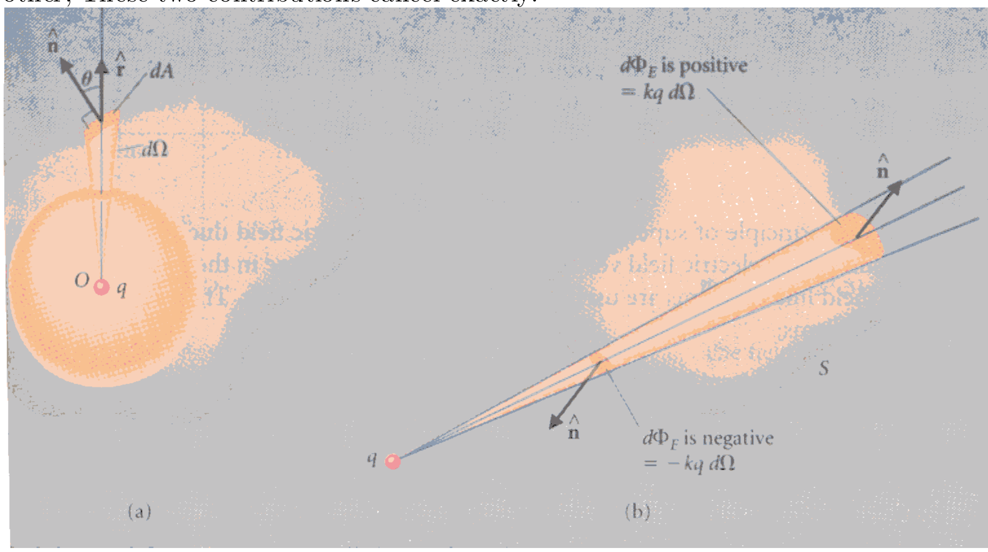
$$\begin{aligned}\Phi_E &= \oint_S \vec{E}\cdot\hat{n}dA = \oint_S \frac{kq}{r^2}\hat{r}\cdot\hat{n}dA \\ &= kq\oint_S d\quad = 4\pi kq = \frac{q}{\varepsilon_0}\end{aligned}$$

Here we used equation (1) and the fact that any closed surface subtends solid angle $4\pi$ at any interior point. Now if our electric field is due to a superposition of fields due to $N$ point charges, we have

$$\begin{aligned}\Phi_E &= \oint_S \sum_{i=1}^{N}\vec{E}_i\cdot\hat{n}dA = \oint_S \sum_{i=1}^{N}\frac{kq_i}{r_i^2}\hat{r}_i\cdot\hat{n}dA \\ &= \sum_{i=1}^{N} kq_i \oint_S d\quad = \frac{\sum_{i=1}^{N} q_i}{\varepsilon_0} = \frac{Q}{\varepsilon_0}\end{aligned}$$

where $Q$ is the total charge inside $S$.

Now suppose that $\vec{E}$ is the superposition of fields due to charges both inside and outside $S$. For any charge that is outside $S$, there is no contribution to the flux, This is because the fields lines run through the surface, in at one side and out at the other, giving a positive flux at one side and a negative flux at the other, These two contributions cancel exactly!




■ FIGURE 23.29
(a) We may express the
due to a point charge t
element of surface in t
solid angle subtended b
at the position of the c
cone of field lines from
outside a closed surfac
the surface twice. The f
uted by the element of
to the charge is negativ
r̂ are almost opposite
$d\Phi_E = -kq\, d\Omega$. The
of area on the far side
contributes a positive f
and the total is zero.

Thus only charge inside the surface contributes to the net flux. This result is Gauss' Law:

$$\oint_S \vec{E} \cdot \hat{n} dA = \frac{Q_{\text{inside}}}{\varepsilon_0}$$

Since we used no information other than geometry and Coulomb's law to get this result, there is no new physics here. However, by expressing the same physics in a different mathematical form, we have a useful new tool. This is a *global* statement about how electric field behaves over an entire surface. We can still use this result to determine $\vec{E}$ locally (at a point) *if* we have enough symmetry. It is necessary that the surface integral can be reduced to one component of $\vec{E}$ times the area of the surface. That can only be done if $\vec{E}$ is parallel or perpendicular to $\hat{n}$ on each surface patch, so that $\vec{E} \cdot \hat{n}$ is either zero, or equal to one component of $\vec{E}$, and that component has a constant value over the whole surface where $\vec{E} \cdot \hat{n}$ is not zero..

To use Gauss' Law to find electric fields we use the method outlined in LB Chapter 24. Let's start with the infinite line that we just looked at using integration.

MODEL We're going to use Gauss' law and so we start with a field line diagram. The system has (a) rotational symmetry about the line and (b) translational symmetry along the line. If we rotate about the line or slide vertically along the line, the picture can't change. Electric field lines begin at positive charge and end at negative charge. The negative charge in this system is all at "infinity", so the field lines start on the line and point radially outward like spokes of a wheel with the line passing through the axis of the wheel.

SETUP We choose a surface with its pieces either parallel or perpendicular to $\vec{E}$ everywhere. This surface is a cylinder with its axis along the filament and having a radius $r$ and height $h$. Then we choose a coordinate system so that $\vec{E}$ has only one component. In a cylindrical coordinate system,with $z-$axis along the filament, $\vec{E} = E_r \hat{r}$. Now we calculate the flux though this cylinder.

$$\begin{aligned}\Phi &= \oint \vec{E}\cdot\hat{n}dA = \int_{\text{top}} E_r\hat{r}\cdot\hat{z}dA + \int_{\text{bottom}} E_r\hat{r}\cdot(-\hat{z})\,dA + \int_{\text{curved part}} E_r\hat{r}\cdot\hat{r}dA \\ &= 0+0+\int_{\text{curved part}} E_r\ dA\end{aligned}$$

Now here's where the symmetry comes in. Because of the rotational and translational symmetry, the value of the component $E_r$ is the same at each point of the curved surface, and thus we can take it out of the integral.

$$\Phi = E_r \int_{\text{curved part}} dA = E_r A = E_r 2\pi r h$$

The charge inside our surface is $Q_{\text{inside}} = \lambda h$

SOLVE We apply Gauss' Law to get

$$\begin{aligned}E_r 2\pi r h &= \frac{\lambda h}{\varepsilon_0} \\ E_r &= \frac{\lambda}{2\pi r\varepsilon_0}\end{aligned}$$

ANALYZE There are several important things to note about this result. First, the height $h$ of the cylinder cancelled out, as it must because of the symmetry. Second, we have found a component of $\vec{E}$, not the magnitude $\left|\vec{E}\right| = E$. The charge density $\lambda$ could be positive or negative. (We assumed it was positive when drawling the FLD, but did not use its sign anywhere in our analysis.) Thus $E_r$ could also be positive or negative, while the magnitude of a vector is always positive. Thus our result tells us that field lines point outward from a positive line but run inward toward a negative line.

The result is identical to the result we obtained by integration, so we needn't repeat the dimension checks etc that we did there. Note, however, that the math is much easier here! When Gauss' law can be used to find $\vec{E}$ it is by far the easiest method. But there are only a few simple cases where it can be used at all. These are systems that have complete spherical, cylindrical or plane symmetry.

Gauss' law in integral form is a global statement about $\vec{E}$. Now we'd like to obtain an equivalent local statement– that is, something that is true at every point in space. We can use the divergence theorem to do this. First we express the charge inside the volume using the charge density $\rho$.

$$\oint \vec{E}\cdot\hat{n}\ dA = \frac{Q_{\text{inside}}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho\ dV$$

The we use the divergence theorem to express the left hand side as a volume integral:

$$\int \vec{\nabla}\cdot\vec{E}\ dV = \int \frac{\rho}{\varepsilon_0}\ dV$$

Both integrals are over the same volume $V$, so

$$\int \left( \vec{\nabla} \cdot \vec{E} - \frac{\rho}{\varepsilon_0} \right) dV = 0 \tag{3}$$

Now in general we cannot conclude that an integrand is zero because the integral is zero. For example,

$$\int_{-1}^{+1} x dx = 0 \qquad \text{but} \qquad x \text{ is not} \equiv 0$$

However, (3) is true for absolutely any volume $V$ at all, so the only way this can happen is if the integrand is zero:

$$\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\varepsilon_0} = 0$$

and thus

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{4}$$

at every point of space. This is the differential form of Gauss' law, and is the first Maxwell Equation.

*Aside*: at this point we can get the second Maxwell equation almost for free. Remember that there is no such thing as magnetic charge, so Gauss' law for the magnetic field says that the magnetic flux through any closed surface is identically zero. This leads directly to the differential Gauss' law for $\vec{B}$

$$\vec{\nabla} \cdot \vec{B} = 0$$

We'll return to this later in the semester.