

Conductors

So far we have been calculating electric fields due to charge distributions in otherwise empty space (vacuum). Now we need to pay attention to the actual material in which the charges exist. If we are not in vacuum, the material must be one of the following:

- conductor.
- insulator
- semiconductor

A conductor is a material that contains charges that are free to move under the influence of applied forces. Most conductors are metals, like copper or steel, and usually the charges that move are electrons. The interesting thing about conductors is that they do not allow any electric fields to exist within them in a static situation. Suppose an electric field is applied to a conducting material. Then each free charge will experience a force $\vec{F} = -e\vec{E}$ and will accelerate rapidly in the direction opposite \vec{E} . Thus the charges will rearrange themselves until there is no longer any \vec{E} inside the material, and the acceleration stops. Each charge need only move a tiny distance (a few atomic diameters) to achieve this rearrangement. Thus the whole process is completed very rapidly. As the charges move, the fields have to change correspondingly. It is the timescale for the fields to change that determines how long the whole process takes. Since the electric fields can change at the speed of light, the time scale may be estimated as

$$\tau \sim \frac{L}{c}$$

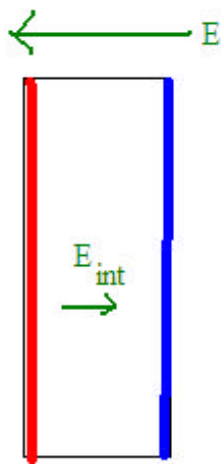
where L is a characteristic dimension of the conductor. If $L = 1$ m, then

$$\tau \sim \frac{1 \text{ m}}{3 \times 10^8 \text{ m/s}} = 3 \text{ ns}$$

To get an idea of how this works, let's take an infinite slab of copper, 1 m wide. We apply a field of 10 V/m to the slab. Then each electron accelerates at a rate

$$a = \frac{(1.6 \times 10^{-19} \text{ C})(10 \text{ N/C})}{9 \times 10^{-31} \text{ kg}} \simeq 2 \times 10^{12} \text{ m/s}^2$$

This is a huge acceleration! After each electron has moved one atomic diameter (which takes a time roughly given by $s = \frac{1}{2}at^2$ or $t = \sqrt{2(10^{-10}\text{m})/2 \times 10^{12} \text{ m/s}^2} = 10^{-11} \text{ s}$), there is a layer of negative charge on one surface of the slab and a layer of positive charge on the other.



These layers produce an internal electric field that is opposite the original applied field. The electrons continue to accelerate until the internal field exactly balances the applied field so that the sum is zero. The surface charge density then is σ , where

$$E_{\text{int}} = \frac{\sigma}{\epsilon_0} = E_{\text{app}}$$

and thus the σ we have is

$$\begin{aligned} \sigma &= \epsilon_0 E_{\text{app}} = 8.85 \times 10^{-12} \text{ F/m} \times 10 \text{ V/m} \\ &= 8.85 \times 10^{-11} \text{ C/m}^2 \\ &= \frac{8.85 \times 10^{-11} \text{ C}}{1.6 \times 10^{-19} \text{ C/e}} / \text{m}^2 = 5.5 \times 10^8 \text{ e/m}^2 \\ &\sim 5.5 \times 10^8 \text{ e/m}^2 \times 10^{-20} \text{ m}^2 \text{ atom} \\ &5.5 \times 10^{-12} \text{ e/atom.} \end{aligned}$$

So you can see that it really takes very few electrons to make this happen.

Once the electric field in the interior is zero, the charge density in the interior must be zero too, because if $\vec{E} \equiv 0$, then $\vec{\nabla} \cdot \vec{E} = 0$ and thus $\rho = 0$. Notice here that the divergence is zero because the electric field is zero at *all* interior points. If \vec{E} were zero at one point, we could then have a non-zero divergence. So all of the net charge density is on the surfaces of the conductor, and not in the interior.

Finally, if $\vec{E} \equiv 0$ inside the conductor, then the potential difference between any two points in the interior is

$$\Delta V = \int_A^B \vec{E} \cdot d\vec{s} = 0$$

Thus the whole conductor is an equipotential. Since we have already established that the field lines are perpendicular to the equipotentials, this means that outside the conductor the electric field is perpendicular to the surface.

Another way to understand this result is to note that any tangential field would cause electrons to flow around the surface until that tangential component were reduced to zero.

Insulators, sometimes called dielectrics, are materials that do not have charges free to move under the influence of applied fields. The charges can move a little tiny bit, all the same, leading to the production of little dipoles of atomic scale inside the material- the material is polarized. These dipoles also produce layers of charge density on the surface- called bound charge density, because the charges are still bound to their respective atoms. Thus we still get some internal field, but it is not large enough to render the internal field zero.

$$\vec{E}_{\text{net}} = \vec{E}_{\text{applied}} + \vec{E}_{\text{int}}; \quad \left| \vec{E}_{\text{net}} \right| = \left| \vec{E}_{\text{applied}} \right| - \left| \vec{E}_{\text{int}} \right| > 0$$

Finally, semiconductors are an interesting intermediate case in which there are some charges that can move, and we can increase their numbers by appropriate doping. The most well-known semiconductor is silicon, but carbon is also a semiconductor.

Boundary conditions

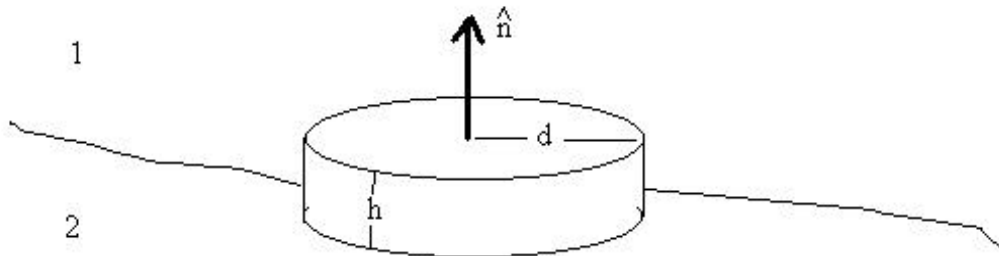
We will need to understand what happens to the electric field at a boundary between two different materials. We have already seen above that the value of \vec{E} is different inside a conductor versus outside, for example. The two equations that we already have for \vec{E} provide the answer.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

and

$$\vec{\nabla} \times \vec{E} = 0$$

To make use of the first equation, we construct a small, flat box (a pillbox or tuna can) that straddles the boundary between the two regions. For example we could choose a cylinder with height h and radius $d \gg h$. We place the box with its vertical sides perpendicular to the boundary surface, and the other sides parallel to the surface. To do this we have to make d much less than any dimension over which such things as the slope of the boundary change. If R is the smallest radius of curvature of the surface, then $h \ll d \ll R$. Now we integrate Gauss' law over the volume of the box.



$$\int_{\text{pillbox}} \vec{\nabla} \cdot \vec{E} \, d\tau = \int_{\text{pillbox}} \frac{\rho}{\epsilon_0} d\tau$$

Next we use the divergence theorem to convert the integral on the left to a surface integral

$$\int_{\text{surface of pillbox}} \vec{E} \cdot d\vec{A} = \int_{\text{pillbox}} \frac{\rho}{\epsilon_0} dA dh$$

On the right we integrate over the vertical dimension, and use the definition of the surface charge density:

$$\sigma = \lim_{h \rightarrow 0} \int \rho dh$$

(Strictly h cannot go all the way to zero: the charge layer will be a few atomic diameters thick.) On the left, the surface has three pieces: The top side (in medium 1, with $d\vec{A} = \hat{n}dA$), the bottom side (in medium 2, with $d\vec{A} = -\hat{n}dA$) and the vertical sides of height $h \ll d$, with $d\vec{A}$ tangent to the surface. Thus

$$\int_{\text{top}} \vec{E}_1 \cdot \hat{n} \, dA + \int_{\text{sides}} \vec{E} \cdot d\vec{A} + \int_{\text{bottom}} \vec{E}_2 \cdot (-\hat{n}) \, dA = \frac{1}{\epsilon_0} \int \sigma \, dA$$

The integral over the sides is neglected because its area $2\pi dh$ is much less than the area πd^2 of the top and bottom, and the tangential electric field component is not expected to be vastly larger than the normal component to compensate. (Indeed, if one medium is a conductor, $\vec{E}_{\text{tan}} = 0$.) Thus we have

$$\int_{\text{top}} \vec{E}_1 \cdot \hat{n} \, dA + \int_{\text{bottom}} \vec{E}_2 \cdot (-\hat{n}) \, dA = \frac{1}{\epsilon_0} \int \sigma \, dA$$

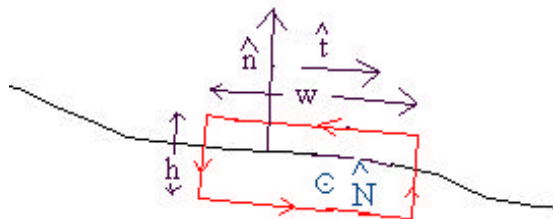
Now since our pillbox is so small, by construction, the integral is basically just one differential element, so we remove the integral sign and cancel the dA which is the same in each integral, so finally we have our boundary condition:

$$\left(\vec{E}_1 - \vec{E}_2 \right) \cdot \hat{n} = \frac{\sigma}{\epsilon_0} \quad (1)$$

This is a very useful result, but do be careful using it. Signs matter! The surface charge density could be positive or negative, and there is a dot product on the left that can take either sign. Always draw the pillbox, define your normal vector \hat{n} , and apply Gauss' law carefully. It only takes a few minutes and it will save you lots of trouble. If you just "grab" the formula (1) without an accompanying diagram, you have a 50% chance of getting the sign wrong. Of course if σ happens to be zero there is no sign problem, and we have the simpler result that the normal component of \vec{E} is continuous across the boundary.

ASIDE Remember that we previously obtained Gauss' law for \vec{B} , $\vec{\nabla} \cdot \vec{B} = 0$, so the same analysis gives us the result that the normal component of \vec{B} is continuous across any boundary.

The second equation we have for \vec{E} is $\vec{\nabla} \times \vec{E} = 0$. To make use of this, we are going to use Stokes' theorem, so we need to construct a skinny rectangle across the surface. There is considerable freedom in how we do this, because there are infinitely many ways that we can orient the rectangle. So we just pick one to start.



The rectangle is oriented with its long sides parallel to the vector \hat{t} in the surface and its normal \hat{N} is also parallel to the surface, but perpendicular to \hat{t} . The normal \hat{n} to the surface is parallel to the short sides of the rectangle. We choose $h \ll w \ll R$ as before. Now we integrate over the surface of this rectangle:

$$\int_{\text{rectangle}} (\vec{\nabla} \times \vec{E}) \cdot \hat{N} dA = 0$$

Now use Stokes' theorem to convert the integral to a line integral around the rectangle:

$$\oint_{\text{rectangle}} \vec{E} \cdot d\vec{l} = 0 = \vec{E}_1 \cdot (-\hat{t}) w + \int_{\text{left side}} \vec{E} \cdot (-\hat{n}) dh + \vec{E}_2 \cdot \hat{t} w + \int_{\text{right side}} \vec{E} \cdot \hat{n} dh$$

The two integrals over the short sides are negligible because $h \ll w$. (And since we took $w \ll R$, the tiny values almost cancel anyway.) Thus

$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{t} = 0$$

Remember that the direction of \hat{t} is arbitrary so long as it lies in the surface. We could repeat the analysis for any number of different orientations of the rectangle, obtained by rotating the picture about \hat{n} . Thus we conclude that

$$\vec{E}_{\text{tangential}} \text{ is continuous across the boundary} \quad (2)$$

While we have proved this result for static fields, it is also true when things are changing in time, as we will see next semester.

Summarizing, we have the conditions that tell us how \vec{E} changes at a boundary between two different materials:

$$\text{Normal component:} \quad (\vec{E}_1 - \vec{E}_2) \cdot \hat{n} = \frac{\sigma}{\epsilon_0}$$

and

$\vec{E}_{\text{tangential}}$ is continuous across the boundary

Finally we can learn something about the potential. The potential difference across the boundary is

$$\Delta V = V_1 - V_2 = \int_1^2 \vec{E} \cdot d\vec{l}$$

We are going to argue that the integral on the right goes to zero as the path length from 1 to 2 goes to zero, so that the potential is continuous across the surface. This is clearly true unless \vec{E} has some kind of nasty singularity at the surface, but it doesn't. Our results (1) and (2) show that \vec{E} has at most a finite discontinuity at the surface under normal circumstances. It would take a weird boundary layer (such as a dipole layer) to make the potential change. We won't need to worry about those odd things this semester.

More properties of conductors

If we apply the results we have just obtained to conductors, with $\vec{E} = 0$ inside the conducting material, we have for the field just outside the conductor:

$$\begin{aligned} \vec{E}_{\text{tan}} &= 0 \\ E_{\perp} &= \frac{\sigma}{\epsilon_0} \end{aligned} \quad (3)$$

These results are consistent with our previous results, obtained qualitatively above.

Distribution of charge on the surface.

The charge will not be uniformly distributed on a conductor unless the surface shape is very symmetrical- like a perfect sphere or a flat slab. We can see qualitatively what happens by remembering that the conductor is an equipotential, and the field lines are perpendicular to the equipotentials. If we draw an odd shape with hills and valleys, it is hard to get the field lines into the valleys, and they tend to congregate on the hills. Through (3) this tells us that the charge density is big near "spikey" points and small in depressions. (See LB 25.6.4)

Holes inside conductors.

We have shown that \vec{E} must be zero inside the conducting material, but what if there is a hole inside the conductor? Can we have \vec{E} inside the hole? If there is no charge inside the hole, the answer is no. We need to use both of our equations for \vec{E} to show this. First, Gauss' law tells us that no field lines can begin or end inside the hole, so the only possibility is that field lines run across the cavity from positive charge in one place on the inner surface to negative in another. But then if we compute

$$\Delta V = \int_1^2 \vec{E} \cdot d\vec{l}$$

where the path of integration lies right along the field line, we get a non-zero result. But this violates the known fact that the whole conductor is an equipotential, and so no such field lines can exist. This is the idea behind the Faraday

cage: a room lined with a conducting material excludes all external electric fields from its interior.

Life gets interesting when we put charge q inside the hole. If we put a Gaussian surface around the charge, but inside the hole, there is a net flux through the surface and so field lines emerge from the charge. However, there can be no field inside the conducting material, so a Gaussian surface lying entirely within the conducting material and surrounding the hole has no flux through it, and so there can be no net charge inside that volume. Thus there must be a charge $-q$ distributed over the inside surface of the conductor, around the hole. Of course if we pull charges to this surface, they have to come from somewhere, and so there must be a net charge of $+q$ on the outer surface of the conductor. (Remember—there can be no net charge density inside the conducting material.) Then this outer charge density gives rise to a field outside the conductor through (3). In this way the outside world learns of the existence of the charge inside. The field outside depends on the value of q and on the shape of the conductor's outer surface, but does not depend on the details of the charge distribution inside.

Force on a conductor

Since there is charge on the surface of a conductor, and there is an electric field present, we would expect that there would be a force acting on the conductor. But the field has a discontinuous jump right at the position of the charge, so which field should we use? The correct result is $\vec{E}_{\text{ave}} = \frac{1}{2}(\vec{E}_{\text{in}} + \vec{E}_{\text{out}})$ so that

$$\vec{F} = \frac{\sigma \vec{E}_{\text{out}}}{2} = \frac{\sigma^2}{2\epsilon_0} \hat{n}$$

The force is outward no matter what the sign of σ , because when σ changes sign, so does \vec{E}_{out} .

Now let's see why this is the right thing to do. We assert that charges do not exert forces on themselves. We have already used this idea in computing the energy of a system of charges. So when computing the force on a patch of surface with charge $dq = \sigma dA$, we have to carefully determine how much of the field \vec{E}_{out} is due to that charge, and how much due to the charge on the rest of the conductor's surface. We can find the field due to the patch itself because we know it is perpendicular to the patch, so we can put a small cylinder around it and apply Gauss' law. By symmetry, the patch produces equal field on either side, and so we get

$$\begin{aligned} \vec{E}_{\text{patch}} &= \frac{\sigma}{2\epsilon_0} \hat{n} \text{ on the outside} \\ &= -\frac{\sigma}{2\epsilon_0} \hat{n} \text{ on the inside} \end{aligned}$$

just as for an infinite sheet. Then the field due to the rest of the surface, which changes negligibly over the little cylinder, must exactly cancel the inward-pointing field due to the patch itself, so it is

$$\vec{E}_{\text{rest}} = \frac{\sigma}{2\epsilon_0} \hat{n} \text{ outside and inside}$$

giving a total field on the outside of

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{n} + \frac{\sigma}{2\epsilon_0} \hat{n} = \frac{\sigma}{\epsilon_0} \hat{n}$$

as required. Then

$$\vec{F} = \sigma \vec{E}_{\text{rest}} = \frac{\sigma^2}{2\epsilon_0} \hat{n} \quad (4)$$

Capacitors

A simple capacitor is made from two conductors, and, in normal use, we put equal and opposite charges on the two conductors. Since each conductor is an equipotential, the potential difference between them

$$\Delta V = \int_1^2 \vec{E} \cdot d\vec{l}$$

may be computed using any path from any point on one conductor to any point on the other. Now we know that the charge will reside on the surface of each conductor, and the charge density depends on the shape of the conductor (greater on the top of hills and smaller at the bottom of valleys). So when we increase the total charge, we change the charge density everywhere by a corresponding factor. This causes \vec{E} everywhere, and thus the potential difference, to increase by that same factor. We'll prove this assertion more rigorously later in the semester. Then

$$|\Delta V| \propto |Q|$$

The constant of proportionality is $1/C$ where C is the capacitance of the configuration. It is a geometrical quantity that depends on the size and shape of the conductors. Then we have the relation between the absolute values:

$$|\Delta V| = \frac{|Q|}{C}; \quad |Q| = C |\Delta V|$$

It is possible for the second conductor to be at infinity. As an example, we can calculate the capacitance of a single spherical conductor of radius a (with the second one at infinity). The field at radius $r > a$ is

$$\vec{E} = k \frac{Q}{r^2} \hat{r}$$

We choose a radial path from any point on the sphere out to infinity, to get

$$\Delta V = \int_a^\infty \frac{kQ}{r^2} dr = \frac{kQ}{a} = \frac{Q}{C}$$

Thus

$$C = \frac{a}{k} = 4\pi\epsilon_0 a$$

This result also shows us how to estimate the capacitance of any configuration of conductors. C is roughly ϵ_0 times an appropriate physical dimension of

the system. For a single sphere, that dimension is twice the circumference at the equator, but we could get a reasonable estimate by using the diameter. The earth is mostly covered in salt water, a pretty good conductor, so we can estimate its capacitance:

$$C_{\text{Earth}} = (8.85 \times 10^{-12} \text{ F/m}) 4\pi (6 \times 10^6 \text{ m}) = 7 \times 10^{-4} \text{ F}$$

This should convince you that a farad is a very big unit!

The primary use of a capacitor in a circuit is as a place to store energy. To see how much energy is stored, we invent a process for charging it up. In a circuit, charge is moved from one conductor to the other through the battery. (As usual, each electron travels a tiny distance to make this happen.) But we can imagine moving the charge directly across from one plate to another to achieve the same end result. If the capacitor is partially charged with charge q , there is already a potential difference $\Delta V = q/C$ between the plates, so we have to do work

$$dW = \Delta V dq = \frac{q}{C} dq$$

to move an additional amount dq of charge from one conductor to the other. Thus the total work done to charge the capacitor is

$$W = \int dW = \int_0^Q \frac{q}{C} dq = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} \frac{(C\Delta V)^2}{C} = \frac{1}{2} C (\Delta V)^2$$

The result is of the usual $(1/2) \times \text{constant} \times (\text{variable})^2$ form.

Of course there is another way to get the energy, and that is to start with the energy density $u = \frac{1}{2}\epsilon_0 E^2$ and integrate.

$$W = \int \frac{1}{2} \epsilon_0 E^2 d\tau$$

Let's do this for our sphere:

$$u = \frac{1}{2} \epsilon_0 \left(\frac{kQ}{r^2} \right)^2$$

and so

$$W = \frac{1}{2} \epsilon_0 k^2 Q^2 \int_a^\infty \frac{1}{r^4} 4\pi r^2 dr = \frac{1}{2} k Q^2 \left(-\frac{1}{r} \right) \Big|_a^\infty = \frac{1}{2} \frac{Q^2}{4\pi \epsilon_0 a} = \frac{1}{2} \frac{Q^2}{C}$$

as required.

We can also use this relation to compute C as

$$C = \frac{1}{2} \frac{Q^2}{W}$$