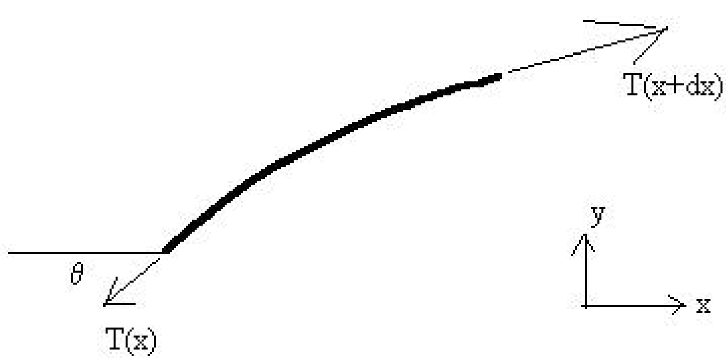
1 Waves– basics

1.1 The wave equation for a string

Waves occur in many physical systems. A wave is a disturbance that propagates through the system at a well-determined speed that depends on the physical properties of the system. All wave disturbances may be decomposed into a sum of simpler waves- sinusoidal waves. In these waves, the shape of the disturbance, captured by taking a snapshot of the disturbance in the medium at a fixed time, has the shape of a sine wave. Every point of the system oscillates in time. Thus we might expect that the system has restoring forces that try to return the system to its initial equilibrium. We'll begin by studying a simple mechanical system– a string. The restoring force is the tension in the string.

In equilibrium the string lies along the x−axis. We look at a differential piece of the displaced string, as shown in the diagram.



The net force on the string has components:

$$\begin{aligned} dF_x &= T(x+dx)\cos(\theta+d\theta) - T(x)\cos\theta \\ dF_y &= T(x+dx)\sin\theta - T(x)\sin\theta \end{aligned}$$

Now let the displacement of the string be small everywhere, fo that $\theta \ll 1$ everywhere. Then

$$\begin{aligned} \cos\theta &= 1 - \frac{\theta^2}{2} + \cdots \simeq 1 \\ \sin\theta &= \theta - \frac{\theta^3}{6} + \cdots \simeq \theta \end{aligned}$$

and

$$\begin{aligned} dF_x &= T(x+dx) - T(x) \\ dF_y &= T(x+dx)(\theta+d\theta) - T(x)\theta \end{aligned}$$

Each piece of the string moves vertically but not horizontally, so

$$dF_x = 0 \Rightarrow T(x + dx) = T(x)$$

The tension is constant along the string. Then the $y-$component is

$$dF_y = Td\theta = (dm) \frac{\partial^2 y}{\partial t^2}$$

The time derivatives are partial derivatives: we are looking at a fixed value of x. The mass of the string segment is

$$dm = \mu dx$$

The string has to stretch a bit in order to curve, but the mass per unit length μ is measured along the undisturbed string. Thus

$$Td\theta = \mu dx \frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial \theta}{\partial x}$$

The $x-$derivatives are also partial because the picture shown is a snapshot taken at a fixed time. Now we work on the angle θ. The slope of the string is

$$\tan \theta = \frac{\partial y}{\partial x} \simeq \theta \text{ for } \theta \ll 1.$$

Thus

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} \tag{1}$$

This is the wave equation for the string. The wave speed is given by

$$v^2 = \frac{T}{\mu}$$

and v is roughly equal to the square root of the restoring force/inertia. The general solution to this equation has the form

$$y = f(x \pm vt)$$

as you can easily check by differentiating and stuffing in. With the minus sign, the solution represents a wave propagating in the direction of increasing x, with the plus sign the pulse propagates in the direction of decreasing x. Since the wave equation is linear, we can have superpositions of such solutions, for example

$$y = f(x + vt) + g(x - vt)$$

Sinusoidal waves occur when f is a sine (or cosine) wave.

$$y = A \cos [k(x - vt) + \phi] = A \cos (kx - \omega t + \phi)$$

At fixed t, the cosine repeats after a distance Δx where

$$k\Delta x = 2\pi$$

or

$$\Delta x = \frac{2\pi}{k} = \lambda$$

where λ is the wavelength of the wave. Similarly, at fixed x, the cosine repeats after a time

$$\Delta t = \frac{2\pi}{\omega} = T$$

where T is the wave period. Also

$$\frac{\omega}{k} = v$$

A is the wave amplitude. It is the maximum value of the disturbance anywhere in the wave. ϕ is a phase constant. It tells us where the maximum displacement occurs at $t = 0$.

The speed v is the *phase speed* of the wave. The wave phase is the argument of the cosine. So a fixed phase ϕ_0 corresponds to

$$kx - \omega t + \phi = \phi_0$$

or

$$x = \frac{\omega}{k}t + \frac{\phi_0 - \phi}{k} = vt + \frac{\phi_0 - \phi}{k}$$

so if k is positive, the value of x corresponding to phase ϕ_0 increases at speed v.

Standing waves occur when equal amplitude waves travel in opposite directions:

$$\begin{aligned} y_{\text{stand}} &= A\cos(kx - \omega t) + A\cos(kx + \omega t + \phi) \\ &= 2A\cos\left(kx + \frac{\phi}{2}\right)\cos\omega t \end{aligned}$$

Now we have a fixed spatial function that oscillates in time.

1.1.1 More math makes it easier

We can use complex numbers to make the math easier. Euler's formula tells us

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus

$$\begin{aligned} A\cos(kx - \omega t + \phi) &= \operatorname{Re} A\exp(i[kx - \omega t + \phi]) \\ &= \operatorname{Re}\left\{\left(Ae^{i\phi}\right)\left(e^{i(kx-\omega t)}\right)\right\} \\ &= \operatorname{Re}\left(\widetilde{A}e^{ikx - i\omega t}\right) \end{aligned}$$

where

$$\widetilde{A} = Ae^{i\phi}$$

is the complex amplitude of the wave. With this notation, the general solution of the wave equation may be written:

$$f(x,t) = \int_{-\infty}^{+\infty} \widetilde{A}(k)\, e^{ikx-i\omega t} dk$$

where

$$\omega = kv$$

The "Re" is implied but is not always written. Real physical quantities are always the real part of complex numbers.

1.2 Boundary conditions: reflection and transmission

When a wave reaches a point where the properties of the system change, part of the wave energy will be reflected and part transmitted into the new region. In the case of 1-D waves on a string, the mass/unit length μ is the critical system property, As μ changes, so does v :

$$v_1 = \sqrt{\frac{T}{\mu_1}}; \qquad v_2 = \sqrt{\frac{T}{\mu_2}}$$

and thus

$$\frac{\omega_1}{k_1} = v_1; \qquad \frac{\omega_2}{k_2} = v_2$$

The disturbance on the string has three components: the incident wave $y_i = A\exp(ik_i x - i\omega t)$, the reflected wave travelling in the negative $x-$direction, $y_r = A_r \exp(-ik_r x - i\omega_r t)$ and the transmitted wave $y_t = A_t \exp(ik_t x - i\omega_t t)$. The first two travel at speed v_1 while the transmitted wave travels at speed v_2. The amplitudes of the reflected and transmitted waves are determined by the *boundary conditions* at the junction between the strings. To determine the two amplitudes, we need two boundary conditions. They are:

(1) The string displacement is continuous across the junction. If this were not true, the string would have an unphysical "break" at the junction. For simplicity, we place the origin at the junction between the strings.

$$\begin{aligned} y_i(0,t) + y_r(0,t) &= y_t(0,t) \\ A\exp(-i\omega t) + A_r \exp(-i\omega_r t) &= A_t \exp(-i\omega_t t) \end{aligned}$$

This boundary condition must hold for all times t. Thus it must be true that

$$\omega = \omega_r = \omega_t$$

The frequency of all the waves is the same. Then it follows that

$$A + A_r = A_t \tag{2}$$

and also

$$k_r = \frac{\omega}{v_1} = k_i; \qquad k_t = \frac{\omega}{v_2}$$

(2) The slope of the string is continuous across the junction. If this were not true, there would be unbalanced forces on the junction that would cause rapid acceleration, rapidly restoring the force balance and the continuity of the slope. (Be careful here— if the strings are tied together so that there is a massive knot at the junction, then there may be a discontinuity of slope. This is not the usual case.)

$$ik_iAe^{-i\omega t} - ik_rA_te^{-i\omega t} = ik_tA_te^{-i\omega t}$$

The exponentials cancel, leaving

$$\frac{A}{v_1} - \frac{A_r}{v_1} = \frac{A_t}{v_2} \tag{3a}$$

From equations (2) and (3a) we can solve for the amplitudes:

$$\begin{aligned} A - A_r &= \frac{v_1}{v_2}(A + A_r) \\ A\left(1 - \frac{v_1}{v_2}\right) &= A_r\left(1 + \frac{v_1}{v_2}\right) \\ A_r &= A\frac{v_2 - v_1}{v_2 + v_1} \end{aligned} \tag{4}$$

and

$$\begin{aligned} A_t &= A\left(1 + \frac{v_2 - v_1}{v_2 + v_1}\right) \\ &= A\left(\frac{2v_2}{v_2 + v_1}\right) \end{aligned} \tag{5}$$

Note that if $v_2 = v_1$, there is no junction, and $A_r = 0$, $A_t = A$, as expected. If the second string is heavier than the first, $\mu_2 > \mu_1$, then $v_2 < v_1$ and A_r is negative. Recall that these amplitudes are complex numbers, so

$$A_r = -A\frac{v_1 - v_2}{v_2 + v_1} = e^{i\pi}A\frac{v_1 - v_2}{v_2 + v_1}$$

The reflected amplitude has a phase difference of π compared with the incident amplitude. Taking the real part,

$$y_r = |A|\frac{v_1 - v_2}{v_2 + v_1}\cos(-ik_rx - \omega t + \pi)$$

and the wave has a phase change of π. The reflected wave is flipped "upside down".

Boundary conditions like this (continuity of function, continuity of derivative) apply in most physical systems.

1.3 Polarization

The waves on the string are transverse waves: the string itself moves perpendicular (in the $y-$direction) to the direction that the wave itself moves (the $x-$direction). The wave is said to be polarized in the vertical (y) direction. We could also set the string vibrating so that it moves in the $z-$direction and we would say the wave is polarized in the $z-$direction. Or we could choose any direction in the $y-z$ plane. These are linearly polarized waves. Now suppose we set the string vibrating in the $y-$direction, and a quarter period later we also set it going with equal amplitude in the $z-$direction. The total displacement is

$$\begin{aligned}\vec{s} &= A\left\{\hat{y}\exp\left(ikx-i\omega t\right)+\hat{z}\exp\left[\left(ikx-i\omega\left(t-\frac{T}{4}\right)\right)\right]\right\}\\ &= Ae^{ikx-i\omega t}\left(\hat{y}+\hat{z}e^{i\pi/2}\right)\\ &= Ae^{ikx-i\omega t}\left(\hat{y}+i\hat{z}\right)\end{aligned}$$

Now we take the real part:

$$\vec{s}=A\left[\hat{y}\cos\left(kx-\omega t\right)-\hat{z}\sin\left(kx-\omega t\right)\right]$$

Fix attention at one point of the string – say $x=0$, then we have

$$\vec{s}\left(0,t\right)=A\left(\hat{y}\cos\omega t+\hat{z}\sin\omega t\right)$$

Now the string moves around a circle of radius A in the $y-z-$plane. This is circular polarization.

Contrast the string with waves on a spring– these are longitudinal waves– the spring coils travel back and forth along the spring- the same direction that the wave moves. The concept of polarization does not arise for these waves.

1.4 Energy transmission

As the wave travels along the string, it transmits energy. We can see how the energy is related to the wave properties by looking at the rate at which the string to the left of a point $x=x_0$ does work on the string to its right. The string to the left exerts a force $\vec{T}$ on the string to the right, and at the point of contact the string has velocity

$$\vec{v}=\hat{y}\frac{\partial y\left(x_0,t\right)}{\partial t}$$

Thus

$$P=\vec{T}\cdot\vec{v}=-T\sin\theta\frac{\partial y\left(x_0,t\right)}{\partial t}$$

But since the string has small displacement, $\theta\ll 1$,

$$\sin\theta\simeq\tan\theta=\frac{\partial y}{\partial x}$$

So, with $y = A\cos(kx - \omega t)$

$$\begin{aligned} P &= -T\left.\frac{\partial y}{\partial x}\right|_{x_0}\frac{\partial y(x_0,t)}{\partial t} \\ &= TkA\sin(kx_0 - \omega t)\,\omega A\sin(kx_0 - \omega t) \\ &= \mu v\omega A^2\sin^2(kx_0 - \omega t) \end{aligned}$$

Since the sine function is squared, the power is always positive; energy is transmitted to the right continuously– this is the direction in which the wave is propagating. The power depends on the square of the wave ampliude– doubling the amplitude quadruples the energy transmitted–and also depends on the wave speed and the wave frequency.

2 Electromagnetic waves in vacuum

Maxwell's equations in vacuum are ($\rho = 0$, $\vec{j} = 0$)

$$\vec{\nabla}\cdot\vec{E} = 0 \tag{6}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{7}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{8}$$

$$\vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \tag{9}$$

Even though ρ and $\vec{j}$ are zero, the fields are not necessarily zero, because changing $\vec{B}$ acts as a source of $\vec{E}$, and changing $\vec{E}$ acts as a source of $\vec{E}$. The resulting fields are wave fields that travel at the speed of light. To demonstrate this, let's try to eliminate one of the fields. Start with Ampere's law (9) and take the curl:

$$\begin{aligned} \vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) &= \frac{1}{c^2}\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t} \\ \vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right) - \nabla^2\vec{B} &= \frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right) \end{aligned}$$

Now use equation (7) on the left and Faraday's law (8) on the right:

$$-\nabla^2\vec{B} = -\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial^2 t}$$

so we obtain the wave equation with wave speed c :

$$\nabla^2\vec{B} = \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial^2 t} \tag{10}$$

The equation for waves on a string (1) was 1-dimensional in the space variables, but here we have the ∇^2 operator, so we have derivatives in all 3 space variables.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{B}=\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial^2 t}$$

The waves are not constrained to move in one direction, but can move along any line in space, described by the wave vector $\vec{k}$. The magnitude of this vector is related to the wavelength, as with $1-$D waves.

$$k=\left|\vec{k}\right|=\frac{\omega}{c}=\frac{2\pi}{\lambda}$$

The speed of the waves in vacuum is

$$c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}=3\times 10^8 \text{ m/s}$$

EM waves occur at all frequencies. The names we give the waves are historical and reflect the methods we use to detect them. They range from gamma rays at very high frequencies to low frequency radio. See G page 377 and LB page 533. Visible light occupies a small fraction of the spectrum around $\nu = 10^{14}-10^{15}$ Hz (or, $\lambda =$ 400-700 nm)

The electric field satisfies the same equation, as we can see by starting with Faraday's law:

$$\begin{aligned}
\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) &= -\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}\\
\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\nabla^2\vec{E} &= -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)=-\frac{\partial}{\partial t}\left(\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\right) \qquad (11)\\
\nabla^2\vec{E} &= \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}
\end{aligned}$$

2.1 Monochromatic plane waves

The term "monochromatic" or "single-color" means that the wave has a single frequency ω. This is an idealization, of course, but it greatly simplifies our math. The wave is plane if a surface of constant phase is a flat plane. Let's choose to put the $z-$axis along the vector $\vec{k}$, so that the wave is travelling in the $z-$direction. Then we can write the waves as:

$$\begin{aligned}
\vec{B} &= \vec{B}_0\exp\left(ikz-i\omega t\right)\\
\vec{E} &= \vec{E}_0\exp\left(ikz-i\omega t\right)
\end{aligned}$$

Stuffing into Gauss' law, we have

$$\vec{\nabla}\cdot\vec{E}=ikE_{0,z}\exp\left(ikz-i\omega t\right)=0$$

This can be satisfied only if $k = 0$, which makes no sense, or $E_{0,z} = 0$. Thus $\vec{E}_0$ has no $z-$component, and $\vec{E}$ is perpendicular to $\vec{k}$. Since $\vec{\nabla} \cdot \vec{B}$ is also zero, we get the same result for $\vec{B}_0$. Since both field vectors are perpendicular to the direction of wave propagation, EM waves in vacuum are transverse waves. Now let's put the $x-$axis along the directionof $\vec{E}_0$. From Faraday's law, we get

$$\begin{aligned}
\vec{\nabla} \times \vec{E} &= \vec{\nabla} \times \left[E_0 \hat{x} \exp\left(ikz - i\omega t\right)\right] \\
&= \hat{y}\frac{\partial}{\partial z}\left[E_0 \exp\left(ikz - i\omega t\right)\right] = ikE_0 \hat{y} \exp\left(ikz - i\omega t\right) \\
&= -\frac{\partial}{\partial t}\vec{B}_0 \exp\left(ikx - i\omega t\right) = i\omega \vec{B}_0 \exp\left(ikx - i\omega t\right)
\end{aligned}$$

Thus we get

$$\vec{B}_0 = \frac{k}{\omega}E_0 \hat{y} = \frac{E_0}{c}\hat{y} \tag{12}$$

Thus the magnetic field is perpendicular to the electric field, has magnitude E_0/c, and the wave amplitude has the same phase as the $\vec{E}-$wave amplitude. Since $\vec{E}$ is perpendicular to $\vec{B}$, by convention we choose the electric field vector to give the direction of polarization. This wave is polarized in the $x-$direction.

We can write all of this in a coordinate-free form. If the wave propagates in direction $\hat{k}$, the dot product $\hat{k} \cdot \vec{r}$ gives the distance along the direction of $\hat{k}$, so we replace kz with $k\hat{k} \cdot \vec{r} = \vec{k} \cdot \vec{r}$. Thus

$$\begin{aligned}
\vec{E} &= \vec{E}_0 \exp\left(i\vec{k} \cdot \vec{r} - i\omega t\right) \\
\vec{B} &= \vec{B}_0 \exp\left(i\vec{k} \cdot \vec{r} - i\omega t\right)
\end{aligned}$$

where, from the divergence equations,

$$\vec{k} \cdot \vec{E}_0 = \vec{k} \cdot \vec{B}_0 = 0$$

and from the curl equation

$$\begin{aligned}
\vec{\nabla} \times \vec{E}_0 \exp\left(i\vec{k} \cdot \vec{r} - i\omega t\right) &= \hat{x}\left(\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y\right) + \hat{y}\left(\frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z\right) + \hat{z}\left(\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x\right) \\
&= \left[\hat{x}\left(ik_y E_{0,z} - -k_z E_{0y}\right) + \hat{y}\left(ik_z E_{0,x} - ik_x E_{0,z}\right)\right. \\
&\quad \left. + \hat{z}\left(ik_x E_{0,y} - ik_y E_{0x}\right)\right] \exp\left(i\vec{k} \cdot \vec{r} - i\omega t\right) \\
&= i\vec{k} \times \vec{E}_0 \exp\left(i\vec{k} \cdot \vec{r} - i\omega t\right) = -\frac{\partial}{\partial t}\vec{B}_0 \exp\left(i\vec{k} \cdot \vec{r} - i\omega t\right) \\
&= i\omega \vec{B}_0 \exp\left(i\vec{k} \cdot \vec{r} - i\omega t\right)
\end{aligned}$$

Thus

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0 \tag{13}$$

2.2 Energy and momentum in EM waves

The energy density in the waves fields is

$$\begin{aligned} u &= \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \\ &= \frac{\varepsilon_0}{2}\left(E^2 + c^2 B^2\right) \end{aligned}$$

But since we already showed that $B = E/c$, there is equal energy in the electric and magnetic components of the wave.

$$u = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 \cos^2\left(\vec{k}\cdot\vec{r} - \omega t + \phi\right)$$

The Poynting vector describes the flux of energy:

$$\begin{aligned} \vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\vec{E}\times\left(\frac{\vec{k}\times\vec{E}}{\omega}\right) \\ &= \frac{1}{\omega\mu_0}\left[\vec{k}E^2 - \vec{E}\left(\vec{k}\cdot\vec{E}\right)\right] \end{aligned}$$

But $\vec{k}\cdot\vec{E} = 0$, so

$$\vec{S} = \frac{E^2}{c\mu_0}\hat{k} = \frac{\varepsilon_0 E^2}{c\mu_0\varepsilon_0}\hat{k} = cu\hat{k}$$

This shows explicitly that the energy density in the wave is carried along at the wave speed c.

If we average over several (or many) wave periods, we get the time averaged transmitted power per unit area of the wavefront:

$$< \mathcal{P} > \ = \frac{1}{2c}\varepsilon_0 E_0^2$$

As with waves on a string, the power goes as the square of the wave amplitude, depends on the wave speed c, and also on the properties of the medium, here ε_0. Physicists call this quantity the wave intensity. Astronomers beware – the term intensity in astronomy means something different.

The waves carry momentum as well as energy: From Notes 2 equation 6, the average momentum flux density is

$$< \vec{\mathcal{P}}_{\mathrm{EM}} > \ = \frac{< \vec{S} >}{c}$$

Now if the wave energy is absorbed at a flat screen, momentum is also aborbed, and thus there is a force exerted on the screen. The force, from Newton's second law, is

$$\vec{F} = \frac{d\vec{p}}{dt}$$

and the pressure on the screen is the normal force per units area. If the wave strikes the screen at normal incidence, then

$$P = \frac{|F|}{A} = \frac{\left|<\vec{S}>\right|}{c} = \frac{E^2}{2c^2\mu_0} = \frac{1}{2}\varepsilon_0 E^2$$

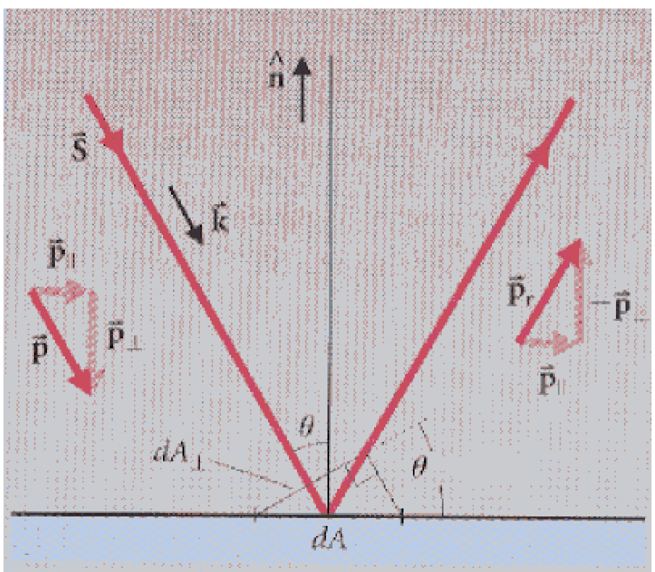
If the screen reflects all the energy and momentum, then the momentum imparted to the screen is twice as big:

	Before reflection	after reflection
Momentum in wave	$\vec{p}_{in}$	$\vec{p}_{out} = -\vec{p}_{in}$
Momentum in screen	0	$\vec{p}_{\text{screen}}$
Total	$\vec{p}_{in}$	$\vec{p}_{\text{screen}} - \vec{p}_{in}$

Setting the total before equal to the total after, we get

$$\vec{p}_{\text{screen}} = 2\vec{p}_{in}$$

Things get more interesting when the wave is incident at an angle θ. (See LB "Digging Deeper" pg 1049.)



Momentum in direction $\hat{k}$ at angle θ to the surface normal $\hat{n}$ is carried by the wave. Only the normal component is absorbed or reflected:

$$p_{\text{normal}} = p\cos\theta$$

The momentum in area $dA_{\perp}$ of the wave impacts area dA of the surface, where

$$dA_{\perp} = dA\cos\theta$$

Thus the momentum absorbed per unit area is

$$\frac{\Delta p_{\text{norm}}}{\Delta A} = \frac{p\cos\theta}{dA_\perp/\cos\theta} = \frac{p}{dA_\perp}\cos^2\theta$$

Thus the radiation pressure is

$$P = \frac{f}{2}\varepsilon_0 E^2\cos^2\theta$$

where the factor $f = 1$ for total absorption and $= 2$ for total reflection.

3 Electromagnetic waves in matter

When waves propagate in matter, there are non-zero sources ρ and $\vec{j}$ that must be included in Maxwell's equations. These sources affect the speed and other properties of the waves. Here we will discuss *LIH* materials that can be described by the fields $\vec{D}$ and $\vec{H}$, thus allowing us to "bury" the bound charges. Then if the free charge density and free current density are zero, the equations are:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{D} &= 0 = \varepsilon\vec{\nabla}\cdot\vec{E} \qquad (14)\\
\vec{\nabla}\cdot\vec{B} &= 0\\
\vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}\\
\vec{\nabla}\times\frac{\vec{B}}{\mu} &= \vec{\nabla}\times\vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad (15)
\end{aligned}$$

We find the wave equation using the same methods as before. Comparing these equations with equations (6) to (9), we see that the only difference is that ε_0 has been replaced by ε and μ_0 has been releced by μ. Thus the wave equation for $\vec{E}$ takes the form

$$\nabla^2\vec{E} = \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2} \qquad (16)$$

where the wave phase speed is

$$v = \frac{1}{\sqrt{\varepsilon\mu}}$$

In most ordinary materials like glass, $\mu \simeq \mu_0$ and $\varepsilon > \varepsilon_0$, so that

$$\frac{v}{c} = \sqrt{\frac{\varepsilon_0\mu_0}{\varepsilon\mu}} \simeq \sqrt{\frac{\varepsilon_0}{\varepsilon}} < 1$$

and light travels more slowly than in vacuum. The index of refraction of the material is

$$n = \frac{v}{c} \simeq \sqrt{\frac{\varepsilon}{\varepsilon_0}} > 1$$

The Poynting vector is now

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}$$

The relation between the amplitudes of the $\vec{E}$ and $\vec{B}$ fields may be found from Faraday's law:

$$\vec{k}\times\vec{E}_0 = \omega\vec{B}_0 = \frac{k\vec{B}_0}{v}$$

so that

$$B_0 = \frac{E_0}{v} \tag{17}$$

The fields are still perpendicular to each other and to $\vec{k}$, and vary in phase.

4 Reflection and transmission of waves at a boundary

4.1 Normal incidence

When a wave reaches a boundary between two different, LIH media, the wave is partially transmitted and partially reflected. As with waves on a string, each wave has the same frequency. This ensures that the boundary conditions are satisfied at all times, not just at one time. The boundary conditions on the fields are the same ones that we found in Physics 360:

$$\text{normal } \vec{D} \text{ is continuous} \tag{18}$$

$$\text{tangential } \vec{E} \text{ is continuous} \tag{19}$$

$$\text{normal } \vec{B} \text{ is continuous} \tag{20}$$

$$\text{tangential } \vec{H} \text{ is continuous} \tag{21}$$

We begin by writing expressions for $\vec{E}$ and $\vec{B}$ in all three waves. For simplicity, let the boundary be the $x,y-$plane, with medium 1 occupying the region $z<0$. The wave travels in the $+\hat{z}$ direction (normal incidence). First $\vec{E}$. We put the $x-$axis parallel to $\vec{E}$, so

$$\begin{aligned}
\vec{E}_\text{i} &= \vec{E}_{\text{i}0}\exp\left(ik_1z - i\omega t\right) = E_{\text{i}0}\hat{x}\exp\left(ik_1z - i\omega t\right)\\
\vec{E}_\text{t} &= \vec{E}_{\text{i}0}\exp\left(ik_2z - i\omega t\right) = E_{\text{t}0}\hat{x}\exp\left(ik_2z - i\omega t\right)\\
\vec{E}_\text{r} &= \vec{E}_{\text{r}0}\exp\left(-ik_1z - i\omega t\right) = E_{\text{r}0}\hat{x}\exp\left(-ik_1z - i\omega t\right)
\end{aligned}$$

The minus sign in front of the k_1z term in the reflected wave signifies that this wave is going in the $-z$ direction The wave numbers are

$$k_1 = \frac{\omega}{v_1} = \omega\sqrt{\varepsilon_1\mu_1}$$

and

$$k_2 = \frac{\omega}{v_2} = \omega\sqrt{\varepsilon_2\mu_2}$$

Now in writing the magnetic field vectors, we must remember that

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0$$

So

$$\begin{aligned}
\vec{B}_{\rm i} &= \vec{B}_{\rm i0} \exp(ik_1 z - i\omega t) = B_{\rm i0}\hat{z} \times \hat{x} \exp(ik_1 z - i\omega t) = \frac{E_{\rm i0}}{v_1}\hat{y} \exp(ik_1 z - i\omega t) \\
\vec{B}_{\rm t} &= \vec{B}_{\rm t0} \exp(ik_2 z - i\omega t) = \frac{E_{\rm t0}}{v_2}\hat{y} \exp(ik_2 z - i\omega t) \\
\vec{B}_{\rm r} &= \vec{B}_{\rm r0} \exp(-ik_1 z - i\omega t) = B_{\rm r0}(-\hat{z}) \times \hat{x} \exp(ik_1 z - i\omega t) = -\frac{E_{\rm r0}}{v_1}\hat{y} \exp(-ik_1 z - i\omega t)
\end{aligned}$$

Now we apply the boundary conditions (18) to (21).

Continuity of normal $\vec{D}$:

$$0 = 0$$

Continuity of tangential $\vec{E}$:

$$E_{\rm i0} + E_{\rm r0} = E_{\rm t0} \tag{22}$$

Continuity of normal $\vec{B}$:

$$0 = 0$$

Continuity of tangential $\vec{H}$:

$$\begin{aligned}
\frac{B_{\rm i0} + B_{\rm r0}}{\mu_1} &= \frac{B_{\rm to}}{\mu_2} \\
\frac{E_{\rm i0} - E_{\rm r0}}{v_1 \mu_1} &= \frac{E_{\rm t0}}{v_2 \mu_2} \qquad (23)
\end{aligned}$$

Now we solve. First multiply eqn (23) by $v_1 \mu_1$:

$$E_{\rm i0} - E_{\rm r0} = v_1 \mu_1 \frac{E_{\rm r0}}{v_2 \mu_2} = E_{\rm t0} \frac{n_2}{n_1} \frac{\mu_1}{\mu_2}$$

and now add to equation (22):

$$2E_{\rm i0} = E_{\rm t0}\left(1 + \frac{n_2}{n_1}\frac{\mu_1}{\mu_2}\right)$$

Thus

$$E_{\rm t0} = 2E_{\rm i0} \frac{n_1 \mu_2}{n_1 \mu_2 + n_2 \mu_1} \tag{24}$$

and then from (22),

$$\begin{aligned}
E_{\rm r0} &= E_{\rm i0}\left(2\frac{n_1 \mu_2}{n_1 \mu_2 + n_2 \mu_1} - 1\right) \\
&= E_{\rm i0}\left(\frac{n_1 \mu_2 - n_2 \mu_1}{n_1 \mu_2 + n_2 \mu_1}\right) \qquad (25)
\end{aligned}$$

This result is interesting, because it can be negative if $n_2\mu_1 > n_1\mu_2$. As in the case of the waves on a string, it means that the $\vec{E}$ vector has a phase change of π, or, equivalently, it changes direction. We may simplify these results if, as is usual, $\mu_1 \simeq \mu_2 \simeq \mu_0$. Then

$$\begin{aligned} E_{\mathrm{t0}} &= E_{\mathrm{i0}} \frac{2n_1}{n_1+n_2} \\ E_{\mathrm{r0}} &= E_{\mathrm{i0}} \left(\frac{n_1-n_2}{n_1+n_2}\right) \end{aligned}$$

From these results we may calculate the reflected and transmitted intensities:

$$< S_{\mathrm{inc}} >= \frac{1}{2\mu_1} E_{\mathrm{i0}} B_{\mathrm{i0}} = \frac{E_{\mathrm{i0}}^2}{2\mu_1 v_1} = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}} E_{\mathrm{i0}}^2$$

$$\begin{aligned} < \quad S_{\mathrm{tr}} >&= \frac{1}{2}\sqrt{\frac{\varepsilon_2}{\mu_2}} E_{\mathrm{t0}}^2 = \frac{1}{2}\sqrt{\frac{\varepsilon_2}{\mu_2}} 4E_{\mathrm{i0}}^2 \left(\frac{n_1\mu_2}{n_1\mu_2+n_2\mu_1}\right)^2 \\ < \quad S_{\mathrm{ref}} >&= \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}} E_{\mathrm{r0}}^2 = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}} E_{\mathrm{i0}}^2 \left(\frac{n_1\mu_2-n_2\mu_1}{n_1\mu_2+n_2\mu_1}\right)^2 \end{aligned}$$

and

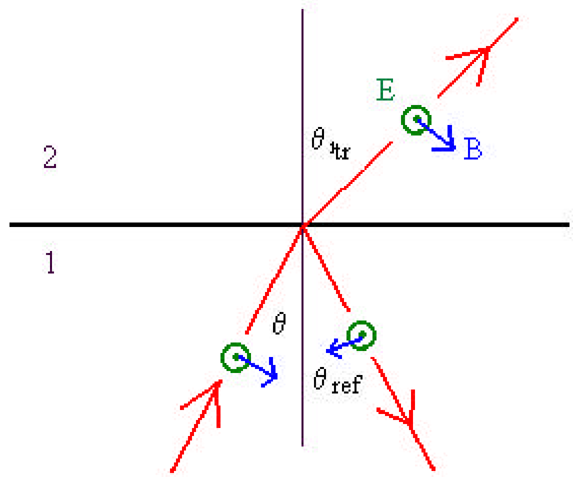
$$\begin{aligned} < \quad S_{\mathrm{tr}} > + < S_{\mathrm{ref}} >&= \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}} E_{\mathrm{i0}}^2 \left[\sqrt{\frac{\varepsilon_2\mu_1}{\mu_2\varepsilon_1}} 4\left(\frac{n_1\mu_2}{n_1\mu_2+n_2\mu_1}\right)^2 + \left(\frac{n_1\mu_2-n_2\mu_1}{n_1\mu_2+n_2\mu_1}\right)^2\right] \\ = \quad & \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}} E_{\mathrm{i0}}^2 \left[\frac{\mu_1}{\mu_2}\sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}} 4\left(\frac{n_1\mu_2}{n_1\mu_2+n_2\mu_1}\right)^2 + \left(\frac{n_1\mu_2-n_2\mu_1}{n_1\mu_2+n_2\mu_1}\right)^2\right] \\ = \quad & \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}} E_{\mathrm{i0}}^2 \left[\frac{n_2\mu_1}{n_1\mu_2} \frac{4\left(n_1\mu_2\right)^2}{\left(n_1\mu_2+n_2\mu_1\right)^2} + \left(\frac{n_1\mu_2-n_2\mu_1}{n_1\mu_2+n_2\mu_1}\right)^2\right] \\ = \quad & \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}} E_{\mathrm{i0}}^2 \left[\frac{4n_1\mu_2n_2\mu_1}{\left(n_1\mu_2+n_2\mu_1\right)^2} + \frac{\left(n_1\mu_2\right)^2 - 2n_1\mu_2n_2\mu_1 + \left(n_2\mu_1\right)^2}{\left(n_1\mu_2+n_2\mu_1\right)^2}\right] \\ = \quad & \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}} E_{\mathrm{i0}}^2 \left[\frac{\left(n_1\mu_2\right)^2 + 2n_1\mu_2n_2\mu_1 + \left(n_2\mu_1\right)^2}{\left(n_1\mu_2+n_2\mu_1\right)^2}\right] = \frac{1}{2}\sqrt{\frac{\varepsilon_1}{\mu_1}} E_{\mathrm{i0}}^2 = \quad < S_{\mathrm{inc}} > \end{aligned}$$

This result is required by energy conservation.

4.2 Non-normal incidence

When a wave is incident at an angle $\theta \neq 0$, we have to tbe aware that the polarization of the wave enters into the boundary conditions. The plane that contains the normal to the surface and the incident ray is called the plane of incidence. (The ray is normal to the wavefront- see LB p547) If the electric field vector is perpendicular to the plane of incidence, as in the diagram below, we say that the wave is polarized perpendicular to the plane of incidence. If the electric field vector lies in the plane of incidence, then the wave is polarized in the plane of incidence.

4.2.1 Polarization perpendicular to the plane of incidence



In this case the electric field vector is entirely tangential to the boundary. We choose coordinates so that the boundary is the $x - y$ plane. Then

$$\vec{E}_{\rm inc} = -E_0 \hat{y} \exp\left(i\vec{k} \cdot \vec{r} - i\omega t\right)$$

$$\vec{E}_{\rm ref} = -E_{\rm r0} \hat{y} \exp\left(i\vec{k}_{\rm r} \cdot \vec{r} - i\omega t\right)$$

and

$$\vec{E}_{\rm trans} = -E_{\rm t0} \hat{y} \exp\left(i\vec{k}_{\rm t} \cdot \vec{r} - i\omega t\right)$$

As usual each wave has the same frequency, and

$$\left|\vec{k}_{\rm r}\right| = \left|\vec{k}\right| = \frac{\omega}{v_1}\ ; \qquad \left|\vec{k}_t\right| = \frac{\omega}{v_2}$$

Further, for points on the boundary,

$$\vec{k} \cdot \vec{r} = kr \sin\theta$$

Thus the boundary condition (19–continuity of tangential $\vec{E}$) becomes

$$\begin{aligned} \vec{E}_{\rm inc} + \vec{E}_{\rm ref} &= \vec{E}_{\rm trans} \\ -E_0 \exp\left(i\vec{k} \cdot \vec{r} - i\omega t\right) - E_{\rm r0} \exp\left(i\vec{k}_{\rm r} \cdot \vec{r} - i\omega t\right) &= -E_{\rm t0} \exp\left(i\vec{k}_{\rm t} \cdot \vec{r} - i\omega t\right) \end{aligned}$$

The factors $\exp(-i\omega t)$ cancel, leaving

$$-E_0 \exp(ikr\sin\theta) - E_{\rm r0} \exp(ik_{\rm r} r \sin\theta_{\rm r}) = -E_{\rm t0} \exp(ik_{\rm t} r \sin\theta_{\rm t}) \qquad (26)$$

This relation must hold at all points on the boundary. Thus we must have

$$k\sin\theta = k_{\rm r} \sin\theta_{\rm r} = k_{\rm t} \sin\theta_{\rm t}$$

Since $k = k_{\mathrm{r}}$, we must have $\sin\theta = \sin\theta_{\mathrm{r}}$, and further, since the angles of incidence and reflection always lie in the range $\leq \theta \leq \pi/2$, we must have

$$\theta = \theta_{\mathrm{r}} \tag{27}$$

This is the law of reflection. Similarly, since $k = \omega/v_1$ and $k_{\mathrm{t}} = \omega/v_2$,

$$\frac{\sin\theta}{v_1} = \frac{\sin\theta_{\mathrm{t}}}{v_2}$$

or, equivalently,

$$n_1 \sin\theta = n_2 \sin\theta_{\mathrm{t}} \tag{28}$$

This is Snell's Law.

It is important to note that these relations arise from the fact that we have to satisfy a boundary condition at every point of the boundary. They do not depend on the details of the boundary condition at all. Thus they hold for plane waves of any kind, not just EM waves.

With equations (27) and (28) satisfied, boundary condition (26) becomes:

$$E_0 + E_{\mathrm{r}0} = E_{\mathrm{t}0} \tag{29}$$

Now we look at the conditions on $\vec{B}$ and $\vec{H}$. These cannot give 2 independent relations, because we only need two equations to find the two unknowns $E_{\mathrm{r}0}$ and $E_{\mathrm{t}0}$. Let's look at them:

$$\begin{aligned}
\vec{B}_{\mathrm{inc}} &= \vec{k} \times \vec{E}_{\mathrm{inc}} = -\vec{k} \times \vec{y} E_0 \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right) \\
&= -k\left(\cos\theta\ \hat{z} + \sin\theta\ \hat{x}\right) \times \vec{y} E_0 \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right) \\
&= k\left(\cos\theta\ \hat{x} - \sin\theta\ \hat{z}\right) E_0 \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right)
\end{aligned}$$

$$\begin{aligned}
\vec{B}_{\mathrm{ref}} &= \vec{k}_{\mathrm{r}} \times \vec{E}_{\mathrm{ref}} = -\vec{k}_{\mathrm{r}} \times \hat{y} E_{\mathrm{r}0} \exp\left(i\vec{k}_{\mathrm{r}}\cdot\vec{r} - i\omega t\right) \\
&= -k\left(-\cos\theta\ \hat{z} + \sin\theta\ \hat{x}\right) \times \vec{y} E_{\mathrm{r}0} \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right) \\
&= k\left(-\cos\theta\ \hat{x} - \sin\theta\ \hat{z}\right) E_{\mathrm{r}0} \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right)
\end{aligned}$$

and

$$\begin{aligned}
\vec{B}_{\mathrm{trans}} &= \vec{k}_{\mathrm{t}} \times \vec{E}_{\mathrm{trans}} = -E_{\mathrm{t}0}\vec{k}_{\mathrm{t}} \times \hat{y} E_{\mathrm{t}0} \exp\left(i\vec{k}_{\mathrm{t}}\cdot\vec{r} - i\omega t\right) \\
&= k_{\mathrm{t}}\left(\cos\theta_{\mathrm{t}}\ \hat{x} - \sin\theta_{\mathrm{t}}\ \hat{z}\right) E_0 \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right)
\end{aligned}$$

Normal $\vec{B}$:

$$-k\sin\theta E_0 - k\sin\theta E_{\mathrm{r}0} = -k_{\mathrm{t}}\sin\theta_{\mathrm{t}} E_{\mathrm{t}0}$$

Using Snell's law, we get back equation (29). We do get an independent equation from the boundary condition for tangential $\vec{H}$:

$$\begin{aligned}
\frac{k}{\mu_1}\cos\theta\,(E_0 - E_{\mathrm{r}0}) &= \frac{k_{\mathrm{t}}}{\mu_2}\cos\theta_{\mathrm{t}}\ E_{\mathrm{t}0} \\
(E_0 - E_{\mathrm{r}0}) &= \frac{v_1\mu_1}{v_2\mu_2} E_{\mathrm{t}0}\frac{\sqrt{1-\sin^2\theta_{\mathrm{t}}}}{\cos\theta} \\
(E_0 - E_{\mathrm{r}0}) &= \frac{n_2\mu_1}{n_1\mu_2} E_{\mathrm{t}0}\frac{\sqrt{1-n_1^2\sin^2\theta/n_2^2}}{\cos\theta} \qquad (30)
\end{aligned}$$

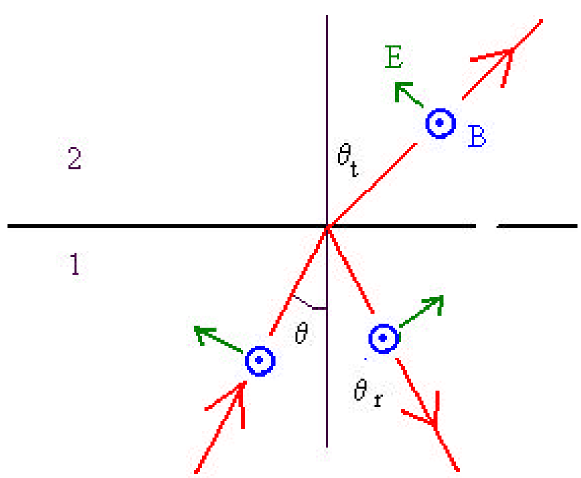
We will simplify by taking $\mu_1 = \mu_2 = \mu_0$. Then combining equations (29) and (30), we have

$$\begin{aligned}
2E_0 &= \left(1 + \frac{\sqrt{n_2^2 - n_1^2\sin^2\theta}}{n_1\cos\theta}\right) E_{\mathrm{t}0} \\
E_{\mathrm{t}0} &= \frac{2E_0 n_1\cos\theta}{n_1\cos\theta + \sqrt{n_2^2 - n_1^2\sin^2\theta}} \qquad (31)
\end{aligned}$$

and then

$$E_{\mathrm{r}0} = E_0\frac{n_1\cos\theta - \sqrt{n_2^2 - n_1^2\sin^2\theta}}{n_1\cos\theta + \sqrt{n_2^2 - n_1^2\sin^2\theta}} \qquad (32)$$

4.2.2 Polarization parallel to the plane of incidence



Now the boundary condition on tangential $\vec{H}$ (again with all μs equal) is

$$\begin{aligned} B_0 + B_{\rm r0} &= B_{\rm t0} \\ n_1 \left(E_0 + E_{\rm r0}\right) &= n_2 E_{\rm t0} \end{aligned} \tag{33}$$

while for tangential $\vec{E}$ we get

$$\begin{aligned} E_0 \cos\theta - E_{\rm r0} \cos\theta &= E_{\rm t0} \cos\theta_{\rm t} \\ E_0 - E_{\rm r0} &= E_{\rm t0} \frac{\sqrt{1-\sin^2\theta_{\rm t}}}{\cos\theta} \end{aligned} \tag{34}$$

Again the third non-trivial condition (for normal D) does not give an independent relation. Combining these, we get

$$\begin{aligned} 2E_0 &= E_{\rm t0} \left(\frac{\sqrt{1-n_1^2\sin^2\theta/n_2^2}}{\cos\theta} + \frac{n_2}{n_1} \right) \\ E_{\rm t0} &= \frac{2E_0 n_1 \cos\theta}{n_2\cos\theta + n_1\sqrt{1-n_1^2\sin^2\theta/n_2^2}} \\ &= \frac{2E_0 n_1 n_2 \cos\theta}{n_2^2\cos\theta + n_1\sqrt{n_2^2 - n_1^2\sin^2\theta}} \end{aligned} \tag{35}$$

and then

$$E_{\rm r0} = E_0 \frac{n_2^2\cos\theta - n_1\sqrt{n_2^2 - n_1^2\sin^2\theta}}{n_2^2\cos\theta + n_1\sqrt{n_2^2 - n_1^2\sin^2\theta}} \tag{36}$$

In both polarizations we find that the reflected and transmitted amplitudes depend on the angle of incidence, as well as the properties of the two materials. In fact, the refelected amplitude (36) may actually be zero at an angle given by

$$n_2^2 \cos\theta = n_1 \sqrt{n_2^2 - n_1^2\sin^2\theta}$$

square both sides:

$$n_2^4 \cos^2\theta = n_1^2 \left(n_2^2 - n_1^2 \sin^2\theta\right)$$

Now use a trick: write $1 = \cos^2\theta + \sin^2\theta$

$$\begin{aligned} \frac{n_2^4}{n_1^2}\cos^2\theta &= n_2^2\left(\cos^2\theta + \sin^2\theta\right) - n_1^2\sin^2\theta \\ \frac{n_2^2}{n_1^2}\left(n_2^2 - n_1^2\right)\cos^2\theta &= \left(n_2^2 - n_1^2\right)\sin^2\theta \end{aligned}$$

Then, provided $n_2 \neq n_1$,

$$\tan\theta_{\rm B} = \frac{n_2}{n_1} \tag{37}$$

This angle is called Brewster's angle. At this angle of incidence, in this polarization, all of the wave energy is transmitted and none is reflected. (We'll discuss why this happens later- see also LB §33.3.2). Note, however, that nothing special happens at this angle for waves polarized perpendicular to the plane of incidence. In fact, E_{r0} in equation (32) is *never zero for any* θ. Check this for yourself. What this means is that if we have unpolarized incident light, equal amplitudes in the two polarizations, at Brewster's angle only one of the two polarizations is reflected. The reflected light is completely polarized perpendicular to the plane of incidence. This phenomenon is called "polarization by reflection".

Under what conditions do we get a phase change of the reflected wave? That is, when is the sign of E_{r0} opposite that of E_0? Starting from equation (32), we want to find when

$$\begin{aligned} n_1 \cos\theta &< \sqrt{n_2^2 - n_1^2 \sin^2\theta} \\ n_1^2 \cos^2\theta &< n_2^2 - n_1^2 \sin^2\theta \end{aligned}$$

This is true for

$$n_1 < n_2$$

independent of θ, as we also found for normal incidence.

For the other polarization, Start with equation (36)

$$n_2^2 \cos\theta - n_1 \sqrt{n_2^2 - n_1^2 \sin^2\theta} < 0$$

or, equivalently,

$$n_2^4 \cos^2\theta < n_1^2 \left(n_2^2 - n_1^2 \sin^2\theta\right)$$

Using the same trick as in finding Brewster's angle, we find this reduces to

$$\frac{n_2^2}{n_1^2}\left(n_2^2 - n_1^2\right)\cos^2\theta < \left(n_2^2 - n_1^2\right)\sin^2\theta$$

or

$$\theta < \theta_B \quad \text{if } n_2 > n_1$$

and

$$\theta > \theta_B \quad \text{if } n_1 < n_2$$

4.2.3 Reflection and transmission coefficients

The incident power striking unit area of the interface is

$$\vec{S}_{\text{in}} \cdot \hat{z} = \left(\vec{E}_{\text{in}} \times \vec{H}_{\text{in}}\right) \cdot \hat{z}$$

Now we time average to get

$$\begin{aligned}
I_{\rm in} &= \frac{1}{2}E_0^2\left[\hat{e}\times\left(\frac{\vec{k}}{\omega}\times\hat{e}\right)\right]\cdot\hat{z} \\
&= \frac{1}{2v_1\mu_1}E_0^2\ \hat{k}\cdot\hat{z} \\
&= \frac{1}{2}\varepsilon_1 v_1 E_0^2\cos\theta
\end{aligned}$$

where

$$\vec{E}_0 = E_0\hat{e}$$

and $\hat{e}$ is the polarization vector. We can compute the reflected and transmitted intensities similarly. Then the reflection coefficient is defined as

$$R = \frac{I_{\rm R}}{I_{\rm in}} = \frac{E_{\rm r0}^2}{E_0^2}$$

and the transmission coefficient is

$$T = \frac{I_{\rm T}}{I_{\rm in}} = \frac{v_1}{v_2}\frac{E_{\rm t0}^2}{E_0^2}\frac{\cos\theta_{\rm t}}{\cos\theta} = \frac{n_2}{n_1}\frac{E_{\rm t0}^2}{E_0^2}\frac{\cos\theta_{\rm t}}{\cos\theta}$$

For polarization perpendicular to the plane of incidence, we get

$$R = \left(\frac{n_1\cos\theta - \sqrt{n_2^2 - n_1^2\sin^2\theta}}{n_1\cos\theta + \sqrt{n_2^2 - n_1^2\sin^2\theta}}\right)^2$$

and

$$\begin{aligned}
T &= \left(\frac{2n_1\cos\theta}{n_1\cos\theta + \sqrt{n_2^2 - n_1^2\sin^2\theta}}\right)^2\frac{n_2\cos\theta_{\rm t}}{n_1\cos\theta} \\
&= \frac{4n_1 n_2\cos\theta\cos\theta_{\rm t}}{\left(n_1\cos\theta + \sqrt{n_2^2 - n_1^2\sin^2\theta}\right)^2}
\end{aligned}$$

and

$$\begin{aligned}
R+T &= \frac{\left(n_1\cos\theta - \sqrt{n_2^2 - n_1^2\sin^2\theta}\right)^2 + 4n_1n_2\cos\theta\cos\theta_{\rm t}}{\left(n_1\cos\theta + \sqrt{n_2^2 - n_1^2\sin\theta}\right)^2} \\
&= \frac{n_1^2\cos^2\theta + n_2^2 - n_1^2\sin^2\theta - 2n_1\cos\theta\sqrt{n_2^2 - n_1^2\sin^2\theta} + 4n_1\cos\theta\sqrt{n_2^2 - n_1^2\sin^2\theta}}{\left(n_1\cos\theta + \sqrt{n_2^2 - n_1^2\sin^2\theta}\right)^2} \\
&= 1
\end{aligned}$$

Verify that this result also holds, as it must, for the other polarization.

5 Waves in conductors

An electric field in a conductor drives a current.

$$\vec{j} = \sigma\vec{E} = \frac{\vec{E}}{\rho}$$

and thus some of the field energy is converted to kinetic energy of electrons in the conductor. This leads to absorption of the wave energy. In order to investigate this, we must include the current in Maxwell's equations. The only equation to change is the Ampere-Maxwell law.

$$\vec{\nabla} \times \vec{B} = \mu\vec{j} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t} = \mu\sigma\vec{E} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}$$

We may still take the free charge density ρ_f to be zero. For suppose ρ_f is not zero. Then from charge conservation,

$$\begin{aligned}\frac{\partial\rho_f}{\partial t} &= -\vec{\nabla}\cdot\vec{j} = -\vec{\nabla}\cdot\sigma\vec{E} \\ &= -\sigma\left(\frac{\rho_f}{\varepsilon}\right)\end{aligned}$$

This differential equation has the solution

$$\rho_f = \rho_0 \exp\left(-\frac{\sigma}{\varepsilon}t\right)$$

so the charge density dies away with a time scale

$$\tau = \frac{\varepsilon}{\sigma}$$

which is very small for good conductors. As we discussed before, even if $\sigma \to \infty$, the timescale does not go to zero, as the dominant factor is then the timescale for the fields to change, of order (scale of system)/(speed of light).

Now we may rederive the wave equation with the extra term in Ampere's law. Starting from equation (11), we have

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right) = -\frac{\partial}{\partial t}\left(\mu\sigma\vec{E} + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

which becomes

$$\nabla^2\vec{E} = \frac{\partial}{\partial t}\mu\sigma\vec{E} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

The wave equation now has an extra term. We may still find a plane wave soution of the form:

$$\begin{aligned}\vec{E} &= \vec{E}_0 \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right) \\ \vec{B} &= \vec{B}_0 \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right)\end{aligned}$$

Stuffing these expressions into the wave equation and Maxwell's equations, we have

$$-k^2 = -i\omega\mu\sigma - \mu\varepsilon\omega^2 \tag{38}$$

$$\begin{aligned}\vec{k}\cdot\vec{E}_0 &= 0\\ \vec{k}\cdot\vec{B}_0 &= 0\\ \vec{k}\times\vec{E}_0 &= \omega\vec{B}_0 \end{aligned}\tag{39}$$

Thus the wave has the same transverse nature as before. The only thing that has changed is the relation of k to ω. Equation (38) requires that k be complex if ω is real. So let $k = \kappa + i\gamma$. Then

$$k^2 = \kappa^2 - \gamma^2 + 2i\kappa\gamma = i\omega\mu\sigma + \mu\varepsilon\omega^2$$

So we have two equations obtained by setting the real part on the left equal to the real part on the right, and also by setting the two imaginary parts equal.

$$\kappa^2 - \gamma^2 = \mu\varepsilon\omega^2$$

$$2\kappa\gamma = \omega\mu\sigma$$

From the second, we get

$$\gamma = \frac{\omega\mu\sigma}{2\kappa} \tag{40}$$

and then from the first

$$\kappa^2 - \left(\frac{\omega\mu\sigma}{2\kappa}\right)^2 - \mu\varepsilon\omega^2 = 0$$

This is a quadratic equation for κ^2 with solution

$$\begin{aligned}\kappa^2 &= \frac{\mu\varepsilon\omega^2 \pm \sqrt{\left(\mu\varepsilon\omega^2\right)^2 + \left(\omega\mu\sigma\right)^2}}{2}\\ &= \mu\varepsilon\omega^2\frac{1 \pm \sqrt{1 + \left(\sigma/\omega\varepsilon\right)^2}}{2}\end{aligned}$$

We must get back our previous result $k = \omega\sqrt{\mu\varepsilon}$ as $\sigma \to 0$, so we need the plus sign:

$$\kappa = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{1 + \sqrt{1 + \left(\sigma/\omega\varepsilon\right)^2}}{2}}$$

and then from (40), we have

$$\gamma = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{2}{1 + \sqrt{1 + \left(\sigma/\omega\varepsilon\right)^2}}}$$

The imaginary part of k shows that the wave is damped:

$$\exp i\vec{k}\cdot\vec{r} = \exp\left[i\hat{k}\cdot\vec{r}\,(\kappa+i\gamma)\right] = \exp\left(i\kappa\hat{k}\cdot\vec{r}\right)\exp\left(-\gamma\hat{k}\cdot\vec{r}\right)$$

and taking the real part, we have

$$\cos\left(\kappa\hat{k}\cdot\vec{r}\right)\exp\left(-\gamma\hat{k}\cdot\vec{r}\right)$$

The wave attenuates exponentially as it propagates. The distance travelled before the wave amplitude drops to $1/e$ of its original value is called the skin depth

$$\delta = \frac{1}{\gamma}$$

Faraday's law (equation (39)) shows that $\vec{E}$ and $\vec{B}$ no longer oscillate in phase. We may write $\vec{k}$ in exponential form as

$$\vec{k} = \hat{k}\,(\kappa+i\gamma) = \hat{k}\sqrt{\kappa^2+\gamma^2}\exp i\phi, \qquad \tan\phi = \frac{\gamma}{\kappa}$$

where

$$\begin{aligned} k &= \sqrt{\kappa^2+\gamma^2} = \sqrt{\mu\varepsilon\omega^2\frac{1+\sqrt{1+(\sigma/\omega\varepsilon)^2}}{2}+\frac{\sigma^2}{2}\frac{\mu}{\varepsilon}\frac{1}{1+\sqrt{1+(\sigma/\omega\varepsilon)^2}}} \\ &= \omega\sqrt{\mu\varepsilon}\sqrt{\frac{2+(\sigma/\omega\varepsilon)^2+2\sqrt{1+(\sigma/\omega\varepsilon)^2}+\sigma^2/\omega^2\varepsilon^2}{2\left(1+\sqrt{1+(\sigma/\omega\varepsilon)^2}\right)}} \\ &= \omega\sqrt{\mu\varepsilon}\left(1+(\sigma/\omega\varepsilon)^2\right)^{1/4} \end{aligned}$$

and

$$\tan\phi = \frac{\sigma}{\varepsilon\omega\left(1+\sqrt{1+(\sigma/\omega\varepsilon)^2}\right)}$$

Then equation (39) becomes

$$\begin{aligned} \hat{k}\times\hat{e}E_0\left(ke^{i\phi}\right) &= \vec{B}_0 \\ B_0 &= \omega\sqrt{\mu\varepsilon}\left(1+(\sigma/\omega\varepsilon)^2\right)^{1/4}E_0e^{i\phi} \end{aligned}$$

A material is a good conductor as far as wave propagation is concerned if

$$\frac{\sigma}{\omega\varepsilon} \gg 1$$

in which case we find

$$B_0 \simeq \sqrt{\sigma\omega\mu}E_0e^{i\phi}$$

so that

$$B_0 \gg \frac{E_0}{v}$$

and the phase shift δ is almost $\pi/4$. ($\tan\delta \to 1$). The fields in the conductor are primarily magnetic in character. In this limit we also have

$$\kappa \simeq \omega\sqrt{\mu\varepsilon}\sqrt{\frac{1+\sigma/\omega\varepsilon}{2}} \simeq \sqrt{\frac{\omega\sigma\mu}{2}} \simeq \gamma$$

and the wave damps within one wavelength. On the other hand, if $\sigma/\omega\varepsilon \ll 1$, we have a poor conductor, the phase shift is small, and the amplitudes of the fields are not much different from those in a non-conducting medium.

5.1 Reflection at a conducting surface

There may be a free *surface* charge density on the surface of a conductor. In fact, there usually is. But there cannot be a free surface current density as this would require an infinite electric field ($\vec{j} = \sigma\vec{E}$, and if $\vec{j} = \vec{K}\delta\left(z - z_{\text{surface}}\right)$ then $\vec{E} = \frac{\vec{K}}{\sigma}\delta\left(z - z_{\text{surface}}\right)$. Such an electric field cannot exist. Thus we choose to use the boundary conditions on tangential $\vec{E}$, normal $\vec{B}$ and tangential $\vec{H}$. Let's consider normal incidence for simplicity. Then we have

$$E_0 + E_{\text{r}} = E_{\text{t}}$$

and

$$\begin{aligned} H_0 - H_{\text{r}} &= H_{\text{t}} \\ \frac{\omega}{v_1}\left(E_0 - E_{\text{r}}\right) &= \sqrt{\frac{\sigma\omega}{\mu_2}}E_{\text{t}}e^{i\phi} \\ \left(E_0 - E_{\text{r}}\right) &= v_1\sqrt{\frac{\sigma}{\omega\mu_2}}E_{\text{t}}e^{i\phi} \end{aligned}$$

Thus we have

$$\begin{aligned} 2E_0 &= E_{\text{t}}\left(1 + v_1\sqrt{\frac{\sigma}{\omega\mu_2}}e^{i\phi}\right) \\ E_{\text{t}} &= \frac{2E_0}{\left(1 + v_1\sqrt{\frac{\sigma}{\omega\mu_2}}e^{i\phi}\right)} \end{aligned}$$

and then

$$E_{\text{r}} = E_0\left(\frac{1 - v_1\sqrt{\frac{\sigma}{\omega\mu_2}}e^{i\phi}}{1 + v_1\sqrt{\frac{\sigma}{\omega\mu_2}}e^{i\phi}}\right) \tag{41}$$

As $\sigma \to \infty$, $E_{\text{t}} \to 0$ and $E_{\text{r}} \to -1$.

5.2 Note on computing power using complex numbers

When our amplitudes are complex, as in (41), we must be careful when evaluating physical quantities that involve the square of these numbers. Let's look at the power.

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

Since

$$\mathrm{Re}\,(z_1 z_2) \neq \mathrm{Re}\,(z_1)\,\mathrm{Re}\,(z_2)$$

we must take the real part before multiplying. Thus

$$\begin{aligned}\vec{S}_{\text{phys}} &= \frac{1}{\mu_0}\vec{E}_0\cos\left(\vec{k}\cdot\vec{r}-\omega t+\phi_E\right)\times\vec{B}_0\cos\left(\vec{k}\cdot\vec{r}-\omega t+\phi_B\right)\\ &= \frac{1}{2\mu_0}\vec{E}_0\times\vec{B}_0\left[\cos\left(2\vec{k}\cdot\vec{r}-2\omega t+\phi_E+\phi_B\right)+\cos\left(\phi_E-\phi_B\right)\right]\end{aligned}$$

Now we time average, to get

$$<\vec{S}_{\text{phys}}> \; = \frac{1}{2\mu_0}\vec{E}_0\times\vec{B}_0\cos\left(\phi_E-\phi_B\right)$$

We can get the same result more easily as follows:

$$\frac{1}{2\mu_0}\vec{E}\times\vec{B}^* = \frac{1}{2\mu_0}\vec{E}_0\times\vec{B}_0\exp\left(i\phi_E-i\phi_B\right)$$

and so

$$<\vec{S}_{\text{phys}}> \; = \mathrm{Re}\left(\frac{1}{2\mu_0}\vec{E}\times\vec{B}^*\right) \tag{42}$$

6 Dispersive media

In general the properties of a medium, such as ε and μ, are frequency dependent. This means that waves of different frequencies travel at different speeds. A plane wave carries no information. To transmit a signal we have to vary either the amplitude of the frequency of the wave- the resulting signal is called AM or FM (as on your radio dial). The wave crests travel at the phase speed $v_\phi = \omega/k$ but the envelope that carries the information travels at the group speed $v_g = d\omega/dk$. It is v_g, not v_ϕ, that must be less than c. In order to get the signal, many different frequencies make up the wave packet, and because they do not all travel at the same speed, over time the signal becomes less well defined– the pulse is *dispersed*. Those of you who plan to take Phys 704 in the spring will learn more about these effects then. For now, let's see if we can understand how the frequency dependence arises.

We begin with a simple classical model of an electron in an atom. The electron behaves as an oscillator with a restoring force

$$\vec{F}_{\text{res}} = -m\omega_0^2\vec{s}$$

where ω_0 is the natural oscillation frequency and $\vec{s}$ is the electron displacement. There is also damping due to radiation reaction, which we may write as

$$\vec{F}_{\rm damp} = -m\gamma\frac{d\vec{s}}{dt}$$

Now when an incident EM wave reaches the electron, it will experience a force

$$\vec{F}_{\rm EM} = -e\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

We have already seen that, for waves in a medium with phase speed v_ϕ,

$$B_0 = \frac{E_0}{v_\phi}$$

so the magnetic force is smaller than the electric force by a factor v/v_ϕ which is typically $\ll 1$ since $v_\phi \sim c$. So we will ignore the magnetic force. Now we may write the equation of motion for the electron:

$$\begin{aligned} \vec{F}_{\rm tot} &= m\vec{a} \\ -e\vec{E}_0\cos\omega t - m\omega_0^2\vec{s} - m\gamma\frac{d\vec{s}}{dt} &= m\frac{d^2\vec{s}}{dt^2} \end{aligned}$$

The electron oscillates with $\vec{s}$ parallel to $\vec{E}_0$. We may put the $x-$axis along this direction, to get

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = -eE_0\cos\omega t = \mathrm{Re}\left(-\frac{e}{m}E_0\exp\left(-i\omega t\right)\right)$$

The solution will be of the form

$$x = \mathrm{Re}\left(x_0 e^{-i\omega t}\right)$$

Stuff this in and check:

$$\left(-\omega^2 - i\gamma\omega + \omega_0^2\right)x_0 = -\frac{e}{m}E_0$$

so

$$x_0 = \frac{eE_0/m}{\omega^2 + i\gamma\omega - \omega_0^2} \tag{43}$$

and the electron contributes a dipole moment about its original equilibrium position of

$$\vec{p} = -e\vec{s} = -\frac{e^2}{m}\frac{\vec{E}_0}{\omega^2 + i\gamma\omega - \omega_0^2}$$

Summing up over all the particles in the material, where there are n molecules per unit volume, each of which has f_j electrons with frequency ω_j and damping γ_j, we get a polarization

$$\vec{P} = n\vec{p} = -n\frac{e^2}{m}\sum_j \frac{f_j\vec{E}_0}{\omega^2 + i\gamma_j\omega - \omega_j^2}$$

So we now have a model for the susceptibilty of the material

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

Thus

$$\chi_e = -n\frac{e^2}{m}\sum_j \frac{f_j}{\omega^2 + i\gamma_j\omega - \omega_j^2}$$

Because χ is a complex number, the polarization is out of phase with the driving field $\vec{E}$. The dielectric constant is than

$$\varepsilon = \varepsilon_0 \left(1 + \frac{ne^2}{\varepsilon_0 m}\sum_j \frac{f_j}{\omega_j^2 - i\gamma_j\omega - \omega^2}\right) \tag{44}$$

and consequently the wave equation (16) has plane wave solutions of the form

$$\vec{E} = \vec{E}_0 \exp\left(i\vec{k}\cdot\vec{r} - i\omega t\right)$$

where

$$-k^2\vec{E}_0 = \varepsilon\mu\omega^2\vec{E}_0 = \varepsilon_0\mu\omega^2\vec{E}_0\left(1 + \frac{ne^2}{\varepsilon_0 m}\sum_j \frac{f_j\left(\omega_j^2 + i\gamma_j\omega - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \left(\gamma_j\omega\right)^2}\right)$$

and because ε is complex, k must be complex too.

$$k = k_{\mathrm{r}} + ik_{\mathrm{i}}$$

Putting the $z-$axis along $\hat{k}$, we have

$$\vec{E} = \vec{E}_0 \exp\left(-k_{\mathrm{i}} z\right)\exp\left(ik_{\mathrm{r}} z - i\omega t\right)$$

The wave is damped. The wave intensity is proportional to E^2, and so it decreases as $\exp\left(-2k_{\mathrm{i}} z\right) = \exp\left(-\alpha z\right)$ where α is the absorption coefficient.

Let's look at the special case where the second term in equation (44) is small, for example, in gases. Then we can approximate the square root using the binomal series:

$$\sqrt{1+x} \simeq 1 + \frac{x}{2} \quad \text{if } x \ll 1$$

So

$$k = \sqrt{\varepsilon\mu}\omega = \frac{\omega}{c}\left(1 + \frac{ne^2}{2\varepsilon_0 m}\sum_j \frac{f_j\left(\omega_j^2 + i\gamma_j\omega - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \left(\gamma_j\omega\right)^2}\right)$$

so

$$\alpha = \frac{\omega^2}{c}\frac{ne^2}{\varepsilon_0 m}\sum_j \frac{f_j\gamma_j}{\left(\omega_j^2 - \omega^2\right)^2 + \left(\gamma_j\omega\right)^2} \tag{45}$$

and the refective index is

$$n = \frac{ck_{\mathrm{r}}}{\omega} = 1 + \frac{ne^2}{2\varepsilon_0 m}\sum_j \frac{f_j\left(\omega_j^2 - \omega^2\right)}{\left(\omega_j^2 - \omega^2\right)^2 + \left(\gamma_j\omega\right)^2} \tag{46}$$

The denominator of the fraction in these equations becomes very small when $\omega \simeq \omega_j$ because $\gamma_j \ll \omega_j$, and thus the fraction becomes large. There are *resonances* at the fundamental frequencies of the system. These correspond to atomic and molecular line transition frequencies. To display the behavior more clearly, look at a system with only one resonance, and let

$$\frac{ne^2}{\varepsilon_0 m} = \omega_{\mathrm{p}}^2; \qquad \omega/\omega_j = x; \qquad \gamma_j/\omega_j = \eta$$

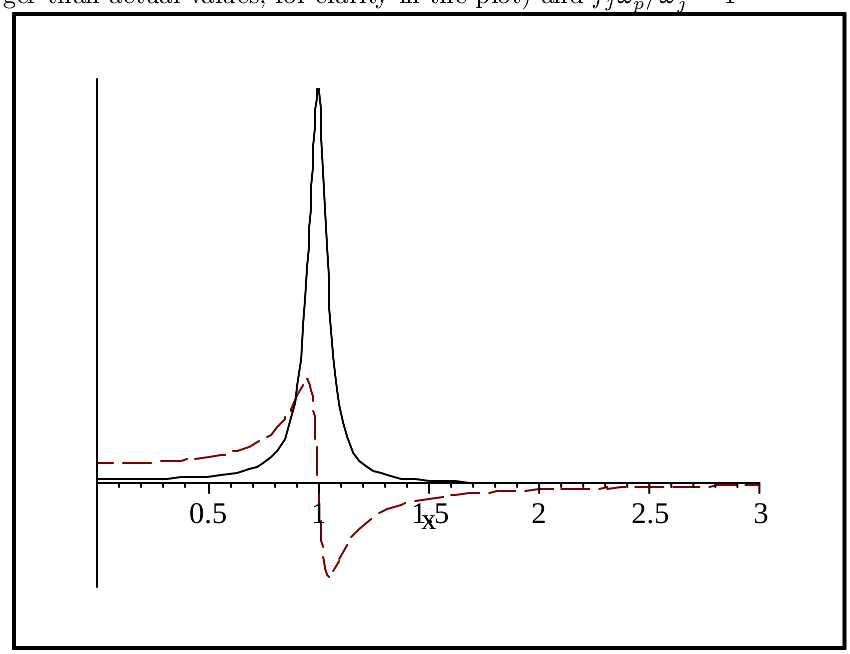
Then

$$\alpha = \frac{\omega^2}{\omega_j^2 c}\frac{\omega_p^2}{\omega_j^2}\frac{f_j\gamma_j}{\left(1-\omega^2/\omega_j^2\right)^2 + \left(\gamma_j\omega\right)^2/\omega_j^4} = \frac{\omega_p^2}{c\omega_j}\frac{\eta f_j}{\left(1-x^2\right)^2 + (\eta x)^2}$$

and

$$n = 1 + \frac{\omega_p^2}{2\omega_j^2}\frac{f_j\left(1-x^2\right)}{\left(1-x^2\right)^2 + (\eta x)^2}$$

The plot shows $c\alpha/\omega_j$ (solid line) and $n-1$ (dashed line) with $\eta = 0.1$ (much larger than actual values, for clarity in the plot) and $f_j\omega_p^2/\omega_j^2 = 1$



The absorption coefficient peaks strongly around $x = 1$, and is very small away from that region. The width of the peak is determined by η, which is usually much less than the value of $1/10$ used in these plots. The index of refraction n is greater than 1 for $x < 1$ but becomes <1 for $x > 1$. Thus the phase speed of the waves is actually greater than c for frequencies slightly greater than resonance. The group speed, however, remains less than c.The refractive index n drops steeply at the resonance– this effect is called anomalous dispersion. For real materials, we have to sum the contributions from all the resonances, but graphs like this describe the behavior near each one.

For frequencies well away from resonance, the effect of damping is negligible and we have the simpler relation

$$n = 1 + \frac{\omega_p^2}{2\omega_j^2}\frac{f_j}{(1 - x^2)}$$

Further, if we are well below resonance, $x = \omega/\omega_j \ll 1$ (in transparent materials like glass ω_j is usually in the UV), we have

$$n = 1 + \frac{\omega_p^2}{2\omega_j^2} f_j \left(1 + x^2\right)$$

and we see that n increases with frequency. (See LB §16.5.5).

7 Wave guides

We'll close this discussion of waves by looking at happens when a wave is confined within a set of boundaries. The wave fields have to satisfy boundary conditions on the walls as well as the wave equation. Consider a wave guide made of of a long, straight, perfectly conducting tube with a constant cross-section, whose shape, for the moment, is arbitrary. The electric field has to be zero inside the perfectly conducting walls, and since wave fields are time-varying, that means that $\vec{B}$ must also be zero. Otherwise, the non-zero $\partial\vec{B}/\partial t$ would create an $\vec{E}$ through Faraday's law, and we know there can be no $\vec{E}$. Then making use of the boundary conditions, we must have

$$\vec{B} \cdot \hat{n} = 0$$

on the guide walls, and

$$\hat{n} \times \vec{E} = 0$$

as well. We do not use the boundary condition on $\vec{D} \cdot \hat{n}$ because there could be a non-zero surface charge density on the wall.

We are interested in waves propagating along the guide, in the $z-$direction. The waves can start off propagating at an angle to the $z-$axis, in which case they will reflect at the wall, and propagate down the guide by bouncing back and forth. The superposition of these waves forms the total disturbance in the guide. As the waves superpose, they will interfere, and only the waves that interfere constructively contribute. It is easier to solve a boundary-value problem than to try to sum the appropriate set of reflecting waves. (But see LB Digging Deeper, page 1060 for a simple example of how it goes.) We should note that each of the reflecting waves has the form we have come to expect ($\vec{E}$ perpendicular to $\vec{B}$ perpendicular to $\vec{k}$), but when we superpose them to get a disturbance travelling parallel to the guide (z) axis, the resulting fields can have a $z-$component. (See LB Figure 33.28, for example.) So we have to allow for this possibility. We start off with the following assumed form for $\vec{E}$ and $\vec{B}$ (as usual, real part is implied).

$$\begin{aligned}\vec{E} &= \vec{E}_0 \exp(ikz - i\omega t) = \left[\vec{E}_\perp(x,y) + E_z(x,y)\hat{z}\right]\exp(ikz - i\omega t) \\ \vec{B} &= \vec{B}_0 \exp(ikz - i\omega t) = \left[\vec{B}_\perp(x,y) + B_z(x,y)\hat{z}\right]\exp(ikz - i\omega t)\end{aligned}$$

The amplitudes must depend on x, y so as to satisfy the boundary conditions. Now we stuff these into Maxwell's equations. For simplicity let the guide interior be vacuum.

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= \left(\vec{\nabla}_\perp\cdot\vec{E}_\perp + ikE_z\right)\exp(ikz - i\omega t) = 0 \\ \vec{\nabla}\cdot\vec{B} &= \left(\vec{\nabla}_\perp\cdot\vec{B}_\perp + ikB_z\right)\exp(ikz - i\omega t) = 0\end{aligned}$$

From these first two equations we can already see that the $z-$components of the fields act as sources for the perpendicular components:

$$\vec{\nabla}_\perp\cdot\vec{E}_\perp = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = -ikE_z \tag{47}$$

$$\vec{\nabla}_\perp\cdot\vec{B}_\perp = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = -ikB_z \tag{48}$$

From the last two equations, we have

$$\left(\vec{\nabla}\times\vec{E}\right)_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega B_x = \frac{\partial E_z}{\partial y} - ikE_y \tag{49}$$

$$\left(\vec{\nabla}\times\vec{E}\right)_y = ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y \tag{50}$$

$$\left(\vec{\nabla}\times\vec{E}\right)_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \tag{51}$$

and similarly

$$\left(\vec{\nabla}\times\vec{B}\right)_x = \frac{\partial B_z}{\partial y} - ikB_y = -i\frac{\omega}{c^2}E_x \tag{52}$$

$$\left(\vec{\nabla}\times\vec{B}\right)_y = ikB_x - \frac{\partial B_z}{\partial x} = -i\frac{\omega}{c^2}E_y \tag{53}$$

$$\left(\vec{\nabla}\times\vec{B}\right)_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\frac{\omega}{c^2}E_z \tag{54}$$

We can solve these equations to get the $x-$ and $y-$components in terms of the $z-$components. From (49)

$$E_y = \frac{1}{ik}\left(\frac{\partial E_z}{\partial y} - i\omega B_x\right)$$

and then from (53)

$$B_x = \frac{1}{ik}\left(\frac{\partial B_z}{\partial x} - \frac{i\omega}{c^2}E_y\right)$$

Combining these, we have

$$\begin{aligned} E_y &= \frac{1}{ik}\left[\frac{\partial E_z}{\partial y} - i\omega\frac{1}{ik}\left(\frac{\partial B_z}{\partial x} - \frac{i\omega}{c^2}E_y\right)\right] \\ &= \frac{1}{ik}\frac{\partial E_z}{\partial y} + i\omega\frac{1}{k^2}\frac{\partial B_z}{\partial x} + \frac{\omega^2}{k^2c^2}E_y \end{aligned}$$

So

$$\begin{aligned} E_y\left(1 - \frac{\omega^2}{c^2k^2}\right) &= i\left(\frac{\omega}{k^2}\frac{\partial B_z}{\partial x} - \frac{1}{k}\frac{\partial E_z}{\partial y}\right) \\ E_y &= \frac{i}{\omega^2/c^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right) \qquad (55) \end{aligned}$$

Similarly, we get for the other components:

$$E_x = \frac{i}{\omega^2/c^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right) \qquad (56)$$

$$B_x = \frac{i}{\omega^2/c^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right) \qquad (57)$$

$$B_y = \frac{i}{\omega^2/c^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right) \qquad (58)$$

Now if we can just find the $z-$components, we are done! So go back to equations (47) and (48) and insert the results for the $x-$and $y-$components. We simplify the notation by writing

$$\gamma^2 \equiv \frac{\omega^2}{c^2} - k^2$$

$$\begin{aligned} \frac{\partial}{\partial x}\left[\frac{i}{\gamma^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\frac{i}{\gamma^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)\right] &= -ikE_z \\ \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} &= -\gamma^2 E_z \qquad (59) \end{aligned}$$

and we get the same equation for B_z:

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} = -\gamma^2 B_z \qquad (60)$$

These are decoupled equations: the solution for E_z is independent of B_z and vice versa. That means we can solve for E_z assuming $B_z = 0$ – these are the TM or transverse magnetic modes, and then solve for B_z assuming $E_z = 0$ – these are the transverse electric modes, and then get a general solution as a linear combination

$$\vec{E} = \alpha \vec{E}_{\text{TE}} + \beta \vec{E}_{\text{TM}}$$

If both E_z and B_z are zero we have TEM, or transverse electro-magnetic modes. For these modes, equation (51) becomes

$$\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{z} = 0 = \vec{\nabla} \times \vec{E} \Rightarrow \vec{E} = -\vec{\nabla} \Psi$$

and equation (47) becomes

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \vec{\nabla} \cdot \vec{E} = 0 = -\nabla^2 \Psi$$

We know that the maxima and minima of the function Ψ, a solution of Laplace's equation, must occur on the boundary of the region. But $\vec{E} \times \hat{n}$ is zero on the boundary, which means that Ψ is constant on the boundary, and thus it is constant throughout the guide. But if Ψ is constant then $\vec{E} = 0$. The only exception to this is if the guide has two separate pieces to its boundary– as in a coaxial cable.

Now it is time to tackle the boundary conditions. Tangential $\vec{E}$ must be zero, and one of the tangential components is E_z, so for the TM mode we must have

$$E_z = 0 \text{ on the walls.} \tag{61}$$

The boundary condition for B_z is a bit more complicated, so we'll wait for a specific example.

7.1 Waves in a rectangular guide

Let the guide have a rectangular cross-section measuring $a \times b$.

7.1.1 TM modes

The E_z component satisfies the differential equation (59) together with the boundary condition (61). We may solve using separation of variables:

$$E_z = X(x) Y(y)$$

Then the differential equation is

$$X''Y + XY'' = -\gamma^2 XY$$

or

$$\frac{X''}{X} + \frac{Y''}{Y} = -\gamma^2$$

The two terms on the left of this equation must each be constant if we are to satisfy the equation for all values of x and y. Further, we want solutions for X that are zero at two values of x, $x = 0$ and $x = a$. Thus we need a negative separation constant so that the solution is a sine:

$$X'' = -\alpha^2 X \Rightarrow X = \sin\alpha x \quad \text{and} \quad \alpha = \frac{n\pi}{a}$$

similarly for Y :

$$Y'' = -\beta^2 Y \Rightarrow Y = \sin\beta y \quad \text{and } \beta = \frac{m\pi}{b}$$

Then

$$\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 = \gamma_{mn}^2 = \frac{\omega^2}{c^2} - k^2 \tag{62}$$

At a given frequency ω, the corresponding wave number is

$$k = \sqrt{\frac{\omega^2}{c^2} - \gamma_{mn}^2}$$

Clearly ω must be greater than γ_{mn} for the wave to propagate ($k > 0$). Thus for each mode (value of m and n) there is a minimum frequency at which that mode will propagate. This is the cut-off frequency for that mode.

$$\omega_{\text{cutoff},mn} = c\gamma_{mn} = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

There is also a cutoff frequency for the guide, corresponding to the lowest cutoff frquency for any mode.

$$\omega_{\text{cutoff, guide}} = \omega_{\text{cutoff,lowest mode}} = c\gamma_{\min}$$

Considering only TM modes, we have a cutoff frequency corresponding to $m = n = 1$

$$\omega_{\text{cutoff,TM}} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

The wave phase speed is

$$v_\phi = \frac{\omega}{k} = \frac{\omega}{\sqrt{\omega^2/c^2 - \gamma_{mn}^2}} = c\frac{1}{\sqrt{1 - c^2\gamma_{mn}^2/\omega^2}}$$

and is always greater than c. However the group speed is

$$v_g = \frac{d\omega}{dk}$$

To get this most easily, we start with the expresson for k^2 : .

$$2k\frac{dk}{d\omega} = 2\frac{\omega}{c^2}$$

So

$$\frac{d\omega}{dk} = c^2\frac{k}{\omega} = c\frac{c}{v_\phi}$$

Thus if $v_\phi > c$, $v_g < c$. Information travels down the guide more slowly than in vacuum. Sometimes the guides are called delay lines, indicating that the guide slows the propagation of information down the guide.

7.1.2 TE modes

For the TE modes, we use the boundary condition on normal $\vec{B}$. Thus at $x = 0$ and $x = a$ we must have $B_x = 0$, while at $y = 0$ and $y = b$ we must have $B_y = 0$. We need to express these results in terms of B_z, so we use relations (57) and (58), remembering that $E_z \equiv 0$ in this mode, so

$$\begin{aligned} B_x &= \frac{i}{\omega^2/c^2 - k^2}\left(k\frac{\partial B_z}{\partial x}\right) \\ B_y &= \frac{i}{\omega^2/c^2 - k^2}\left(k\frac{\partial B_z}{\partial y}\right) \end{aligned}$$

and so are boundary conditons are

$$\begin{aligned} \frac{\partial B_z}{\partial x} &= 0 \text{ at } x = 0, a \\ \frac{\partial B_z}{\partial y} &= 0 \text{ at } y = 0, b \end{aligned}$$

These are Neumann conditions. The solutions are then:

$$B_z = \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)$$

with

$$\gamma_{mn}^2 = \frac{n^2}{a^2} + \frac{m^2}{b^2}$$

except that now either n or m may be zero (but not both!) giving a lowset frequency for the TE mode of

$$\omega_{\text{cutoff,TE}} = \frac{c\pi}{a}$$

for the case $a > b$. Since this is a lower frquency than for the TM modes, this is the cutoff frequency for the guide.