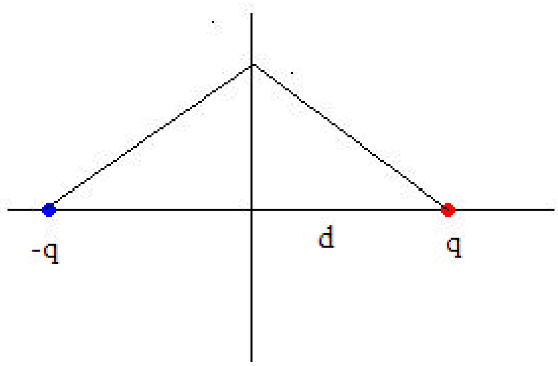
**Method of images**

Suppose that our region $R$ is the half-universe $z > 0$, and the bounding surface $S$ comprises an infinite, grounded, conducting plane at $z = 0$ as well as a surface at infinfity on the positive $z$ side. We do not know, nor care, what is happening at $z < 0$. Now suppose that there is a point charge $q$ at point $P$ in our region, distance $d$ from the conducting plane. What is the resulting potential in $R$?

Well, we can guess what is going to happen. Charge of sign opposite $q$ will be drawn along the conducting plane, and the field lines emanating from $q$ will attach to these charges on the plane, meeting the plane at right angles. We can calculate the resulting potential by imagining a problem that extends into the region $z < 0$, and is mirror symmetric about the plane, with a negative charge $-q$ at distance $d$ from the plane, but on the negative $z$ side. Now every point on the plane is at the same distance from each of the charges, and thus has potential zero.




Put the $z-$axis through the charge $q$, and let $P$ on the plane be a distance $s$ from the origin. Then

$$V(P) = \frac{kq}{\sqrt{d^2 + s^2}} - \frac{kq}{\sqrt{d^2 + s^2}} = 0$$

Since this system gives the correct potential on the plane, and at $z \to \infty$ (where it is zero), and it satisfies the differential equation $\nabla^2 V = -\rho/\varepsilon_0$ in $R$ it must be the correct solution. The charge $-q$ does not contribute to $\rho$ since it is not in $R$. We don't care what equation $V$ satisfies for $z < 0$ since that region is not in $R$! Using this model, the potential at a point in $R$ is

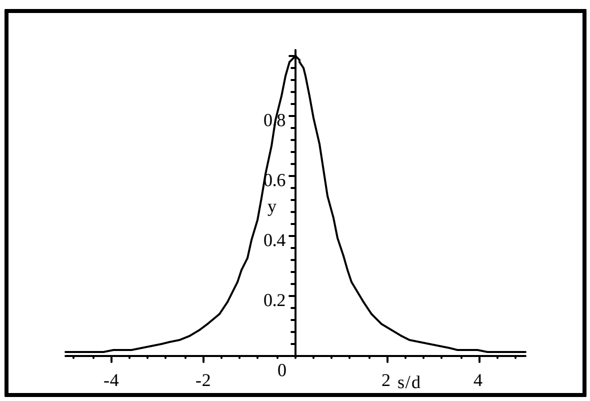
$$V(\vec{r}) = \frac{kq}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{kq}{\sqrt{x^2 + y^2 + (z+d)^2}}$$

This model cannot give us a solution for the potential for $z < 0$.

Once we have this solution, we can find it to get the charge density on the plane at $z = 0$.

$$
\begin{aligned}
\sigma &= \varepsilon_0 E_\perp = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0} \\
&= -\varepsilon_0 kq \left[ -\frac{1}{2}\frac{2(z-d)}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} - \left(-\frac{1}{2}\right)\frac{2(z+d)}{\left(x^2+y^2+(z+d)^2\right)^{3/2}} \right]_{z=0} \\
&= -\varepsilon_0 kq \left[ \frac{d}{(x^2+y^2+d^2)^{3/2}} + \frac{d}{(x^2+y^2+d^2)^{3/2}} \right] \\
&= -\frac{2\varepsilon_0 kqd}{(x^2+y^2+d^2)^{3/2}} = -\frac{qd}{2\pi (s^2+d^2)^{3/2}}
\end{aligned}
$$

The charge density is negative and is circularly symmetric about the $z-$axis, as expected. It is also maximum at $s = 0$, the closest point to the charge $q$. The plot shows $2\pi\sigma/qd^2$ versus $s/d$.




The total charge on the plane is

$$Q = \int_0^\infty \sigma(s)\, 2\pi s ds = -\int_0^\infty \frac{qd}{(s^2+d^2)^{3/2}} s ds$$

Let $u = s^2 + d^2$, with $du = 2sds$ so that

$$Q = -\frac{qd}{2}\int_{d^2}^\infty \frac{du}{u^{3/2}} = -\frac{qd}{2}\left.\frac{-2}{u^{1/2}}\right|_{d^2}^\infty = -q$$

Now we see that the image charge $-q$ represents the effect of the induced charge, also $-q$, on the plane.

The negative charge on the plane attracts the charge $q$ toward the plane. We can compute the force on the charge $q$ at $z > 0$ using the image charge $-q$ that represents the charge on the plane.

$$F_z = -\frac{kq^2}{(2d)^2}$$

You can verify this, if you have the stomach for it, by integrating $\sigma^2/2\varepsilon_0$ over the plane. Or you can take my word for it.

Now let's look at the energy. The energy of the two-charge system is

$$U = -\frac{kq^2}{2d}$$

But remember that the energy is stored in the field throughout space. But our real space $R$ is only half as big as the space in whch this energy exists. So we might expect that the energy in the "plane plus one charge" system is

$$U = -\frac{kq^2}{4d} \tag{1}$$

We can verify this in the usual way, by computing the work done to assemble the system. We bring our charge $q$ in from infinity. When the charge is at coordinate $z$, its image is at $-z$ to make the potential on the plane zero, and thus the force attracting $q$ to the plane is

$$F(z) = \frac{kq^2}{(2z)^2}$$

Now we actually have to pull on the charge with a force of this magnitude to prevent it from accelerating toward the plane. So the work done is negative. The displacement $d\vec{s} = dz\hat{z}$ is in the negative $z-$ direction, because each $dz$ is negative as we reduce the value of $z$ from $\infty$ to $d$.

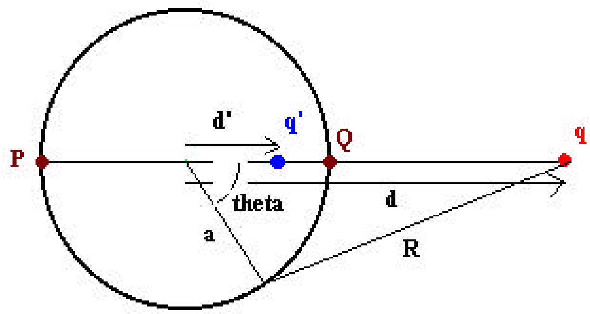
$$\begin{aligned} W &= \int \vec{F} \cdot d\vec{s} = \int_\infty^d \frac{kq^2}{(2z)^2}\hat{z} \cdot dz\hat{z} = \int_\infty^d \frac{kq^2}{(2z)^2} dz \\ &= -\left.\frac{kq^2}{4z}\right|_\infty^d = -\frac{kq^2}{4d} \end{aligned}$$

which agrees with (1) above.

How would the solution change if the potential on the plane is $V_0 \neq 0$?

**Images in a sphere**

Here our region $R$ is the region outside a grounded, conducting sphere of radius $a$. We have a point charge $q$ outside the sphere at a distance $d$ from the center of the sphere. Can we solve this problem with images? The system of sphere plus point has rotational symmetry about a line throught he center of the sphere and the charge, so if there is an image in the sphere, it must be on that same line. We don't know the magnitude or position of the charge, so let's let the charge be $q'$ at a distance $d'$ from the center of the sphere. With two unknowns, we need two equations to find them, so we pick two points on the sphere. The easiest points to work with are the two points $P$ and $Q$ on the line of symmetry:




$$V(P) = \frac{kq'}{a+d} + \frac{kq}{d+a} = 0$$

and, with $d > a > d'$,

$$V(Q) = \frac{kq'}{a-d'} + \frac{kq}{d-a} = 0$$

Thus

$$\begin{aligned} q'(d+a) + q(a+d') &= 0 \\ q'(d-a) + q(a-d') &= 0 \end{aligned}$$

Add the two equations to get

$$2q'd + 2qa = 0 \Rightarrow q' = -q\frac{a}{d} \tag{2}$$

and subtract the two equations to get

$$2q'a + 2qd' = 0 \Rightarrow d' = -\frac{q'a}{q} = \frac{a^2}{d} \tag{3}$$

These results show that $q$ and $q'$ have opposite signs, as expected, and $q'$ is inside the sphere if $q$ is outside.

Now we need to check that this solution makes the whole surface of the sphere have zero potential: At an arbitrary point with polar angle $\theta$, we have

$$\begin{aligned} V &= \frac{kq'}{\sqrt{a^2 + (d')^2 - 2ad'\cos\theta}} + \frac{kq}{\sqrt{a^2+d^2-2ad\cos\theta}} \\ &= \frac{-kqa/d}{\sqrt{a^2 + (a^2/d)^2 - 2(a^3/d)\cos\theta}} + \frac{kq}{\sqrt{a^2+d^2-2ad\cos\theta}} \\ &= \frac{-kq/d}{\sqrt{1 + (a/d)^2 - 2(a/d)\cos\theta}} + \frac{kq}{\sqrt{a^2+d^2-2ad\cos\theta}} \\ &= \frac{-kq}{\sqrt{d^2+a^2-2ad\cos\theta}} + \frac{kq}{\sqrt{a^2+d^2-2ad\cos\theta}} = 0 \end{aligned}$$

So it works!

Once again we note that this model gives the correct potential outside the sphere, but not inside. Of course we already know the potential inside– it is zero! (Make sure that you understand why.)