Laplace's equation

We have expressions for $\vec{E}$ and V as integrals over the charge density ρ, but these expressions are useful only if (a) we actually know ρ eveywhere in the universe and (b) we can do the integrals. These are not trivial conditions. So we shall find it useful to work with the differential equations for $\vec{E}$ and V, especially if we are interested in a solution in a limited region of space, say, inside our experimental apparatus. We have already found the equation for V : it is Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

and this is even simpler at points where the charge density is zero:

$$\nabla^2 V = 0$$

This is Laplace's equation, and it is actually very useful. For example, when we have a system of conductors, we know that all the charge resides on the surfaces of the conductors and nowhere else, so ρ is zero almost everywhere. Functions that satisfy Laplace's equation are called Harmonic functions.

Physics principles are usually expressed in the form of differential equations, like Laplace's equation, but the solution to the physics problem does not depend on the differential equation alone, but also on boundary conditions. We are used to this idea from mechanics. If a particle moves under the influence of a constant force, then it has constant acceleration and its position satisfies the differential equation:

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}$$

We integrate once to get

$$\frac{d\vec{r}}{dt} = \vec{a}t + \vec{v}_0$$

where the integration constant $\vec{v}_0$ is an initial condition- a boundary condition in time. Integrating again, we get

$$\vec{r} = \frac{1}{2}\vec{a}t^2 + \vec{v}_0 t + \vec{r}_0$$

where $\vec{r}_0$ is the intial position. The easiest solution is obtained when we know both $\vec{v}_0$ and $\vec{r}_0$. For example, if $\vec{v}_0 = 0$, the particle will move along a straight line parallel to $\vec{a}$, but if $\vec{v}_0$ is perpendicular to $\vec{a}$, the path will be a parabola. We could also solve the problem if we knew the particle's position at two different times, and then we would be able to find the constant $\vec{v}_0$. Suppose $\vec{r}(t_1) = \vec{r}_1$. Then

$$\vec{r}_1 = \frac{1}{2}\vec{a}t_1^2 + \vec{v}_0 t_1 + \vec{r}_0$$

so

$$\vec{v}_0 = \frac{\vec{r}_1 - \vec{r}_0}{t_1} - \frac{1}{2}\vec{a}t_1$$

and so the solution is

$$\begin{aligned}\vec{r} &= \frac{1}{2}\vec{a}t^2 + \left(\frac{\vec{r}_1 - \vec{r}_0}{t_1} - \frac{1}{2}\vec{a}t_1\right)t + \vec{r}_0 \\ &= \frac{1}{2}\vec{a}t\left(t - t_1\right) + \frac{\vec{r}_1 t}{t_1} + \vec{r}_0\left(1 - \frac{t}{t_1}\right)\end{aligned}$$

So you can see that the boundary conditions are just as important as the differential equation.

Let's see what kind of solutions we get to Laplace's equation.

Laplace's Equation -general properties:

In Cartesian coordinates we have

$$\nabla^2 V = \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0$$

Now let's suppose that V depends only on x, (the problem is one-dimensional) so the equation simplifies:

$$\frac{d^2V}{dx^2} = 0$$

and this looks just like the mechanics equation we solved above (but simpler, because V is a scalar). We integrate twice, getting two integration constants:

$$\frac{dV}{dx} = a \quad (1)$$

$$V = ax + b \quad (2)$$

The solution is linear in x. To proceed, we need the boundary conditions: We will need to know the value of V at two places, or, we could be told the potential at one place and the electric field at another. Of course equation (1) shows us that the electric field is a constant, the same everywhere. This is actually the solution for the electric field between two infinite conducting planes, or the field produced by an infinite charged sheet.

Already we can see some interesting features of the result. First, V has no maxima or minima in the interior of any finite region. This property turns out to be general, and applies to all solutions of Laplace's equation. Second, the potential at any point P is the average of the values at two points equidistant from P:

$$V(x) = \frac{1}{2}\left[V(x + c) + V(x - c)\right]$$

This tells us that Laplace's equation is a sort of "averager", its solutions are nice and smooth. It also gives us a nice method for finding solutions numerically by doing the averaging in a computer. Supposing we know that $V = 1$ V at $x = 0$ and $V = 5$ V at $x = 2$ m. We set up a grid in a spreadsheet program, say having ten cells between $x = 0$ and $x = 2$ m. We put in the known boundary values, and any values we like at the intermediate points. Then we put in an instruction to average the values on either side, and let the spreadsheet do this

over and over until it converges. It converges very fast and gives the solution (2).

We can also look at the solution in spherical coordinates, where the potential depends only on r. Laplace's equation takes the form

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

We integrate once to get

$$r^2\frac{dV}{dr} = c$$

and again to get

$$V = -\frac{c}{r} + b$$

With $c = -q$ and $b = 0$, this is our usual solution for the potential produced by a point charge or outside a charged sphere. With $c = 0$ we get the solution inside a uniformly charged sphere- a constant.

Now let V be a function of the two coordinates x and y. The equation is a partial differential equation, and its solution is more interesting.

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0$$

The Laplacian operator still has the averaging property we discussed above, and the numerical method works very nicely in two dimensions. We can write the averaging analytically like this:

$$V(x,y) = \frac{1}{2\pi a}\oint_{\substack{\text{circle of radius a}\\ \text{centered at x,y}}} Vdl$$

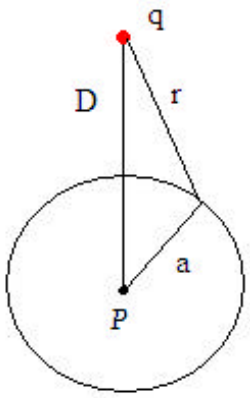
Of course this won't help us solve the problem unless we know the potential on the circle.

The solution is the two-dimensional equivalent of the straight line we got in one dimension. A straight line is the shortest distance between two points. So here we have a surface that has the smallest area for a given perimeter. Thus if our perimeter is in a plane (ie the potential is constant on the perimeter), then the surface is flat (the potential is the same everywhere in the region. As in one dimension, there are no maxima or minima in the region (no mountain peaks or valley floors).

Extending these ideas to three dimensions, the averaging property gives

$$V(x,y.z) = \frac{1}{4\pi a^2}\int_{\text{surface of sphere}} Vda \tag{3}$$

where the sphere is centered at (x, y, z) and has radius a. We can show this by starting with the potential due to a charge q located at a distance D from our point P. For convenience, put the origin at P, and the polar axis through the charge q. Since there is no charge inside the sphere, the potential satisfies Laplace's equation inside (and on) the sphere.



Then a point on the surface of the sphere with polar angle θ is at a distance $r = \sqrt{a^2 + D^2 - 2aD\cos\theta}$ from q, and

$$\begin{aligned}\frac{1}{4\pi a^2}\int_{\text{surface of sphere}} V\,da &= \frac{1}{4\pi a^2}\int_0^{2\pi}\int_0^{\pi}\frac{kq}{\sqrt{a^2+D^2-2aD\cos\theta}}a^2\sin\theta\,d\theta d\phi \\ &= \frac{2\pi}{4\pi}\int_0^{\pi}\frac{kq}{\sqrt{a^2+D^2-2aD\cos\theta}}\sin\theta\,d\theta\end{aligned}$$

Now let $u = a^2 + D^2 - 2aD\cos\theta$ so $du = -2aD\,(-\sin\theta)\,d\theta$ and the integral becomes

$$\begin{aligned}\frac{kq}{4aD}\int_{(D-a)^2}^{(D+a)^2}\frac{du}{u^{1/2}} &= \frac{kq}{4aD}\left.\frac{u^{1/2}}{1/2}\right|_{(D-a)^2}^{(D+a)^2} \\ &= \frac{kq}{2aD}\left[(D+a)-(D-a)\right] \\ &= \frac{kq}{D} = V(P)\end{aligned}$$

Now we can use the principle of superposition to show that for any number of charges outside the sphere, the average property (3) is true.

What happens if the charge q is inside the sphere? Taking the positive square root as before, the integral now gives us

$$\frac{kq}{2aD}\left[(D+a)-(a-D)\right] = \frac{kq}{2a} \neq V(P)$$

The averaging property is not true because Laplace's equation is not satisfied everywhere inside the sphere!

Uniqueness of the solution

What boundary conditions do we need to determine the solution to Laplace's equation? We can get an idea from what we have learned already. In one

dimension we need two conditions: the potential on two planes or the potential on one plane and the field anywhere. In two dimensions we can get a solution if we know V on any circle surrounding our point, and in 3 dimensions if we know V on a sphere surrounding our point How can we generalize these ideas?

We are interested in finding the potential in a region R bounded by a surface S. It is possible, but not necessary, that R is infinite in one or more dimensions. . If R is not infinite, then the boundary conditions we are given stand in for a detailed knowledge of the charge distribution outside the region R, which we either don't know or don't care about. For example, if we hook a battery up to a capacitor, and ground one side, we know that the potentials on the two plates will be V and 0. There will also be a complicated distribution of charge in the battery, in the "ground", and on the connecting wires. We neither know nor care about these charge distributions. Knowing the potential on the two plates is sufficient to find the field inside the capacitor.

In general we will be able to get a solution if we know:

- the potential on all of the bounding surface S, or
- the normal electric field component on all of the bounding surface S, or
- the potential on part of S and the normal component of $\vec{E}$ on the rest.

If the boundary is a conductor, knowing the normal component of $\vec{E}$ is equivalent to knowing the surface charge density on the boundary.

These conditions are necessary and sufficient. If we know this much we can get an answer. If we know less *or more* we will not be able to get an answer. In the first case the problem is underspecified, and the second, overspecified. If we try to fit too many conditions there may be no answer that satisfies all the conditions.

Furthermore, when we can get an answer, it is the only answer – it is unique.

To show uniqueness, we begin by assuming the opposite. Let's assume that there are two different solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ that satisfy the differential equation in R and the boundary conditions that V is known on S. Let's look at the difference between the two solutions: $U = V_1 - V_2$. Then

$$\nabla^2 U = \nabla^2 V_1 - \nabla^2 V_2 = 0 - 0 = 0$$

(Notice that U satisfies Laplace's equation even if V satisifes Poisson's equation, since the charge density on the RHS will subtract away.) Furhermore, if the potential is a specified function $V_0(\vec{r})$ (not necessarily a constant) on the boundary S, then

$$\begin{aligned} U(\vec{r}) &= V_1(\vec{r}) - V_2(\vec{r}) \text{ on } S \\ &= V_0(\vec{r}) - V_0(\vec{r}) = 0 \end{aligned}$$

But we know that the solution for U has no maxima and minima in R, all the extrema are on the boundary S. So both the maximum and the minimum are zero, and so U must be zero everywhere.

If the boundary is a conducting surface, it is enough to know the total charge on each separate conductor. Again we suppose there are two solutions V_1 and V_2 with corresponding fields $\vec{E}_i = -\vec{\nabla}V_i,\ i = 1, 2$. Then again if $U = V_1 - V_2$

$$\nabla^2 U = 0$$

The boundary conditions now are given by the integral form of Gauss' law:

$$\oint_{S_j} \vec{E}_i \cdot d\vec{A} = \frac{Q_j}{\varepsilon_0}$$

on each conductor j. Thus

$$\begin{aligned} \oint_{S_j} -\vec{\nabla}V_1 \cdot d\vec{A} - \oint_{S_j} -\vec{\nabla}V_2 \cdot d\vec{A} &= \frac{Q_j}{\varepsilon_0} - \frac{Q_j}{\varepsilon_0} = 0 \\ \oint_{S_j} \vec{\nabla}U \cdot d\vec{A} &= 0 \end{aligned}$$

We can multiply by U because the potential must be a finite constant on the surface of each conductor, making U constant on each S_j:

$$U \oint_{S_j} \vec{\nabla}U \cdot d\vec{A} = 0 = \oint_{S_j} U\vec{\nabla}U \cdot d\vec{A}$$

Now we can sum over all the surfaces S_j to get the total bounding surface.

$$\oint_{S} U\vec{\nabla}U \cdot d\vec{A} = 0$$

Now let's look at this trick:

$$\vec{\nabla} \cdot \left(U\vec{\nabla}U\right) = U\vec{\nabla}^2 U + \vec{\nabla}U \cdot \vec{\nabla}U = 0 + \left(\vec{\nabla}U\right)^2 \text{ in } R$$

and so, by the divergence theorem,

$$\int_R \left(\vec{\nabla}U\right)^2 d\tau = \int_R \vec{\nabla} \cdot \left(U\vec{\nabla}U\right) d\tau = \oint_S U\vec{\nabla}U \cdot d\vec{A}$$

Thus we have shown that

$$\int_R \left(\vec{\nabla}U\right)^2 d\tau = 0$$

Here the integrand is always positive, so the only way that the integral can be zero is if $\vec{\nabla}U = 0$ everywhere. This means that the two fields $\vec{E}_1$ and $\vec{E}_2$ are the same, and the potentials V_1 and V_2 can differ by at most a constant.

This proof also shows us how to get the general result. Let's start with

$$\oint_S U\vec{\nabla}U \cdot d\vec{A}$$

Now on parts of the surface where V is specified, U is zero, and on parts where $\vec{\nabla}V \cdot \hat{n}$ is specified, $\vec{\nabla}U \cdot \hat{n} = 0$. Thus for the whole surface,

$$\oint_S U\vec{\nabla}U \cdot d\vec{A} = 0 = \int_R \left(\vec{\nabla}U\right)^2 d\tau$$

and again we find that the two potentials can differ by at most a constant.

These theorems are very powerful. If we can get a solution to our problem by any means at all, including an outright guess, and that solution satisfies both the differential equation and the boundary conditions, then it is THE solution.