1 Conservation laws in E&M

Last semester we noted that charge is conserved, and we were able to write both integral

$$\frac{dQ}{dt} = \frac{d}{dt}\int \rho d\tau = -\int \vec{j}\cdot\hat{n}\ dA$$

and differential

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

equations to express charge conservation. Our next task is to do the same thing with energy, momentum and angular momentum.

1.1 Conservation of energy in E&M

We have already found expressions for the energy stored in electric and magnetic fields.The total energy in a volume V is

$$U = \int_V \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}\right) d\tau$$

The work done by electromagnetic forces on a charge q moving through a displacement $d\vec{l}$ is

$$\begin{aligned} dW &= q(\vec{E} + \vec{v}\times\vec{B})\cdot d\vec{l} \\ &= q\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\vec{v}\ dt \\ &= q\vec{E}\cdot\vec{v}\ dt \end{aligned}$$

As usual, the magnetic force does no work. Thus the rate at which work is done is

$$\frac{dW}{dt} = q\vec{E}\cdot\vec{v}$$

Now consider a volume element with charge $dq = \rho d\tau$. The work done on this charge is

$$\frac{dW}{dt} = \rho\vec{v}\cdot\vec{E}\ d\tau = \vec{j}\cdot\vec{E}\ d\tau$$

Now we sum up over the whole volume to find the total rate of doing work:

$$\frac{dW}{dt} = \int_V \vec{j}\cdot\vec{E}\ d\tau$$

We can express this result in terms of the fields alone by using the Ampere-Maxwell law for $\vec{j}$:

$$\vec{j} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t} \tag{1}$$

Thus

$$\frac{dW}{dt} = \int_V \vec{E} \cdot \left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \, d\tau$$

Now from the front cover:

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{B} \right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{E} \right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{B} \right)$$

The last term is what we have in our integrand, and we can use Faraday's law for $\vec{\nabla} \times \vec{E}$ to get:

$$\vec{E} \cdot \left(\vec{\nabla} \times \vec{B} \right) = \vec{B} \cdot \left(-\frac{\partial \vec{B}}{\partial t} \right) - \vec{\nabla} \cdot \left(\vec{E} \times \vec{B} \right)$$

Then

$$\frac{dW}{dt} = -\int_V \left[\frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} + \frac{1}{\mu_0} \vec{\nabla} \cdot \left(\vec{E} \times \vec{B} \right) + \vec{E} \cdot \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \, d\tau$$

Using the divergence theorem on the middle term, we have:

$$\frac{dW}{dt} = -\frac{1}{2} \int_V \left[\frac{1}{\mu_0} \frac{\partial}{\partial t} \left(\vec{B} \cdot \vec{B} \right) + \varepsilon_0 \frac{\partial}{\partial t} \left(\vec{E} \cdot \vec{E} \right) \right] \, d\tau - \int_S \frac{1}{\mu_0} \left(\vec{E} \times \vec{B} \right) \cdot \hat{n} \, dA$$

Or, for a fixed volume V,

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int \left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0} \right) \, d\tau &= -\int_S \frac{1}{\mu_0} \left(\vec{E} \times \vec{B} \right) \cdot \hat{n} \, dA - \frac{dW}{dt} \\ &= -\int_S \frac{1}{\mu_0} \left(\vec{E} \times \vec{B} \right) \cdot \hat{n} \, dA - \int_V \vec{j} \cdot \vec{E} \, d\tau \qquad (2) \end{aligned}$$

This equation should be interpreted as follows:

> rate of change of em energy stored in the volume = energy flow into the volume - rate at which energy is converted to non-em forms as the fields do work..

Remember that $\hat{n}$ points outward from the volume. Mathematically:

$$\frac{dU}{dt} = -\int_s \vec{S} \cdot \hat{n} \, dA - \frac{dW}{dt} \qquad (3)$$

where

$$\vec{S} = \frac{1}{\mu_0} \left(\vec{E} \times \vec{B} \right) \qquad (4)$$

is the Poynting vector that describes energy flow in the fields. Its units are $\mathrm{J/m^2 \cdot s}$. $\vec{S} \cdot \hat{n}$ is positive when energy flows out of the volume.

We can convert equation (2) to differential form in the usual way, using the divergence theorem.

$$\int_V \left[\frac{\partial}{\partial t}\frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) + \vec{j}\cdot\vec{E} + \vec{\nabla}\cdot\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)\right] d\tau = 0$$

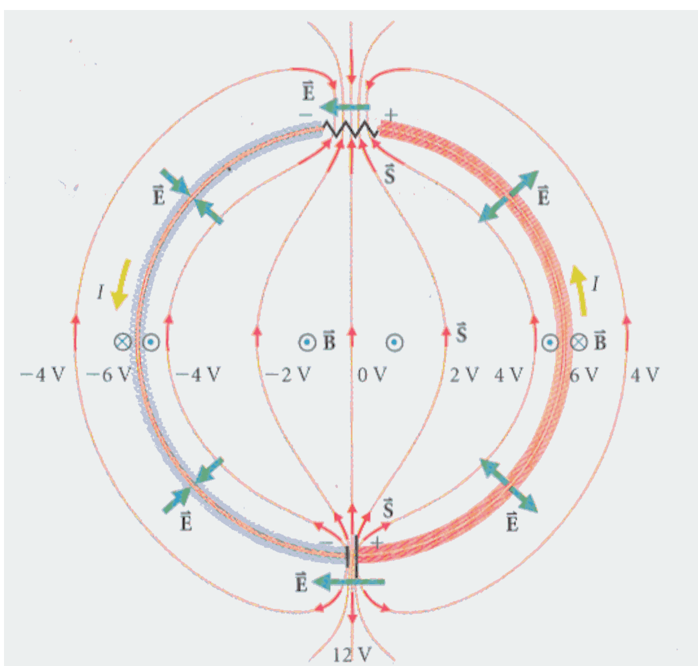
for any volume V, so

$$\frac{\partial}{\partial t}\frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) + \vec{j}\cdot\vec{E} + \vec{\nabla}\cdot\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) = 0$$

or, equivalently,

$$\frac{\partial u_{\mathrm{EM}}}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{j}\cdot\vec{E} \tag{5}$$

It is interesting at this point to consider energy flow in a simple circuit with a battery and a resistor. The battery supplies energy to the circuit, and the resistor converts that energy to non-electro-magnetic forms (primarily thermal energy). So energy is transported from the battery to the resistor. But how? First let's look at the field configuration. There are both electric and magnetic fields. After the switch is closed the battery quickly establishes the necessary charge distribution to create the potential distribution and electric field that drive current through the resistor. The current loop produces a magnetic field, as shown in the diagram. Then there is a non-zero Poynting flux that transmits energy across the empty space between the battery and the resistor. $\left|\vec{S}\right|$ is greatest near (but not *in*) the wires, where the fields are greatest.



1.2 Conservation of momentum

1.2.1 Newton's 3rd law in E&M

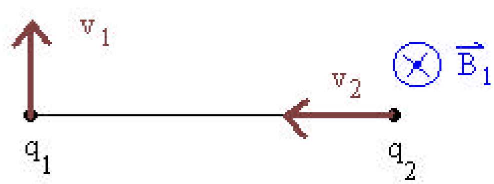
It is easy to show that Newton's third law holds in electrostatics: The force on charge A due to charge B is equal and opposite to the charge on B due to $A-$ this comes directly from Coulomb's law. But things get more interesting when magnetic fields are involved. The current density due to a charge q moving with speed $\vec{v}$ is

$$\vec{j} = \rho \vec{v} = q\delta\left(\vec{r} - \vec{r}_q\left(t\right)\right)\vec{v}$$

and this current density produces a magnetic field according to the Biot-Savart Law. Provided that $v \ll c$, we may write

$$\begin{aligned} \vec{B}\left(\vec{r}\right) &= \frac{\mu_0}{4\pi}\int \frac{\vec{j}\times\vec{R}}{R^3}d\tau' \qquad \text{where } \vec{R} = \vec{r} - \vec{r}' \\ &= \frac{\mu_0}{4\pi}q\int \delta\left(\vec{r} - \vec{r}_q\left(t\right)\right)\frac{\vec{v}\times\vec{R}}{R^3}d\tau \\ &= \frac{\mu_0}{4\pi}q\frac{\vec{v}\times\vec{R}}{R^3} \qquad \text{where } \vec{R} = \vec{r} - \vec{r}_q\left(t\right) \end{aligned}$$

Now consider two charges moving at right angles, as shown:



Charge 1 produces a magnetic field $\vec{B}_1$ at the position of charge 2, with a resulting force exerted on 2 by 1

$$\vec{F}_{\text{on 2 by 1}} = q_2\vec{v}_2\times\vec{B}_1$$

that points downward. But the magnetic field produced by charge 2 is zero at the (instantaneous) position of charge 1 because $\vec{v}\times\vec{R} = 0$, and so the force exerted on 1 by 2 is zero. This result clearly violates Newtons' third law. It then appears that momentum is not conserved, as

$$\frac{d\vec{p}_2}{dt} = \vec{F}_{\text{on 2 by 1}}; \qquad \frac{d\vec{p}_1}{dt} = \vec{F}_{\text{on 1 by 2}} = 0$$

But momentum conservation is one of the most fundamental principles in physics. The problem is resolved by realising that there is momentum stored in the fields that is also changing.

It should not surprise us that momentum is stored in the fields if energy is. Consider a particle moving with velocity $\vec{v}$. It has kinetic energy $K = \frac{1}{2}mv^2$

and momentum $\vec{p} = m\vec{v}$. The ratio of the magnitudes is

$$\frac{K}{|\vec{p}|} = \frac{v}{2}$$

Later this semester, we will see that as the particle's speed approaches the speed of light, the ratio becomes

$$\frac{K}{|\vec{p}|} = c\left(\frac{\gamma - 1}{\gamma}\right)\frac{c}{v} = c\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)\left(\frac{c}{v}\right) \to c \quad \text{as } v \to c$$

We have been using a classical field model of electricity and magnetism, but we could also model the fields using a particle picture. The photon is a massless particle that travels at speed c, and so for the photon $K/p = c$. Thus we might guess that for the EM field, the momentum flux is given by

$$\frac{\vec{S}}{c} = \frac{1}{c\mu_0}\left(\vec{E} \times \vec{B}\right)$$

and then the momentum density

$$\vec{\mathcal{P}}_{EM} = \frac{flux}{c} = \frac{\vec{S}}{c^2} = \varepsilon_0\left(\vec{E} \times \vec{B}\right) \tag{6}$$

Our next task is to prove this conjecture.

1.2.2 The Maxwell Stress tensor

We begin by looking at the force per unit volume $\vec{f}$ in a system of charged particles and fields. The charge in a volume $d\tau$ is $dq = \rho d\tau$, and then

$$\begin{aligned} \vec{f} d\tau &= d\vec{F} = \rho d\tau\left(\vec{E} + \vec{v} \times \vec{B}\right) \\ \vec{f} &= \rho\vec{E} + \vec{j} \times \vec{B} \end{aligned}$$

Now we use Maxwell's equations to eliminate ρ and $\vec{j}$:

$$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E}$$

and $\vec{j}$ from equation (1):

$$\vec{f} = \left(\varepsilon_0 \vec{\nabla} \cdot \vec{E}\right)\vec{E} + \left(\frac{1}{\mu_0}\vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \times \vec{B}$$

Let's work on the last term. Remember: the order of the vectors in a cross product matters!

$$\begin{aligned} \frac{\partial}{\partial t}\left(\vec{E} \times \vec{B}\right) &= \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \vec{E} \times \frac{\partial \vec{B}}{\partial t} \\ &= \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \vec{E} \times \left(-\vec{\nabla} \times \vec{E}\right) \qquad \text{(Faraday's law)} \end{aligned}$$

Thus

$$\frac{\partial \vec{E}}{\partial t} \times \vec{B} = \frac{\partial}{\partial t}\left(\vec{E} \times \vec{B}\right) + \vec{E} \times \left(\vec{\nabla} \times \vec{E}\right)$$

Now we may expand the triple cross product, to get

$$\begin{aligned}
\left[\vec{E} \times \left(\vec{\nabla} \times \vec{E}\right)\right]_i &= \varepsilon_{ijk} E_j \varepsilon_{klm} \frac{\partial}{\partial x_l} E_m \\
&= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) E_j \frac{\partial}{\partial x_l} E_m \\
&= E_j \frac{\partial}{\partial x_i} E_j - E_j \frac{\partial}{\partial x_j} E_i \\
&= \frac{1}{2}\frac{\partial}{\partial x_i} E^2 - \left(\vec{E} \cdot \vec{\nabla}\right) E_i
\end{aligned}$$

with a similar expansion for $\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} = -\vec{B} \times \left(\vec{\nabla} \times \vec{B}\right)$. Putting all this back into $\vec{f}$, we get

$$\begin{aligned}
\vec{f} &= \left(\varepsilon_0 \vec{\nabla} \cdot \vec{E}\right)\vec{E} - \frac{1}{\mu_0}\left[\frac{1}{2}\vec{\nabla} B^2 - \left(\vec{B} \cdot \vec{\nabla}\right)\vec{B}\right] - \varepsilon_0\left[\frac{1}{2}\vec{\nabla} E^2 - \left(\vec{E} \cdot \vec{\nabla}\right)\vec{E}\right] - \varepsilon_0 \frac{\partial}{\partial t}\left(\vec{E} \times \vec{B}\right) \\
&= -\varepsilon_0 \frac{\partial}{\partial t}\left(\vec{E} \times \vec{B}\right) - \vec{\nabla}\left(\frac{B^2}{\mu_0} + \varepsilon_0 E^2\right) \\
&\quad + \varepsilon_0\left[\left(\vec{\nabla} \cdot \vec{E}\right)\vec{E} + \left(\vec{E} \cdot \vec{\nabla}\right)\vec{E}\right] + \frac{1}{\mu_0}\left[\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B} + \vec{B}\left(\vec{\nabla} \cdot \vec{B}\right)\right]
\end{aligned}$$

In the last line I added the term $\vec{B}\left(\vec{\nabla} \cdot \vec{B}\right)$, which is identically zero, to make the last two terms symmetrical in $\vec{E}$ and $\vec{B}$. Tidying up a bit, we get

$$\vec{f} = -\varepsilon_0\mu_0 \frac{\partial \vec{S}}{\partial t} - \vec{\nabla} u + \varepsilon_0\left[\left(\vec{\nabla} \cdot \vec{E}\right)\vec{E} + \left(\vec{E} \cdot \vec{\nabla}\right)\vec{E}\right] + \frac{1}{\mu_0}\left[\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B} + \vec{B}\left(\vec{\nabla} \cdot \vec{B}\right)\right]$$

The first term is, from equation (6),

$$-\frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} = -\frac{\partial \vec{\mathcal{P}}_{\text{EM}}}{\partial t}$$

The remaining terms may be written as the divergence of a tensor T_{ij}. A tensor as a total of $3 \times 3 = 9$ components, labelled with the two indices i and j, each of which ranges over the values 1-3, $1 = x$, $2 = y$, $3 = z$. It is much easier to write equations in index notation when we have tensors. So

$$\begin{aligned}
\frac{\partial}{\partial x_j}T_{ij} &= -\frac{\partial}{\partial x_i}\left[\varepsilon_0\frac{E^2}{2}+\frac{B^2}{\mu_0}\right]+\varepsilon_0\left[\left(\vec{\nabla}\cdot\vec{E}\right)E_i+\left(\vec{E}\cdot\vec{\nabla}\right)E_i\right]+\frac{1}{\mu_0}\left[\left(\vec{B}\cdot\vec{\nabla}\right)B_i+B_i\left(\vec{\nabla}\cdot\vec{B}\right)\right] \\
&= -\delta_{ij}\frac{\partial}{\partial x_j}\left[\varepsilon_0\frac{E^2}{2}+\frac{B^2}{\mu_0}\right]+\varepsilon_0\left[\frac{\partial E_j}{\partial x_j}E_i+E_j\frac{\partial}{\partial x_j}E_i\right]+\frac{1}{\mu_0}\left[B_j\frac{\partial}{\partial x_j}B_i+B_i\frac{\partial B_j}{\partial x_j}\right] \\
&= -\delta_{ij}\frac{\partial}{\partial x_j}\left[\varepsilon_0\frac{E^2}{2}+\frac{B^2}{\mu_0}\right]+\frac{\partial}{\partial x_j}\left(\varepsilon_0 E_j E_i\right)+\frac{\partial}{\partial x_j}\left(\frac{1}{\mu_0}B_j B_i\right) \\
&= \frac{\partial}{\partial x_j}\left[-\delta_{ij}\left[\varepsilon_0\frac{E^2}{2}+\frac{B^2}{\mu_0}\right]+\varepsilon_0 E_j E_i+\frac{1}{\mu_0}B_j B_i\right] \qquad (7)
\end{aligned}$$

We may write the components conveniently as a matrix. The first term contributes only when $i=j$, ie on the diagonal of the matrix.

$$\overleftrightarrow{T}=\begin{pmatrix}
\frac{\varepsilon_0}{2}\left(E_x^2-E_y^2-E_z^2\right)+\frac{1}{2\mu_0}\left(B_x^2-B_y^2-B_z^2\right) & \varepsilon_0 E_x E_y+\frac{1}{\mu_0}B_x B_y & \varepsilon_0 E_x E_z+\frac{1}{\mu_0}B_x B_z \\
\varepsilon_0 E_x E_y+\frac{1}{\mu_0}B_x B_y & \frac{\varepsilon_0}{2}\left(E_y^2-E_x^2-E_z^2\right)+\frac{1}{2\mu_0}\left(B_y^2-B_x^2-B_z^2\right) & \varepsilon_0 E_y E_z+\frac{1}{\mu_0}B_y B_z \\
\varepsilon_0 E_x E_z+\frac{1}{\mu_0}B_x B_z & \varepsilon_0 E_y E_z+\frac{1}{\mu_0}B_y B_z & \frac{\varepsilon_0}{2}\left(E_z^2-E_x^2-E_y^2\right)+\frac{1}{2\mu_0}\left(B_z^2-B_x^2-B_y^2\right)
\end{pmatrix} \qquad (8)$$

Note that this tensor is symmetric, that is:

$$T_{ij}=T_{ji}$$

Finally we write the force density equation as:

$$f_i=-\frac{\partial\vec{\mathcal{P}}_{\mathrm{EM}},i}{\partial t}+\frac{\partial T_{ij}}{\partial x_j} \qquad (9)$$

and integrating over the volume:

$$\int_V f_i\,d\tau=-\int_V\frac{\partial\vec{\mathcal{P}}_{\mathrm{EM}},i}{\partial t}d\tau+\int_V\frac{\partial T_{ij}}{\partial x_j}d\tau$$

Using the divergence theorem, we convert the last term to a surface integral:

$$F_i=-\int_V\frac{\partial\vec{\mathcal{P}}_{\mathrm{EM}},i}{\partial t}d\tau+\int_S T_{ij}n_j\ dA \qquad (10)$$

or equivalently

$$\vec{F}=-\frac{d}{dt}\int\vec{\mathcal{P}}_{\mathrm{EM}}\ d\tau+\int_S\overleftrightarrow{T}\cdot d\vec{A} \qquad (11)$$

or

$$\frac{d}{dt}\int\vec{\mathcal{P}}_{\mathrm{EM}}\ d\tau=\int_S\overleftrightarrow{T}\cdot d\vec{A}-\vec{F}$$

We may interpret this as:

Rate of change of stored EM momentum in the volume=-force exerted by fields in the volume on charges in the volume - force exerted by fields in the volume on the surface of the volume.

The physical interpretation of the tensor component T_{ij} is:

T_{ij} is the force per unit area in the ith direction exerted on an area element with normal in the jth direction.

The diagonal elements such as T_{xx} are *pressures* (normal force per unit area) while the off-diagonal components are *shears.*

Let's see what happens to equation 11 when there is no time dependence. We have

$$\vec{F} = \int_S \overleftrightarrow{T} \cdot d\vec{A}$$

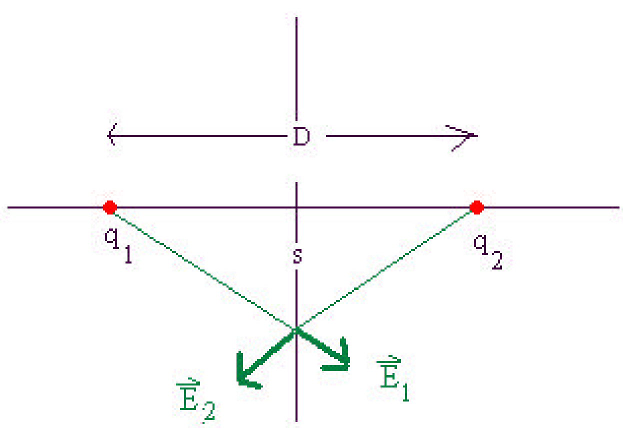
This means that we can calculate the force equally well by calculating the force per unit volume, and integrating over the volume, or the force per unit area and integrating over the area. For example, Suppose we have two point charges separated by a distance D. The volume integral is just Coulomb's law:

$$\vec{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 D^2}\hat{r}$$

But we should be able to get the result by integrating over any surface completely surrounding one of the charges.We need the total electric field due to both charges:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2\right)$$

Let's put the $x-$axis along the line joining the two charges, with the origin half way between them.



The easiest surface to use is a rectangular box, with flat surfaces at $x = 0$, $x \to \infty, y \to \pm\infty$ and $z \to \pm\infty$. This box encloses charge 2, but if we choose

$x \to -\infty$, that box encloses charge 1. The fields go to zero fast enough that the only non-zero contribution is from the surface at finite x. (The fields decrease like $1/\text{distance}^2$, so the product of two field components goes like $1/\text{distance}^4$, but the area only increases as distance^2, so the product $\to 0$ as $1/\text{distance}^2$.) Since the box enclosing charge one has outward normal $\hat{n} = \hat{x}$ while the box enclosing charge 2 has outward normal $\hat{n} = -\hat{x}$, we can see immediately that the forces each charge exerts on the other are equal and opposite. Let's calculate the force on charge 2.

On the plane

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\left(\left(\frac{D}{2}\right)^2 + y^2 + z^2\right)^{3/2}}\left(\frac{D}{2}\hat{x} + y\hat{y} + z\hat{z}\right) + \frac{q_2}{\left(\left(\frac{D}{2}\right)^2 + y^2 + z^2\right)^{3/2}}\left(\frac{-D}{2}\hat{x} + y\hat{y} + z\hat{z}\right)\right)$$

$$\begin{aligned}
F_x &= -\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz T_{xx} \\
&= -\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \frac{\varepsilon_0}{2}\left(E_x^2 - E_y^2 - E_z^2\right) \\
&= -\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \varepsilon_0 \left(E_x^2 - \frac{E^2}{2}\right) \\
&= -\frac{1}{(4\pi)^2\varepsilon_0}\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \left(\begin{array}{c} \left[\frac{q_1 D/2}{(D^2/4+y^2+z^2)^{3/2}} - \frac{q_2 D/2}{(D^2/4+y^2+z^2)^{3/2}}\right]^2 \\ -\frac{1}{2}\left[\frac{q_1\hat{r}_1}{D^2/4+y^2+z^2} + \frac{q_1\hat{r}_2}{D^2/4+y^2+z^2}\right]^2 \end{array}\right)
\end{aligned}$$

We may use polar coordinates on the plane, where $y^2 + z^2 = s^2$

$$\begin{aligned}
\int_0^{2\pi}\int_0^{+\infty} s ds d\theta \frac{q_1^2 D^2/4}{(D^2/4 + s^2)^3} &= \frac{\pi q_1^2 D^2}{4}\int_{D^2/4}^{+\infty}\frac{du}{u^3} \qquad \text{where } u = \frac{D^2}{4} + s^2 \\
&= \frac{\pi q_1^2 D^2}{4}\left(-\frac{1}{2u^2}\right)\Big|_{D^2/4}^{\infty} = \frac{2\pi q_1^2}{D^2}
\end{aligned}$$

and

$$\int_0^{2\pi}\int_0^{+\infty} s ds d\theta \frac{q_1^2}{(D^2/4 + s^2)^2} = \pi q_1^2\left(-\frac{1}{u^2}\right)\Big|_{D^2/4}^{\infty} = \frac{4\pi q_1^2}{D^2}$$

Thus the first term in each of the brackets sum to give

$$\frac{2\pi q_1^2}{D^2} - \frac{1}{2}\frac{4\pi q_1^2}{D^2} = 0$$

Of course we expect this because the result should involve only the product of the two different charges. The same thing happens with the last term in each of the two brackets. Thus we are left with the cross terms.

$$\begin{aligned}
\int_0^{2\pi}\int_0^{+\infty} sdsd\theta \frac{-q_1q_2D^2/2}{\left(D^2/4+s^2\right)^3} &= -\pi q_1q_2\frac{D^2}{2}\int_{D^2/4}^{\infty}\frac{du}{u^3} \qquad \text{where } u=\frac{D^2}{4}+s^2 \\
&= -\pi q_1q_2\frac{D^2}{2}\left(-\frac{1}{2u^2}\right)\Big|_{D^2/4}^{\infty} \\
&= -4\pi\frac{q_1q_2}{D^2}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{2\pi}\int_0^{+\infty} sdsd\theta \frac{2q_1q_2\hat{r}_1\cdot\hat{r}_2}{\left(D^2/4+s^2\right)^2} &= \int_0^{2\pi}\int_0^{+\infty} 2sdsd\theta \frac{q_1q_2\left(r_y^2+r_z^2-r_x^2\right)}{\left(D^2/4+s^2\right)^2 r^2} \\
&= q_1q_2\int_0^{2\pi}\int_0^{+\infty} 2sdsd\theta \frac{\left(s^2-D^2/4\right)}{\left(D^2/4+s^2\right)^3} \\
&= q_1q_2\int_0^{2\pi}\int_0^{+\infty} 2sdsd\theta \frac{\left(s^2+D^2/4-D^2/2\right)}{\left(D^2/4+s^2\right)^3} \\
&= q_1q_2\int_0^{2\pi}\int_0^{+\infty} 2sdsd\theta \left[\frac{1}{\left(D^2/4+s^2\right)^2}-\frac{D^2/2}{\left(D^2/4+s^2\right)^3}\right]
\end{aligned}$$

We havre already done the second integral, so let's work on the first:

$$\int_0^{2\pi}\int_0^{+\infty} 2sdsd\theta\frac{1}{\left(D^2/4+s^2\right)^2} = 2\pi\int_{D^2/4}^{\infty}\frac{1}{u^2}du = \frac{8\pi}{D^2}$$

Thus

$$\begin{aligned}
F_x &= -\frac{q_1q_2}{(4\pi)^2\varepsilon_0}\left(-\frac{4\pi}{D^2}-\frac{1}{2}\left[\frac{8\pi}{D^2}-\frac{8\pi}{D^2}\right]\right) \\
&= \frac{q_1q_2}{4\pi\varepsilon_0D^2}
\end{aligned}$$

as expected.

Finally let's look at F_y.

$$\begin{aligned}
F_y &= -\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}dzT_{yx} \\
&= -\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}dz\varepsilon_0E_xE_y \\
&= -\frac{1}{(4\pi)^2\varepsilon_0}\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}dz\left[\frac{q_1D/2}{\left(D^2/4+y^2+z^2\right)^{3/2}}-\frac{q_2D/2}{\left(D^2/4+y^2+z^2\right)^{3/2}}\right] \\
&\qquad \times\left[\frac{q_1y}{\left(D^2/4+y^2+z^2\right)^{3/2}}-\frac{q_2y}{\left(D^2/4+y^2+z^2\right)^{3/2}}\right]
\end{aligned}$$

The terms look like

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \frac{q_1^2 y D/2}{\left(D^2/4 + y^2 + z^2\right)^3} = 0$$

because the integral over y has an odd integrand integrated over an even interval. The cross terms are zero for the same reason.

$$\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \frac{q_1 q_2 y D/2}{\left(D^2/4 + y^2 + z^2\right)^3} = 0$$

Thus the force only has an $x-$component, as expected.

1.2.3 Conservation of momentum

Let's return to the full force law (11), Since the force acting on the charges in the volume changes their mechanical momentum, we may rewrite the equation as:

$$\frac{d}{dt} \int \left(\vec{\mathcal{P}}_{\text{EM}} + \vec{\mathcal{P}}_{\text{Mech}}\right) d\tau = \int_S \overleftrightarrow{T} \cdot d\vec{A}$$

or, in differential form,

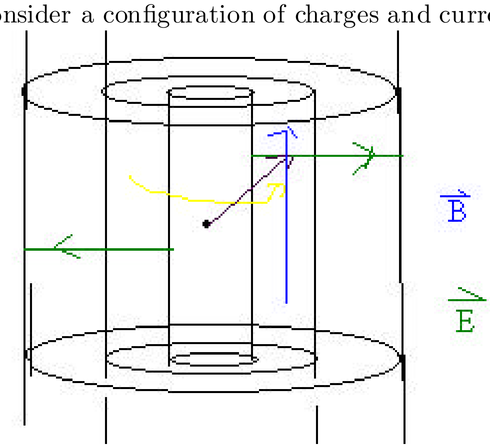
$$\frac{\partial}{\partial t} \left(\vec{\mathcal{P}}_{\text{EM}} + \vec{\mathcal{P}}_{\text{Mech}}\right) = \vec{\nabla} \cdot \overleftrightarrow{T} \qquad (12)$$

1.3 Angular momentum

If the fields carry momentum, they must also carry angular momentum:

$$\begin{aligned} \vec{\ell}_{\text{EM}} &= \vec{r} \times \vec{\mathcal{P}}_{\text{EM}} \\ &= \varepsilon_0 \vec{r} \times \left(\vec{E} \times \vec{B}\right) \end{aligned}$$

As an example, consider a configuration of charges and currents as follows.



We have two concentric cylindrical shells of charge with radii a and $c > a$. The cylinders have equal but opposite charges, so the surface charge densities are σ and $-\sigma c/a$ respectively. There is an electric field

$$\vec{E} = \frac{\sigma a}{\varepsilon_0 s}\hat{s}$$

for $c > s > a$. A solenoid with radius b, $c > b > a$, carries azimuthal current I, so a magnetic field

$$\vec{B} = \mu_0 n I \hat{z}$$

exists in the region $s < b$. Thus for $a < s < b$ we have both electric and magnetic fields.

$$\vec{\mathcal{P}}_{\mathrm{EM}} = \varepsilon_0 \frac{\sigma a}{\varepsilon_0 s}\hat{s} \times \mu_0 n I \hat{z} = -\frac{\sigma a^2}{2s}\mu_0 n I \hat{\theta}$$

and the angular momentum density about the origin is:

$$\begin{aligned}
\vec{\ell}_{\mathrm{EM}} &= \vec{r} \times \left(-\frac{\sigma a}{s}\mu_0 n I \hat{\theta}\right) \\
&= -\frac{\sigma a}{s}\mu_0 n I \left(s\hat{s} + z\hat{z}\right) \times \hat{\theta} \\
&= -\sigma a \mu_0 n I \left(\hat{z} - \frac{z}{s}\hat{s}\right)
\end{aligned}$$

The total angular momentum contained in a cylinder of height h is

$$\begin{aligned}
\vec{L}_{\mathrm{EM}} &= \int \vec{\ell}_{\mathrm{EM}}\, d\tau \\
&= \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\phi \int_a^b s\,ds \left\{\sigma a \mu_0 n I \left(\hat{z} - \frac{z}{s}\hat{s}\right)\right\} \\
&= -\pi\sigma a \mu_0 n I \left(b^2 - a^2\right) h\ \hat{z}
\end{aligned}$$

Now if we turn the current off, $\vec{B} \to 0$ and $\vec{L}_{\mathrm{EM}} \to 0$ also. Since there are no external torques, $\vec{L}$ should be conserved. Aha, but the changing $\vec{B}$ produces an azimuthal $\vec{E}$, by Faraday's law, and the resulting torque on the cylinders causes them to rotate. We have

$$\begin{aligned}
2\pi s E_\theta &= -\frac{d}{dt}\pi s^2 B \\
E_\theta &= -\frac{s}{2}\mu_0 n \frac{dI}{dt}
\end{aligned}$$

and the torque on a height h of the cylinder of radius $a < b$ is

$$\begin{aligned}
\vec{\tau} &= a\sigma\left(2\pi a h\right) E_\theta \hat{z} \\
&= -2\pi\sigma a^2 h \frac{a}{2}\mu_0 n \frac{dI}{dt}\hat{z} = \frac{d\vec{L}_{\mathrm{cylinder}}}{dt}
\end{aligned}$$

Thus the inner cylinder gains angular momentum:

$$\begin{aligned}
\vec{L}_{\mathrm{inner}} &= \int_{t_1}^{t_2} \frac{d\vec{L}_{\mathrm{cylinder}}}{dt} dt = -\pi\sigma a^3 h \mu_0 n\ I\big|_{t_1}^{t_2}\,\hat{z} \\
&= \pi\sigma a^3 h \mu_0 n I\ \hat{z}
\end{aligned}$$

The initial flux through a circle of radius c is $\pi b^2 B$, since B is zero for $b < s < c$, so we have

$$E_\theta(c) = -\frac{b^2}{2c}\mu_0 n \frac{dI}{dt}$$

and, since the outer cylinder has a negative charge density,

$$\vec{\tau} = 2\pi\left(\sigma\frac{a}{c}\right)c^2 h \frac{b^2}{2c}\mu_0 n \frac{dI}{dt}\hat{z}$$

giving

$$\vec{L}_{\text{outer}} = -\pi ach b^2 \mu_0 n I \ \hat{z}$$

The total angular momentum gained by the two cylinders is

$$\vec{L}_{\text{mech}} = -\pi\sigma a \mu_0 n I h \hat{z}\left(b^2 - a^2\right)\hat{z} = \vec{L}_{\text{EM}}$$

The total electromagnetic angular momentum that disappears exactly equals the total mechanical angular momentum gained by the cylinders. Angular momentum is conserved.