Potential

The work done by the electric field on a particle with charge q that moves from a point A to a second point P is

$$W = \int dW$$

where dW is the work done as the particle moves through a displacement $d\vec{s}$

$$dW = \vec{F} \cdot d\vec{s} = q\vec{E} \cdot d\vec{s}$$

Thus

$$W = \int_A^B q\vec{E} \cdot d\vec{s}$$

If we are dealing with static electric fields, the force is *conservative,* that is, the work done is independent of the path we choose between points A and B. We can check this assertion using one of the fields we have already found: say, the field due to an infinite line charge.

$$W = q \int_A^B \frac{\lambda}{2\pi r \varepsilon_0} \hat{r} \cdot d\vec{s}$$

The dot product selects the radial component of $d\vec{s}$, leaving

$$W = q\frac{\lambda}{2\pi\varepsilon_0} \int_A^B \frac{dr}{r} = q\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r_B}{r_A}$$

The result depends only on the radial coordinates of the points A and B, independent of the path taken from one to the other.

Thus we can express the work done as the change in potential energy of the system. The work done by the system reduces the stored energy in the system. Thus

$$W\left(A \to B\right) = U\left(A\right) - U\left(B\right) = q\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r_B}{r_A}$$

where the final expression is for the particular system (line charge) that we used above. Since the value of the test charge that we moved comes out of the integrals, we can use a trick similar to what we did in defining electric field. We define a field quantity called *potential* as follows

$$V\left(A\right) - V\left(B\right) = \frac{1}{q} W\left(A \to B\right) = \int_A^B \vec{E} \cdot d\vec{s} \tag{1}$$

The integral on the far right depends only on the electric field $\vec{E}$, and the two points A and B.

Some people like to write the result with a minus sign, by flipping the limits:

$$V\left(A\right) - V\left(B\right) = -\int_B^A \vec{E} \cdot d\vec{s}$$

It doesn't matter which way you do it, so long as you remember that the integral is a *path integral* along a path that you may choose from one point (the "lower limit") to the other (the "upper limit"). A trick I use to get the signs right is to remember that "Potential decreases along field lines". If we go along a field line in the direction of $\vec{E}$, we are going toward lower potental.

As with potential energy, there is an arbitrary constant in this definition. We can set this constant by choosing a reference point R at which we set $V \equiv 0$. Then we may write our line integral as follows:

$$\begin{aligned} V(A) - V(B) &= \int_A^B \vec{E} \cdot d\vec{s} \\ V(A) - 0 - [V(B) - 0] &= \int_A^R \vec{E} \cdot d\vec{s} + \int_R^B \vec{E} \cdot d\vec{s} \\ &= -\int_R^A \vec{E} \cdot d\vec{s} + \int_R^B \vec{E} \cdot d\vec{s} \end{aligned}$$

and

$$V(P) = -\int_R^P \vec{E} \cdot d\vec{s} \tag{2}$$

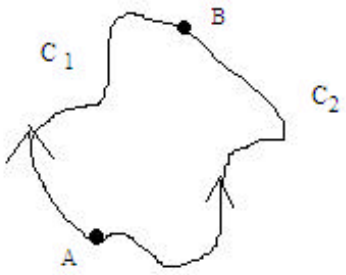
If our system has a finite amount of charge that does not extend out to infinity, we usually choose the reference point to be at infinity. With this choice, the potential at a distance r from a point charge q is

$$V(r) = \frac{kq}{r} \tag{3}$$

However, our line charge extends to infinity, so that choice for R is not available to us. We can see that by noticing that the log function is infinite as $r \to \infty$ (and also negative infinite as $r \to 0$). Thus for this field we must choose our reference point at some finite distance from the line.

Because the potential integral (1) is path independent, we may evaluate the potential difference using two different paths C_1 and C_2 :

$$V(A) - V(B) = \int_{A,C_1}^B \vec{E} \cdot d\vec{s} = \int_{A,C_2}^B \vec{E} \cdot d\vec{s}$$



and so

$$\begin{aligned}\int_{A,C_1}^{B} \vec{E}\cdot d\vec{s} - \int_{A,C_2}^{B} \vec{E}\cdot d\vec{s} &= 0\\ \int_{C=C_1-C_2} \vec{E}\cdot d\vec{s} &= 0\end{aligned}$$

where C is the closed path formed by going from A to B along C_1 and back to A along C_2.

Again this is a global statement about the electric field $\vec{E}$, true for any closed curve C. To get a local statement, we use Stoke's theorem (Griffiths 1.3.5)

$$\int_C \vec{E}\cdot d\vec{s} = 0 = \int_S \left(\vec{\nabla}\times\vec{E}\right)\cdot\hat{n}\ dA$$

where S is any surface spanning the curve C. Since this result must be true for any C (and thus any S), we may conclude that

$$\vec{\nabla}\times\vec{E} = 0 \tag{4}$$

This is our second Maxwell equation, but unlike Gauss' law, it is not yet complete. This result is true only for *static* fields.

Now we have learned something very powerful about static electric fields: The curl of a gradient is always zero (G 1.2.7) so we may express our electric field as the gradient of a scalar function Φ.

$$\vec{E} = -\vec{\nabla}\Phi \Rightarrow \vec{\nabla}\times\vec{E} = 0$$

The minus sign is conventional. Now if $\vec{E} = -\vec{\nabla}\Phi$, then we can compute the path integral as follows:

$$\int_A^B \vec{E}\cdot d\vec{s} = \int_A^B -\vec{\nabla}\Phi\cdot d\vec{s} = -\int_A^B d\Phi = \Phi(A) - \Phi(B)$$

Comparing this result with (1), we can see that this Φ is exactly the potential V that we have already defined!

Note that the relation between $\vec{E}$ and V tells us that V will be an extremum ($\vec{\nabla}V = 0$) wherever $\vec{E} = 0$, but V need not be, and in general is not, zero! Conversely, if $V = 0$, that fact does not tell us anything about the value of $\vec{E}$ at that point. We are free to add any constant to V everywhere in space without changing the value of $\vec{E}$ at all.

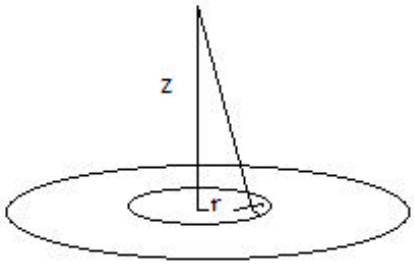
Calculating V.

If we know the electric field, we can use the defining integral (1) to calculate V, as we already did for the infinite line charge case. This method works well when the field is easy to get and is mathematically simple to express (so that the integral is not too hard.) Usually if you have enough symmetry to get $\vec{E}$ using the integral form of Gauss' law, it is pretty easy to get V this way.

If the charge distribution is finite, we may model it as a collection of differential elements (point charges) and use the point charge potential (3). Remember: when you use this method you are putting the reference point at infinity.

Let's compute the potential on its axis due to a disk of radius a with uniform charge density σ.

MODEL We model the disk as a collection of differential rings of width dr. The potential due to one ring is easy to get, because every element of the ring is at the same distance from our point P. Then we sum (integrate) the potentials due to all the rings.



SETUP An element of the ring has length $rd\phi$, thickness dr and charge $dq = \sigma dA = \sigma r dr d\phi$. Its distance from point P is $d = \sqrt{r^2 + z^2}$, the same for each element. The potential due to a ring of radius r is then

$$dV = \int_0^{2\pi} k\frac{\sigma r d\phi}{d} dr = \frac{k\sigma r dr}{\sqrt{r^2 + z^2}} \int_0^{2\pi} d\phi = \frac{2\pi k\sigma r}{\sqrt{r^2 + z^2}} dr$$

Then the potential at P due to all the rings is

$$V(P) = \int_0^a \frac{2\pi k\sigma r}{\sqrt{r^2 + z^2}} dr = \pi k\sigma \int_z^{a^2+z^2} \frac{du}{u^{1/2}}$$

where $u = r^2 + z^2$, $du = 2rdr$

SOLVE The integral is easy, giving

$$V(P) = \pi k\sigma \; 2u^{1/2}\Big|_z^{a^2+z^2} = \pi k\sigma 2\left(\sqrt{a^2 + z^2} - z\right)$$

ANALYZE The surface charge density has dimensions of charge/length^2, so our answer is $k\times$charge/length, as required. As $z \to \infty$, the answer goes to zero, which is correct, but not very informative. The important thing is whether it goes to zero in the right way. We'd expect to get the result for the potential due to a point charge $Q = \pi a^2\sigma$. (This is just RULE 1 again.) Let's investigate:

$$V(P) = \pi k\sigma 2\left(z\sqrt{1 + \frac{a^2}{z^2}} - z\right)$$

Now expand the square root using the binomial expansion:

$$\begin{aligned} V(P) &= \pi k\sigma 2\left[z\left(1+\frac{1}{2}\frac{a^2}{z^2}+\cdots\right)-z\right] \\ &= \pi k\sigma 2\left[z\frac{1}{2}\frac{a^2}{z^2}+\cdots\right] \to k\frac{\pi a^2\sigma}{z} \qquad \text{as } z\to\infty \end{aligned}$$

so we get the correct result.

Now we express the integral in its most general form, as we did previously with $\vec{E}$. The potential at $\vec{r}$ due to a differential element at $\vec{r}'$, with reference point at infinity, is

$$dV = \frac{k\rho\left(\vec{r}'\right)d\tau'}{\left|\vec{r}-\vec{r}'\right|}$$

and so

$$V\left(\vec{r}\right) = k\int\frac{\rho\left(\vec{r}'\right)d\tau'}{\left|\vec{r}-\vec{r}'\right|} \tag{5}$$

The final method for getting V is to solve the differential equation that we can get from Gauss's Law using the relation of V to $\vec{E}$.

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\varepsilon_0} \\ \vec{\nabla}\cdot(-\nabla V) &= \frac{\rho}{\varepsilon_0} \\ \nabla^2 V &= -\frac{\rho}{\varepsilon_0} \end{aligned} \tag{6}$$

This equation is easy to write but is not always so easy to solve, as we shall see.

We can learn something interesting by looking at a point charge again. What is its charge density? It is zero everywhere, except at the position of the charge. For simplicity, let's put the charge at the origin. Then when we integrate over any volume including the origin, we must get

$$q = \int\rho dV$$

We can describe this behavior with a quantity called the delta function. (That's its name, but it is not a real function at all.) It has the properties

$$\begin{aligned} \delta(x) &= 0 \text{ if } x\neq 0 \\ \delta(x) &= \infty \text{ if } x = 0 \\ \int_{-\infty}^{+\infty}\delta(x)\,dx &= 1 \end{aligned}$$

It is actually defined by a property called the sifting property:

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0)$$

Do you see how this follows from the previous properties of $\delta(x)$? Actually all we need is that the range of integration include the origin:

$$\int_a^b f(x)\,\delta(x)\,dx = \begin{cases} f(0) & \text{if} \quad a<0<b \\ 0 & \text{otherwise} \end{cases}$$

Now our charge is at $x=0$ and $y=0$ and $z=0$, so we need three delta functions:

$$\rho(\vec{r}) = q\delta(x)\,\delta(y)\,\delta(z) = q\delta(\vec{r})$$

The final expression is a sort of shorthand for the three delta functions in the middle expression.

Now Poisson's equation becomes

$$\nabla^2\left(\frac{kq}{r}\right) = -\frac{q}{\varepsilon_0}\delta(\vec{r})$$

and so it must be true that

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{r}) \tag{7}$$

This is a very useful result that we will want to use often.

Equipotentials

We can use potential to get a visual picture of electric fields by drawing the surfaces on which V is constant– the equipotential surfaces. Let $d\vec{s}$ be a small displacement along one of these surfaces. Then

$$dV = \vec{E}\cdot d\vec{s} = 0$$

So $d\vec{s}$ must be perpendicular to $\vec{E}$ at that point. Thus:

Equipotential surfaces are perpendicular to electric field lines.

Energy

The potential energy of any system equals the work done to assemble the system. (LB Chapter 8) So if we want to find the energy stored in an electric system, we just have to dream up a convenient procedure for putting the system together. Let's start with a system of point charges.

We bring in the first charge q_1. There are no pre-existing fields, so no force is needed and no work is done. We put the charge at position $\vec{r}_1$.

Now we bring in the second charge q_2. At each point along our path, we have to exert a force on q_2 that is the exact opposite of the force exerted by q_1, so that the net force on q_2 is zero.

$$\vec{F}_{\text{exerted by us}} = -\vec{F}_{\text{exerted by } q1} = -q_2\vec{E}_1$$

Then we can move it at a constant, infinitesimally slow, speed, until we get to the final position $\vec{r}_2$. The work done is

$$W_2 = \int_\infty^{P_2} -q_2\vec{E}_1\cdot d\vec{s} = q_2 V_1(\vec{r}_2) = q_2\frac{kq_1}{|\vec{r}_2-\vec{r}_1|} \tag{8}$$

Now we bring in the third charge. This time the force we need is the exact opposite of the sum of the forces exerted by the first two charges:

$$\begin{aligned} W_3 &= \int_{\infty}^{P_3} -q_3\left(\vec{E}_1+\vec{E}_2\right)\cdot d\vec{s} = -q_3\left[\int_{\infty}^{P_3}\vec{E}_1\cdot d\vec{s}+\int_{\infty}^{P_3}\vec{E}_2\cdot d\vec{s}\right] \\ &= q_3\left[V_1\left(P_3\right)+V_2\left(P_3\right)\right] = k\frac{q_3q_1}{|\vec{r}_3-\vec{r}_1|}+k\frac{q_3q_2}{|\vec{r}_3-\vec{r}_2|} \end{aligned}$$

Now we can see a pattern emerging.

$$\begin{aligned} W &= W_1+W_2+\cdots \\ U &= W = kq_2\frac{q_1}{|\vec{r}_2-\vec{r}_1|}+k\frac{q_3q_1}{|\vec{r}_3-\vec{r}_1|}+k\frac{q_3q_2}{|\vec{r}_3-\vec{r}_2|}+\cdots \\ &= k\sum_{i=1}^{N}\sum_{j<i}\frac{q_iq_j}{|\vec{r}_i-\vec{r}_j|} = \frac{k}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{q_iq_j}{|\vec{r}_i-\vec{r}_j|} \end{aligned}$$

Another way to think of this is that the energy is stored in pairs of charges, and we need to sum over all the distinct pairs.

We can also express the result in terms of potential: Let $V\left(\vec{r}_i\right)$ denote the potential at the position $\vec{r}_i$ of charge q_i due to all of the other charges, $j=1-N$, excluding charge q_i. Then

$$U = \frac{1}{2}\sum_{i=1}^{N}q_iV\left(\vec{r}_i\right)$$

Now what if we have a continuous distribution of charge instead of a set of point charges? Well, we model our distribution as a collection of differential elements, and treat each element as a point charge $dq=\rho d\tau$. But now our sum becomes an integral.

$$U = \frac{1}{2}\int V\left(\vec{r}\right)\rho\left(\vec{r}\right)d\tau \tag{9}$$

(I'm using Griffith's notation τ for volume to avoid confusion with potential V.) However there's a catch here. Notice that we have no way to exclude $i=j$ as we did when summing over point charges. We'll see what that does in a minute.

We are going to express the energy U in terms of the field alone, so we start by expressing ρ in terms of $\vec{E}$ using Gauss' law,

$$\rho = \varepsilon_0\vec{\nabla}\cdot\vec{E}$$

Then

$$U = \frac{\varepsilon_0}{2}\int V\vec{\nabla}\cdot\vec{E}\;d\tau$$

We'd like to use the relation between $\vec{E}$ and V, $\vec{E}=-\vec{\nabla}V$, but how? We need to do an "integation by parts". Note that (G 1.2.6 iii)

$$\vec{\nabla}\cdot\left(\phi\vec{u}\right) = \phi\vec{\nabla}\cdot\vec{u}+\vec{u}\cdot\vec{\nabla}\phi$$

We can use this result with $\phi = V$ and $\vec{u} = \vec{E}$ to get

$$V\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(V\vec{E}\right) - \vec{E}\cdot\vec{\nabla}V = \vec{\nabla}\cdot\left(V\vec{E}\right) + \vec{E}\cdot\vec{E}$$

Thus

$$U = \frac{\varepsilon_0}{2}\int\left[\vec{\nabla}\cdot\left(V\vec{E}\right) + \vec{E}\cdot\vec{E}\right]d\tau$$

Next we can use the divergence theorem to evaluate the first term:

$$\int\vec{\nabla}\cdot\left(V\vec{E}\right)d\tau = \oint\limits_{S_\infty} V\vec{E}\cdot\hat{n}dA$$

The original integral was over all space, so the bounding surface is "at infinity". Now provided that the total charge in our system is finite, we can use RULE 1 again to argue that

$$V \simeq \frac{kQ}{r}; \qquad \vec{E} \simeq \frac{kQ}{r^2}\hat{r}$$

so that on our surface at infinity (which we might as well take to be a sphere centered at the origin, so that $\hat{r}\cdot\hat{n} = 1$)

$$\oint\limits_{S_\infty} V\vec{E}\cdot\hat{n}dA = kQ^2\lim_{r\to\infty}\oint\frac{1}{r^3}r^2 d \quad = kQ\lim_{r\to\infty}\frac{1}{r}\oint d \quad = 4\pi kQ\lim_{r\to\infty}\frac{1}{r} = 0$$

Notice that idealizations like infinite line charges must be excluded here because they extend all the way to infinity.

The remaining term in our energy expression is

$$U = \frac{\varepsilon_0}{2}\int E^2 d\tau = \int u\, d\tau$$

where the lower case u is the electric *energy density*

$$u = \frac{\varepsilon_0}{2}E^2 \tag{10}$$

This expression is of the standard form for energies in physics: 1/2 times (constant) times (variable)^2. (KE = $\frac{1}{2}mv^2$, spring PE=$\frac{1}{2}ks^2$, etc), and, like these energies, it is always positive. Let's see what happens if we apply the result to two point charges. For convenience put the origin on the first charge.

$$\vec{E}\left(\vec{r}\right) = \frac{kq_1}{r^2}\hat{r} + kq_2\frac{\left(\vec{r}-\vec{r}_2\right)}{\left|\vec{r}-\vec{r}_2\right|^3}$$

Thus

$$\begin{aligned} E^2 &= \left[\frac{kq_1}{r^2}\hat{r} + kq_2\frac{\left(\vec{r}-\vec{r}_2\right)}{\left|\vec{r}-\vec{r}_2\right|^3}\right]\cdot\left[\frac{kq_1}{r^2}\hat{r} + kq_2\frac{\left(\vec{r}-\vec{r}_2\right)}{\left|\vec{r}-\vec{r}_2\right|^3}\right] \\ &= \left(k\frac{q_1}{r^2}\right)^2 + \left(\frac{kq_2}{\left|\vec{r}-\vec{r}_2\right|^2}\right)^2 + 2k^2\frac{q_1q_2}{r^2}\hat{r}\cdot\frac{\left(\vec{r}-\vec{r}_2\right)}{\left|\vec{r}-\vec{r}_2\right|^3} \end{aligned}$$

The first term here depends only on q_1, the second only on q_2, and the third on the product $q_1 q_2$. The first term gives the *self-energy* of charge one, and it is infinite if the charge is really a point.

$$\begin{aligned} U_1 &= \frac{\varepsilon_0}{2}\int E_1^2 d\tau = \frac{\varepsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_{r_{\min}}^{\infty}\left(k\frac{q_1}{r^2}\right)^2 r^2 \sin\theta d\theta d\phi dr \\ &= \frac{\varepsilon_0}{2}k^2 q_1^2 \int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta d\theta \int_{r_{\min}}^{\infty}\frac{1}{r^2}dr \\ &= 2\pi\varepsilon_0 k^2 q_1^2 \left(-\frac{1}{r}\right)\Big|_{r_{\min}}^{\infty} \\ &= \frac{k}{2}\frac{q_1^2}{r_{\min}} \to \infty \text{ as } r_{\min} \to 0 \end{aligned}$$

Similarly the second term is the self energy of charge 2. The third term is the interaction energy that depends on both of the charges, and it is finite. In fact, it is identical to the result (8)

$$U_{12} = \int 2k^2 \frac{q_1 q_2}{r^2}\hat{r}\cdot\frac{(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3}d\tau = k\frac{q_1 q_2}{r_{12}}$$

that we obtained for two point charges. This is the only part of the total energy that can ever be changed. The self energy is an intrinsic property of each charge and we cannot get our hands on it. Thus we choose to ignore it, and look only at the finite, interaction energy. The self energy was introduced because we could not exclude "$i = j$" from our integral (9).

This formulation tells us something about where the energy is stored in our system: it is stored in the field itself and extends throughout the whole of space.

Notice that since the energy is quadratic in the fields, we cannot just add the energies of different parts of our system. The interaction terms are very important. The superposition principle for fields tells us that we can add the fields due to different sources:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots$$

but the corresponding energy density is proportional to

$$E^2 = E_1^2 + E_2^2 + \cdots + 2\vec{E}_1\cdot\vec{E}_2 + 2\vec{E}_1\cdot\vec{E}_3 + \cdots$$