Physics 460 Fall 2006 Susan M. Lea

1 The beginnings of relativity

The principle of relativity was first expressed by Galileo in the 17th century:

> If two reference franes move at constant relative velocity with respect to each other, the results of any physics experiment conducted in either frame will be the same.

Equivalently, the laws of physics are independent of reference frame. Galileo used this principle to understand projectile motion (free fall under gravity combined with constant horizontal velocity). But at the beginning of the 20th century, Einstein noticed that there were difficulties in applying this principle to E&M.

1. Einstein imagined riding along with an EM wave by moving to a frame travelling at speed c with respect to the original frame. He noticed that the oscillating fields he would apparently see in this frame were inconsistent with Maxwell's equations.

2. Maxwell's equations predict a fixed speed for light: $c = 1/\sqrt{\mu_0\varepsilon_0} = 3\times 10^8$ m/s. But Galilean relativity predicts that an observer in a frame moving at speed v with respect to the "lab" frame, and parallel to the direction of propagation, should measure wave speed $c - v$. This is what happens with sound waves, for example. (See LB §16.4.)

3. There is a simple experiment that we use to demonstrate Faraday's Law. Take a coil and push a bar magnet toward it (LB §30.1). As the magnetic flux through the coil increases, there is an induced electric field that drives current in the coil. But what happens if we hold the magnet fixed and push the coil toward the magnet? According to the principle of relativity, we get the same current. But in this case the force driving the current is part magnetic, since the electrons in the coil now have a non-zero velocity through the magnetic field, and part electric due to induced electric field. This appears to violate the principle of relativity.

It would appear that we need to throw out either the principle of relativity or Maxwell's equations. But Einstein realized there was another alternative: we can rethink our ideas of space and time. He started with the following *postulates of Special Relativity:*

1. The speed of light is constant and equal to $1/\sqrt{\mu_0\varepsilon_0}$ independent of reference frame.

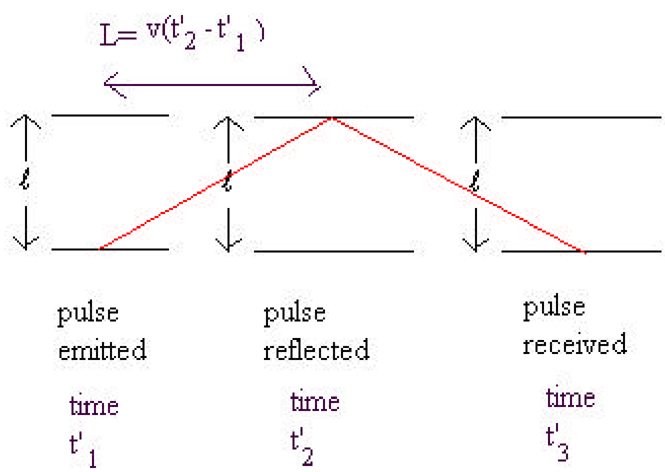
2. The laws of physics are independent of (inertial) reference frame.

The whole theory follows from these two postulates. We shall discover that some physical quantities are *invariant*: that is, they have the same value in all reference frames. These are obviously very important quantities, and we will want to foucus attention on them. Invariant quantities include the speed of light (postulate 1), as well as the mass and charge of a particle. *Relative* quantities have different values in different reference frames. They include electric and magnetic fields, as well as most distances and times.

1.1 Relativity of time intervals

A proper understanding of relativity requires that we understand the process of measurement. Physical processes are defined by a series of *events.* An event is defined by its spatial coordinates and the time that it occurs– its time coordinate. Now consider an experiment to measure time intervals. We design a clock by generating a pulse of light that travels from its source to a mirror and back to a receiver co-located with the source. The distance from source to mirror is ℓ, and so the time interval between the two events (a) pulse is generated and (b) pulse is received is $\Delta t = 2\ell/c$.

Now we place our clock on a train travelling at speed v with respect to the original, or "lab", frame. The picture of the events as seen from the lab is as follows:



The pulse is emitted at time t_1', and reflected at time t_2', both measured in the lab frame. But the mirror has moved a distance $L = v\,(t_2' - t_1')$ during this time, and so the light pulse has to travel a distance $s = \sqrt{\ell^2 + L^2}$ to reach the mirror, and thus takes a time

$$t_2' - t_1' = \frac{s}{c}$$

The return trip is similar, giving an observed clock interval of

$$\Delta t' = t'_3 - t'_1 = 2\,(t'_2 - t'_1) = \frac{2s}{c}$$

Squaring and simplifying, we get

$$\begin{aligned}\left(\frac{\Delta t'}{2}\right)^2 &= \frac{1}{c^2}\left[\ell^2 + v^2\left(\frac{\Delta t'}{2}\right)^2\right] \\ \frac{\Delta t'}{2} &= \frac{\ell}{c\sqrt{1 - v^2/c^2}}\end{aligned}$$

or

$$\Delta t' = \gamma \Delta t \tag{1}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}; \qquad \beta = \frac{v}{c} \tag{2}$$

Griffiths describes equation (1) with the phrase "moving clocks run slow", but this is incredibly misleading, as the phenomenon has nothing to do with the clock per se, and I strongly suggest that you forget it immediately. The important issue is:

> The smallest time interval between two events is measured by an observer who sees both events happen at the same place in her/his reference frame. All other obervers in other frames measure a longer time interval between the two events.

This smallest time interval is called the *proper time interval* between the two events. As we shall see, the proper time interval is an *invariant,* while the coordinate time interval is a relative quantity.

1.2 Length contraction

The measurement of lengths is a little more complicated, but is closely related to the measurement of time intervals. We can see this by setting up an experiment to measure the length of a rocket. Engineers on board the rocket measure its length in the usual way using stationary meter sticks, and get a result ℓ. (An engineer at each end of the rocket reads the meter stick at the same time.) The rocket is moving at speed v past a space station. How can engineers on board the station measure its length? Simple: they note the time t'_1 that the front end passes them, and the time t'_2 that the back end passes, and conclude that

$$\ell' = v\,(t'_2 - t'_1)$$

The engineers on the rocket see the space station fly past them, and also note the times: t_1 when the station passes the nose cone and t_2 when the station passes the tail fins. They conclude that

$$t_2 - t_1 = \frac{\ell}{v}$$

Now which observers see both events at the same place? The space station engineers! So their time interval is the shorter one.

$$t_2 - t_1 = \gamma\,(t_2' - t_1')$$

and thus

$$\frac{\ell}{v} = \gamma\frac{\ell'}{v}$$

and thus

$$\ell' = \frac{\ell}{\gamma} \tag{3}$$

The rocket appears shorter to the observers on the space station, who see the rocket moving by them. This effect is called length contraction. It effects lengths measured parallel to the relative velocity $\vec{v}$. Lengths perpendicular to $\vec{v}$ are not affected. To see why, consider a train moving along rails separated by a distance w. If w were a relative quantity, then we'd have a dilemma. Suppose the observers on the train see the separation of the moving tracks contracted to a distance $w' < w$. Then the train's wheels would fall off the track to the outside as the train accelerates. But if the observers on the ground see the moving train's axle contracted to a distance $w' < w$, the train's wheels would fall off the track to the inside. Both can't be true, so we conclude that $w' = w$.

Again, I strongly suggest that you avoid Griffith's misleading statement that "moving objects are shortened". Nothing happens to the objects: the proper length of an object is an *invariant*, and length contraction is a purely observational effect. i

> The greatest length of an object (its *proper length*) is measured by observers who observe the object to be at rest.

1.3 Relativity of sumultaneity

Along with these effects we find another related, and perhaps stranger, effect. Whether two events are simultaneous depends on the observer. To see why this happens, consider Einstein's example. A train of proper length ℓ is moving at speed v with respect to the ground ("lab frame"). An observer, Oliver, on the ground sees two lightning bolts hit the ends of the train simultaneously, just as the center of the train passes him. How does he measure this? Well, the light from the bolts reaches him at the same time t_1, so he concludes that the bolts hit each end at time

$$t_{\rm hit} = t_1 - \frac{\ell/2}{\gamma c}$$

Remember: he oberves the train's length to be contracted, hence the γ in the denominator.

Now let's consider the events as observed by Penny, sitting on the middle of the train. Light from the front bolt reaches Penny at time Δt_1 after it struck the train. During the travel time $c\Delta t_1$, Penny advances a distance $v\Delta t_1$ toward the bolt strike, and so the distance travelled is not $\ell/2$ but $\ell/2 - v\Delta t_1$, and so

$$c\Delta t_1 = \frac{\ell}{2} - v\Delta t_1$$

and thus

$$\Delta t_1 = \frac{\ell}{2c\,(1+\beta)}$$

Similarly, for the rear bolt we get

$$\Delta t_2 = \frac{\ell}{2c\,(1-\beta)}$$

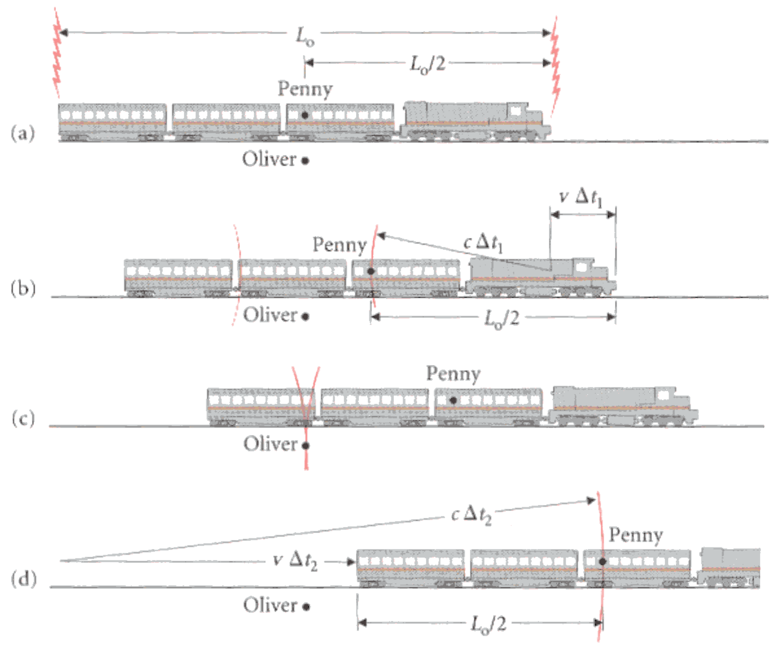
Thus Penny observes the front hit before the rear hit. But if Penny observes the front hit at t_1', and knows that the front end is a distance $\ell/2$ from her, she concludes that the bolt struck at

$$t'_{\text{hit, front}} = t_1' - \frac{\ell}{c}$$

and if she observes the back hit at $t_2' > t_1'$, she concludes that

$$t'_{\text{hit, back}} = t_2' - \frac{\ell}{c}$$

Since these times are not equal, she concludes that the two bolts did not hit simultaneously! The sequence of events is shown in the drawing below.



2 The mathematics of Spacetime

To describe an event in spacetime, we need four coordinates: x, y, z and t. We can call t the fourth coordinate, or, more frequently, we use

$$x_0 = ct, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z$$

where ct is the zeroth coordinate. The values of these coordinates are different for different observers. But some combinations are invariant. For example, consider a light wave that starts at the origin (of spacetime! $ct = x = y = z = 0$). After a time dt it has travelled a distance cdt :

$$d\ell = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = cdt$$

and thus the combination

$$ds^2 = c^2 dt^2 - d\ell^2 = 0 \tag{4}$$

and this must hold in any reference frame because the speed of light is the same in every reference frame (postulate 1). The quantity ds is called the differential *space-time interval* between two events on the light ray. It is an invariant. When $ds = 0$, as here, the interval is lightlike. Timelike intervals have $ds^2 > 0$ and spacelike intervals have $ds^2 < 0$. These distinctions are also invariant. An interval cannot be spacelike in one frame and timelike in another.

We can draw a diagram, called a space-time diagram, in which we represent the events as points. It is necessary to suppress at least one space dimension to draw the diagram. Light rays travel at 45° in such a diagram. These diagrams are useful for getting a visual image of the situation of interest. All events that can be reached from an event E_1 constitute the future of that event, and they lie on or within an upward-opening cone with opening angle 45° and apex at point E_1. Similarly the past is the set of events from which E_1 can be reached. These events lie in a downward-opening cone with apex at E_1.

2.1 The Lorentz transformation

Now we'll look at how the coordinates change as we move from frame to frame. Since we expect an object moving at constant velocity in one frame (which we'll call unprime) to be moving at some constant velocity in any other frame (prime), the transformation must be linear. If this were not true, we would have accelerations (and thus forces) in one frame and not in another. This would violate postulate 2. For simplicity, let's put the $x-$axis along the relative velocity $\vec{v}$. Thus, since we have already established that dimensions perpendicular to $\vec{v}$ are not contracted, we expect

$$y' = y; \quad z' = z$$

But

$$\begin{aligned} ct' &= Act + Bx \\ x' &= Dct + Ex \end{aligned}$$

Now we use the invariance of the space-time interval :

$$\begin{aligned} ds^2 &= (cdt')^2 - (dx')^2 - (dy')^2 - (dz')^2 \\ &= (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \end{aligned}$$

Next insert our assumed linear transformation:

$$(Acdt + Bdx)^2 - (Dcdt + Edx)^2 - (dy)^2 - (dz)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

This statement must be true for all values of dt, dx, dy, dz. So first let $dx = dy = dz = 0$. then

$$A^2 - D^2 = 1 \tag{5}$$

Next let $dt = dy = dz = 0$.

$$B^2 - E^2 = -1 \tag{6}$$

Finally with $dy = dz = 0$, we get

$$\left(A^2 - D^2 - 1\right)(cdt)^2 + \left(B^2 - E^2 + 1\right)dx^2 + (AB - DE)\,2cdtdx = 0$$

and using the results (5) and (6), this simplifies to

$$AB = DE \tag{7}$$

Thus $B = DE/A$ and inserting this into (6) we get

$$E^2\left(\frac{D^2}{A^2} - 1\right) = -1$$

and then using equation (5) we get

$$\frac{E^2}{A^2} = 1$$

so

$$E = \pm A$$

But if A were negative, time would be running backward in the prime frame, and that can't be. Similarly, if E were negative, distances would be increasing in opposite senses in the two frames– this would be inconvenient to say the least. So we require both E and A to be positive, and thus $E = A$. Then we also have $B = D$, and our transformation has simplified:

$$\begin{aligned} ct' &= Act + Bx \\ x' &= Bct + Ax \end{aligned}$$

Now consider an object at rest at the spatial origin of the prime frame: $x' = y' = z' = 0$ for all t'. As measured in the unprime frame, this object has coordinates

$$x = vt, \quad y = z = 0$$

And thus

$$x' = Act + Bvt = 0 \Rightarrow A = -B\beta$$

and then from (5) we get

$$\begin{aligned} A^2\left(1-\beta^2\right) &= 1 \\ A &= \frac{1}{\sqrt{1-\beta^2}} = \gamma \end{aligned}$$

So finally our transformation is:

Lorentz transformation:

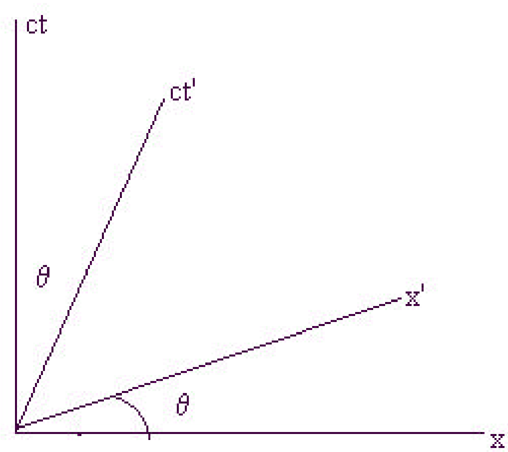
$$\begin{aligned} ct' &= \gamma\left(ct - \beta x\right) \\ x' &= \gamma\left(x - \beta ct\right) \end{aligned} \tag{8}$$

As $\beta \to 0$, $\gamma \to 1$ and we get back the Galilean transformation

$$x' = x - vt$$

In a space-time diagram for the unprime frame, the prime axes are sloping lines making angle θ with the $x-$ and $ct-$axes, where

$$\tan\theta = \beta$$



The new space and time axes coincide along a 45° line as $\beta \to 1$, thus we find that we cannot have a frame travelling at a speed greater than c with respect to the first. Use this diagram to show that events simultaneous in the unprime frame are not simultaneous in the prime frame, and vice versa..

Now let's revisit time dilation Two events have coordinates $(ct_1, x_1, 0, 0)$ and $(ct_2, x_2, 0, 0)$ in the unprime frame. In the prime frame, the coordinates are

$$ct_1' = \gamma\left(ct_1 - \beta x_1\right)$$

and

$$ct'_2 = \gamma\,(ct_2 - \beta x_2)$$

If $x_2 = x_1$, the time interval between events in the prime frame is

$$\Delta t' = t'_2 - t'_1 = \gamma\,(t_2 - t_1) = \gamma \Delta t$$

The events occur at the same place in the unprime frame, and so the time interval in the prime frame is longer. But suppose $x_2 \neq x_1$. Then

$$\begin{aligned} x'_1 &= \gamma\,(x_1 - \beta c t_1) \\ x'_2 &= \gamma\,(x_2 - \beta c t_2) \end{aligned}$$

and now let's require that $x'_2 = x'_1$: the events occur at the same place in the prime frame.

$$x_1 - \beta c t_1 = x_2 - \beta c t_2 \Rightarrow x_2 - x_1 = \beta c\,(t_2 - t_1)$$

Then

$$\begin{aligned} \Delta t' &= t'_2 - t'_1 = \gamma \left(t_2 - t_1 - \frac{\beta}{c}\,(x_2 - x_1) \right) \\ &= \gamma \left(t_2 - t_1 - \frac{\beta}{c} \beta c\,(t_2 - t_1) \right) = \gamma\,(t_2 - t_1) \left(1 - \beta^2\right) = \frac{t_2 - t_1}{\gamma} \end{aligned}$$

Now the interval is shorter in the prime frame, as expected.

2.2 The velocity transformation

The velocity of a particle measured in the prime frame is

$$\vec{u}' = \left(\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'} \right)$$

Let's look at the $x-$component first. Using the Lorentz transformation, we have

$$\begin{aligned} u'_x &= \frac{\gamma\,(dx - \beta c t)}{\gamma\,(dt - \beta dx/c)} = \frac{dx/dt - \beta c}{1 - (\beta/c)\,dx/dt} \\ &= \frac{u_x - v}{1 - \beta u_x/c} = \frac{u_x - v}{1 - \vec{\beta} \cdot \vec{u}/c} \end{aligned} \tag{9}$$

while for the $y-$component, we get

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\,(dt - \beta\,dx/c)} = \frac{u_y}{\gamma \left(1 - \vec{\beta} \cdot \vec{u}/c\right)} \tag{10}$$

It is easy to check that we get back the right Galilean results as $\beta \to 0$. But look at what happens if $u_x \to c$ (and then of course $u'_y = u'_z = 0$)

$$u'_x = \frac{c - v}{1 - \beta} = c$$

This result is required by postulate 1.

It's a little more interesting if $u_y \to c$, $u_x = 0$ Then we get

$$u'_y = \frac{c}{\gamma}; \quad u'_x = -v$$

and

$$|\vec{u}'| = \sqrt{\left(u'_y\right)^2 + \left(u'_x\right)^2} = \sqrt{\frac{c^2}{\gamma^2} + v^2} = \sqrt{c^2 - v^2 + v^2} = c$$

2.3 Metrics and 4-vectors

We can write the coordinates of an event in spacetime as the components of a 4-dimensional vector:

$$\overleftrightarrow{r} = (ct, x, y, z)$$

Then a differential displacement

$$d\overleftrightarrow{r} = (cdt, dx, dy, dz)$$

with magnitude equal to the space-time interval (4):

$$d\overleftrightarrow{r} \cdot d\overleftrightarrow{r} = c^2 dt^2 - \left(dx^2 + dy^2 + dz^2\right)$$

This is almost the usual rule for finding dot products, except for a sign change. The *metric* for spacetime is

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and we have

$$d\overleftrightarrow{r} \cdot d\overleftrightarrow{r} = d\overleftrightarrow{r}\, g\, d\overleftrightarrow{r} = dr^\mu g_{\mu\nu} dr^\nu \tag{11}$$

where we can use matrix multiplication to evaluate the product on the right. We have discovered the rule for taking dot products of four vectors. A useful way to remember this is:

> Dot product = product of time components - sum of products of space components

Just as the dot product of a 3-vector is a scalar, so here the dot product of two 4-vectors is an invariant.

Notice that in equation (11) we wrote the index on the 4-vector above the symbol, and the index on the metric tensor below the symbol. This is important. In the tensor mathematics of spacetime, it is only permissable to sum over a repeated index, if one of the two indices is up and one is down.

The Lorentz transformation can also be done using matrix methods:

$$\overleftrightarrow{r}' = \Lambda \overleftrightarrow{r}$$

where

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

In index notation, we write

$$r^{\mu} = \Lambda^{\mu}{}_{\nu} r^{\nu}$$

Once we have established the correct mathematics for these 4-vectors, we can write all of the physics of SR in a very compact form. The space-time interval for a particle moving at velocity $\vec{v} = v\hat{x}$ is

$$ds^2 = c^2 dt^2 - v^2 dt^2 = \frac{c^2 dt^2}{\gamma^2}$$

and if $v = 0$, $ds = cdt$. So the proper time is just

$$d\tau = \frac{ds}{c} = \frac{dt}{\gamma}$$

where ds is timelike. This motivates the definition of the 4-velocity. If we differentiate the 4-position with respect to the *invariant* proper time, we get

$$\begin{aligned} \overleftrightarrow{v} &= \frac{d\overleftrightarrow{r}}{d\tau} = \frac{\gamma d}{dt}(ct, \vec{r}) \\ &= \gamma (c, \vec{v}) \end{aligned} \tag{13}$$

where $\vec{v}$ is the usual 3-velocity measured in a specific reference frame. Now if we apply the rule we have discovered for taking dot products of 4-vectors, we get

$$\overleftrightarrow{v} \cdot \overleftrightarrow{v} = \gamma^2 \left(c^2 - v^2\right) = c^2$$

which is certainly an invariant!

The Lorentz transformation may be used to transform any 4-vector. Let's do the velocity transformation. Let $\vec{u}$ be the velocity of a particle in the unprime frame. Then

$$\overleftrightarrow{u}' = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c} \quad \text{and} \quad \gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}}$$

$\vec{v} = v\hat{x}$ being the relative velocity of the two frames. Thus

$$\overleftrightarrow{u}' = \begin{pmatrix} \gamma\gamma_u\,(c - \beta u_x) \\ \gamma\gamma_u\,(u_x - \beta c) \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma'_u c \\ \gamma'_u u'_x \\ \gamma'_u u'_y \\ \gamma'_u u'_z \end{pmatrix}$$

So that, from the time component,

$$\gamma'_u = \gamma\gamma_u\,(1 - \beta u_x/c) \tag{14}$$

and from the $x-$component

$$u'_x = \frac{\gamma\gamma_u\,(u_x - \beta c)}{\gamma'_u} = \frac{u_x - v}{(1 - \beta u_x/c)}$$

in agreement with (9). I'll leave it to you to check the perpendicular components.

Once we see how to get the 4-vectors, the rest of physics gets easy. Remembering that the mass of a particle is an invariant, we may form the 4-momentum

$$\overleftrightarrow{p} = m\overleftrightarrow{v} = \gamma m\,(c, \vec{v}) = (\gamma mc,\ \vec{p})$$

where the correct relativistic expression for the 3-momentum is $\vec{p} = \gamma m\vec{v}$. (See LB p1089 for a proof of this.) The invariant magnitude of the 4-momentum, squared, is

$$\overleftrightarrow{p} \cdot \overleftrightarrow{p} = \gamma^2 m^2\left(c^2 - v^2\right) = m^2c^2 = \frac{\mathcal{E}_0^2}{c^2} \tag{15}$$

the square of the rest energy $\mathcal{E}_0 = mc^2$ of the particle, divided by c^2. This is also an invariant, since both m and c are. The 4-momentum is

$$\overleftrightarrow{p} = \left(\frac{\mathcal{E}}{c},\ \vec{p}\right)$$

where the total energy of the particle is

$$\begin{aligned}
\mathcal{E} &= \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \\
&= mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2!}\frac{v^4}{c^4} + \cdots\right) \\
&= mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}\frac{v^4}{c^4} + \cdots \\
&= \text{rest energy} + \text{Newtonian kinetic energy} + \text{relativistic corrections}
\end{aligned}$$

We can verify this result as follows. First define the 4-acceleration

$$\overleftrightarrow{a} = \frac{d\overleftrightarrow{v}}{d\tau}$$

and the 4-force

$$\overset{\leftrightarrow}{F} = m\overset{\leftrightarrow}{a} = \frac{d\overset{\leftrightarrow}{p}}{d\tau}$$

Then the dot product

$$\begin{aligned}
\overset{\leftrightarrow}{F} \cdot \overset{\leftrightarrow}{v} &= m\frac{d\overset{\leftrightarrow}{v}}{d\tau} \cdot \overset{\leftrightarrow}{v} \\
&= \frac{m}{2}\frac{d}{d\tau}\left(\overset{\leftrightarrow}{v} \cdot \overset{\leftrightarrow}{v}\right) = \frac{m}{2}\frac{d}{d\tau}c^2 = 0
\end{aligned}$$

since c is a constant, independent of proper time. But if we do it using components and coordinate time, we get

$$\begin{aligned}
\overset{\leftrightarrow}{F} \cdot \overset{\leftrightarrow}{v} &= m\gamma\frac{d}{dt}\left[\gamma\left(c, \vec{v}\right)\right] \cdot \gamma\left(c, \vec{v}\right) \\
&= m\gamma^2\left(c\frac{d\gamma}{dt}, \frac{d\vec{p}}{dt}\right) \cdot \left(c, \vec{v}\right) \\
&= \gamma^2\left(mc^2\frac{d\gamma}{dt} - \vec{v} \cdot \vec{F}\right)
\end{aligned}$$

Since we have already established that the dot product is zero, we get

$$P = \vec{F} \cdot \vec{v} = \frac{d}{dt}\left(\gamma mc^2\right) = \frac{d\mathcal{E}}{dt} \tag{16}$$

The power expended by the force equals the rate of change of the particle's energy.

The momentum 4-vector allows us to combine two laws of physics– conservation of energy and conservation of momentum– into one conservation law: conservationof 4-momentum. For an isolated system,

$$\overset{\leftrightarrow}{p}_{\text{total,before}} = \overset{\leftrightarrow}{p}_{\text{total, after}}$$

If we combine this with the invariance of $\overset{\leftrightarrow}{p} \cdot \overset{\leftrightarrow}{p}$ we can solve some interesting problems quite easily.

The Bevatron at Berkeley was designed to produce antimatter through the reaction

$$p + p \rightarrow p + p + p + \overline{p}$$

where p is a proton and $\overline{p}$ is an antiproton. To what energy must a proton be accelerated so that this reaction will produce antiprotons in a collision with a stationary proton?

In the collision, both energy and momentum are conserved. In the lab frame a proton with speed v collides with a stationary proton, and the reaction products carry away the initial momentum γmv. For the energy required to be a minimum, the reaction products have no internal energy, that is, each particle has the same velocity $\vec{v}_{\text{final}}$. Viewed in the CM frame (center of *momentum*, in relativistic physics), the two protons approach with equal and opposite velocities, and the reaction products are at rest. We could do the problem using the Lorentz transformation, transforming to the CM frame, but there's an easier way. We can calculate the *invariant* magnitude of the total 4-momentum in the lab frame *before* the collision and in the CM frame *after* the collision.

	energy	momentum	invariant $E^2 - p^2c^2$
incoming proton, lab frame	γmc^2	γmv	
stationary proton, lab frame	mc^2	0	
Total, lab frame	$mc^2(\gamma+1)$	γmv	$\left(mc^2\right)^2(\gamma+1)^2 - \gamma^2 m^2 v^2 c^2$
reaction products, CM frame	$4mc^2$	0	$\left(4mc^2\right)^2$

Thus, setting invariant totals before equal to totals after

$$\left(mc^2\right)^2\left[(\gamma+1)^2 - \gamma^2\beta^2\right] = 16\left(mc^2\right)^2$$

Thus

$$\begin{aligned}
\gamma^2\left(1-\beta^2\right) + 2\gamma + 1 &= 16 \\
2(\gamma+1) &= 16 \\
\gamma &= 7
\end{aligned}$$

With $\gamma = 7$, the kinetic energy of the proton is

$$T = \gamma mc^2 = 7\,(0.938 \text{ Gev}) = 5.6 \text{ GeV}$$

The betatron was designed to produce 6.4 GeV per proton.

2.4 Transformation law for $\vec{F}$

The 4-force is

$$\overleftrightarrow{F} = \frac{d\overleftrightarrow{p}}{d\tau} = \gamma\frac{d}{dt}\left(\frac{\mathcal{E}}{c},\ \vec{p}\right) = \gamma\left(\frac{1}{c}\frac{d\mathcal{E}}{dt},\ \vec{F}\right)$$

where $\vec{F}$ is the 3-force, and, from (16) with $\vec{\beta} = \vec{u}/c$,

$$\frac{1}{c}\frac{d\mathcal{E}}{dt} = \vec{\beta}\cdot\vec{F}$$

So

$$\overleftrightarrow{F} = \gamma\left(\vec{\beta}\cdot\vec{F},\ \vec{F}\right)$$

Applying the Lorentz transformation

$$\overleftrightarrow{F}' = \Lambda\overleftrightarrow{F}$$

So, with $\Gamma = 1/\sqrt{1 - v^2/c^2}$, and $\vec{v} = v\hat{x}$

$$\begin{aligned}
\gamma' F_x' &= \Gamma\gamma\left(F_x - \frac{v}{c}\vec{\beta}\cdot\vec{F}\right) = \Gamma\gamma\left(F_x - \frac{v}{c}\vec{\beta}\cdot\vec{F}\right) \\
\gamma' F_z' &= \gamma F_z
\end{aligned}$$

We have already found γ' (equation 14), so

$$F'_x = \frac{1}{\left(1 - \beta_x \frac{v}{c}\right)} \left(F_x - \frac{v}{c}\vec{\beta} \cdot \vec{F}\right) \tag{17}$$

If $\vec{F}$ is perpendicular to $\vec{\beta}$ in the unprime frame, then

$$F'_x = \frac{F_x}{\left(1 - \beta_x \frac{v}{c}\right)} \tag{18}$$

For the y and z components the transformation is

$$F'_z = \frac{F_z}{\Gamma\left(1 - \beta_x \frac{v}{c}\right)} \tag{19}$$

In the special case that the particle is *instantaneously at rest* in the unprime frame, then

$$F'_x = F_x, \qquad \vec{F}'_\perp = \frac{\vec{F}_\perp}{\Gamma}$$

3 Relativistic E&M

The transformation laws for electric and magnetic fields are interesting. Suppose we have a uniform sheet of charge with uniform charge density σ lying in the $x - z$ plane There is a constant electric field $\vec{E} = \sigma\hat{n}/2\varepsilon_0$ above the sheet. A particle with charge q placed above the sheet experiences a force $\vec{F} = q\vec{E}$ that accelerates it away from the sheet.

If we now move to a reference frame moving with speed v parallel to the sheet, $\vec{v} = v\hat{x}$, we see things differently. In this frame there is a surface current density $\vec{K}' = -\sigma'\vec{v}$, and a magnetic field $\vec{B} = \mu_0\vec{K}' \times \hat{n}/2$. The charge also moves with a velocity $-\vec{v}$ in this frame and thus experiences a force due to both the electric and magnetic fields:

$$\begin{aligned}
\vec{F}' &= q\left(\vec{E}' + \vec{v}' \times \vec{B}'\right) \\
&= q\left(\frac{\sigma'\hat{n}}{2\varepsilon_0} - \vec{v} \times \left(-\frac{\mu_0\sigma'\vec{v} \times \hat{n}}{2}\right)\right) \\
&= q\frac{\sigma'}{2\varepsilon_0}\left\{\hat{n} + \mu_0\varepsilon_0\left[\vec{v}\left(\vec{v} \cdot \hat{n}\right) - \hat{n}v^2\right]\right\} \\
&= q\frac{\sigma'}{2\varepsilon_0}\hat{n}\left(1 - \frac{v^2}{c^2}\right) = \frac{q}{\gamma^2}\frac{\sigma'}{2\varepsilon_0}\hat{n}
\end{aligned}$$

The total charge Q on an area $A = \ell \times w$ in the unprime frame is the charge Q on an area $A' = \ell' \times w = \ell w/\gamma$ in the prime frame. So

$$\sigma' = \frac{Q}{A'} = \gamma\frac{Q}{A} = \gamma\sigma$$

Thus

$$\vec{F}' = \frac{q}{\gamma}\frac{\sigma}{2\varepsilon_0}\hat{n}$$

Since $\vec{F}$ is perpendicular to $\vec{v}$, we may use relation (19) to compute $\vec{F}'$ from $\vec{F}$:

$$\vec{F}' = \frac{q}{\gamma}\frac{\sigma}{2\varepsilon_0}\hat{n}$$

The two expressions agree. What this shows is that an electric field in the unprime frame transforms to a combination of electric and magnetic fields in the prime frame. In this case $\vec{E}$ is perpendicular to $\vec{v}$, and

$$\vec{E}' = \gamma\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{20}$$

$$\vec{B}' = \gamma\left(\vec{B} - \vec{v}\times\vec{E}/c^2\right) \tag{21}$$

It turns out that the components of $\vec{E}$ and $\vec{B}$ parallel to $\vec{v}$ are unchanged.

It is interesting that the components of $\vec{E}$ and $\vec{B}$ get mixed together in the transformation law. We can see from this that neither $\vec{E}$ nor $\vec{B}$ can be parts of a four-vector, since the Lorentz transformation mixes up components of a single vector but does not combine components of different vectors. We need a bigger unit– a field *tensor*. A scalar (invariant) is a single number: $1 = 4^0$. A vector has 4 components, so we need 4^1 numbers. The next object up the scale is a rank 2 tensor with $4^2 = 16$ components. We can write these numbers using a 4×4 matrix. We can figure out what these components are by working from the potentials. The scalar and vector potentials form a 4-vector potential:

$$\overleftrightarrow{A} = \left(\frac{V}{c},\ \vec{A}\right)$$

and the 4-dimensional gradient vector is

$$\overleftrightarrow{\Box} = \left(\frac{\partial}{c\partial t}, -\vec{\nabla}\right)$$

so that the 4-divergence of the 4-vector potential is (using our rule for dot-products)

$$\overleftrightarrow{\Box}\cdot\overleftrightarrow{A} = \frac{\partial V}{c^2\partial t} + \vec{\nabla}\cdot\vec{A} = 0$$

This is the Lorentz gauge condition, and it is an invariant. This is what makes the Lorentz gauge condition so useful: if it is true in one frame it is true in all. Now

$$\begin{aligned} B_x &= \left(\vec{\nabla}\times\vec{A}\right)_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ B_x &= -\Box^2 A^3 + \Box^3 A^2 \end{aligned}$$

while

$$E_x = -\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} = c\left(\Box^1 A^0 - \Box^0 A^1\right)$$

This suggests that we form our field tensor from the antisymmetric components

$$F^{\mu\nu} = \Box^\mu A^\nu - \Box^\nu A^\mu$$

This leads to

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{22}$$

The transformation law for this tensor is

$$F' = \Lambda F \Lambda^T$$

or

$$F'^{\mu\nu} = \Lambda^\mu{}_\alpha F^{\alpha\delta} \Lambda_\delta{}^\nu$$

Verify that this rule reproduces equations (20) and (21).

Let's look at the invariants we can form from this tensor. The dot product of two vectors is

$$A^\mu g_{\mu n} A^\nu = A^\mu A_\mu$$

where g is the metric tensor, and

$$A_\mu = g_{\mu\nu} A^\nu$$

is the so-called *covariant* vector formed from the vector $\overleftrightarrow{A}$.[1] By analogy, we can find the invariants

$$C_1 = F^{\mu\nu} g_{\mu\xi} g_{\nu\lambda} F^{\xi\lambda} = F^{\mu\nu} F_{\mu\nu}$$

or, in matrix notation,

$$F_{\mu\nu} = \left(gFg^T\right)_{\mu\nu}$$

and C_1 is the sum of the products of all the elements of the two matrices.Since the metric tensor is diagonal, $g = g^T$, so

$$\begin{aligned} gFg^T &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix} \\ F_{\mu\nu} &= \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \end{aligned} \tag{23}$$

[1] Note that the space components of the covariant vector have the opposite sign from the space components of the contravariant (or usual) vector. Thus the covariant vector $\Box_\mu = \left(\frac{\partial}{\partial ct}, \vec{\nabla}\right)$.

Comparing with (22), we see that the top row and first column have changed sign. Thus

$$C_1 = 2\left(B^2 - \frac{E^2}{c^2}\right)$$

In an EM wave, $B = E/c$ and so $C_1 = 0$. Some fields are primarily magnetic in character, $C_1 > 0$. On the other hand, if $C_1 < 0$, the fields are of electric character.

A second invariant is found from the dual tensor which is created using the 4-d analogue of the Levi-Civita symbol.

$$\begin{aligned}\mathcal{F}_{\mu\nu} &= \varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \\ C_2 &= \mathcal{F}_{\mu\nu}F^{\mu\nu}\end{aligned}$$

The result is

$$C_2 \propto \vec{E}\cdot\vec{B}$$

For an EM wave, this invariant is also zero.

If $C_1 > 0$ and $C_2 = 0$ there will be some frame in which $\vec{B} = 0$. If $C_1 < 0$ and $C_2 = 0$ there is some frame in which $\vec{E} = 0$. If $C_2 \neq 0$, there is no frame in which either $\vec{E}$ or $\vec{B}$ is zero.

For the charged sheet we considered above, we have, in the lab frame,

$$C_1 = 2\left(0 - \frac{\sigma^2}{4\varepsilon_0^2 c^2}\right) = -\frac{\sigma^2}{2\varepsilon_0^2 c^2} = -\frac{\mu_0\sigma^2}{2\varepsilon_0}$$

$$C_2 = 0$$

and in the prime frame

$$\begin{aligned}C_1 &= 2\left[\frac{\mu_0^2\left(K'\right)^2}{4} - \frac{\left(\sigma'\right)^2}{4\varepsilon_0^2 c^2}\right] \\ &= \frac{\mu_0^2\left(\sigma' v\right)^2}{2} - \mu_0\frac{\left(\sigma'\right)^2}{2\varepsilon_0} \\ &= \left(\frac{\mu_0^2\beta^2 c^2}{2} - \frac{\mu_0}{2\varepsilon_0}\right)\gamma^2\sigma^2 \\ &= -\frac{\mu_0}{2\varepsilon_0}\left(1-\beta^2\right)\gamma^2\sigma^2 = -\frac{\mu_0\sigma^2}{2\varepsilon_0}\end{aligned}$$

and C_2 is also zero since $\vec{E}'$ is perpendicular to $\vec{B}'$.

The charge and current form another 4-vector:

$$\overset{\leftrightarrow}{j} = \left(c\rho,\ \vec{j}\right)$$

In the lab frame, for our sheet,

$$\overset{\leftrightarrow}{j} = \left(c\sigma\delta\left(y\right),\ 0\right)$$

Verify that the Lorentz transformation of this 4-vector correctly gives the charge and current in the prime frame. Now we may write charge conservation compactly as

$$\overset{\leftrightarrow}{\Box} \cdot \overset{\leftrightarrow}{j} = 0 = \frac{\partial c\rho}{\partial ct} + \vec{\nabla} \cdot \vec{j} = 0$$

Finally we get Maxwell's equations:

$$\Box_\mu F^{\mu\nu} = \mu_0 j^\nu$$

and

$$\Box_\mu \mathcal{F}^{\mu\nu} = 0$$

The Lorentz force on a particle with charge q and 4-velocity $\overset{\leftrightarrow}{v}$ is

$$F_L^\mu = qF^{\mu\nu} v_\nu = qF^{\mu\nu} g_{\nu\alpha} v^\alpha$$

Let's verify this

$$\begin{aligned}
qF^{\mu\nu}v_\nu &= q\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix} \\
&= q\begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} \gamma c \\ -\gamma v_x \\ -\gamma v_y \\ -\gamma v_z \end{pmatrix} \\
&= q\begin{pmatrix} \frac{E_x}{c}\gamma v_x + \frac{E_y}{c}\gamma v_y + \frac{E_z}{c}\gamma v_z \\ E_x\gamma + B_z\gamma v_y - B_y\gamma v_z \\ E_y\gamma - B_z\gamma v_x + B_x\gamma v_z \\ E_z\gamma + B_y\gamma v_x - B_x\gamma v_y \end{pmatrix} \\
&= q\gamma\begin{pmatrix} \vec{E}\cdot\vec{\beta} \\ E_x + \left(\vec{v}\times\vec{B}\right)_x \\ E_y + \left(\vec{v}\times\vec{B}\right)_y \\ E_z + \left(\vec{v}\times\vec{B}\right)_z \end{pmatrix}
\end{aligned}$$

Apart from the factor of γ, the 3-vector part of this 4-vector is the non-relativistic expression for the Lorentz force, while the zeroth component is the power, as usual.

4 Motion of a charged particle in a uniform electric field

In Newtonian mechanics, a charged particle in a uniform field experiences a constant force $\vec{F} = q\vec{E}$ and accelerates at a constant rate forever. Its speed increases without limit. But Einstein told us that this can't happen. The particle's speed cannot exceed c. Let's see how the mathematics enforces this result. The secret is to use proper time. Then the force law is

$$F_L^\mu = \frac{dp^\mu}{d\tau}$$

If we put the $x-$axis along the direction of the electric field and the $y-$axis so that the particle's initial velocity is in the $x-y-$plane, then the equations are:

0-component

$$\begin{aligned} q\gamma\vec{E}\cdot\vec{\beta} &= \frac{d}{d\tau}\left(\frac{\mathcal{E}}{c}\right) \\ \gamma q E\frac{v_x}{c} &= \frac{d}{d\tau}(\gamma m c) \end{aligned}$$

1-component

$$q\gamma E = \frac{d}{d\tau}p_x = \frac{d}{d\tau}(\gamma m v_x)$$

2-component

$$0 = \frac{dp_y}{d\tau} = \frac{d}{d\tau}(\gamma m v_y)$$

3-component

$$0 = \frac{dp_z}{d\tau}$$

The last two equations tell us that p_y and p_z are each constant. But since p_z is initially zero, it stays zero and we need not consider it further. The first two equations may be simplified by differentiating again. First, define $\quad = qE/mc$. Then

$$\frac{d^2}{d\tau^2}(\gamma c) = \frac{d}{d\tau}(\quad \gamma v_x) = \quad ^2\gamma c$$

The solution for γ is

$$\gamma = Ae^{\quad \tau} + Be^{-\quad \tau}$$

where

$$A + B = \gamma_0 = \frac{1}{\sqrt{1 - v_0^2/c^2}}$$

Then

$$\gamma v_x = \frac{c}{\quad}\frac{d}{d\tau}\gamma = c\left[Ae^{\quad \tau} - Be^{-\quad \tau}\right]$$

Suppose the particle is moving perpendicular to $\vec{E}$ at $t = \tau = 0$. Then

$$A = B = \frac{\gamma_0}{2}$$

So

$$\begin{aligned} \gamma &= \gamma_0 \cosh \quad \tau \\ v_x &= c \tanh \quad \tau \end{aligned}$$

Then finally

$$\gamma m v_y = \gamma_0 m v_0$$

so

$$v_y = \frac{v_0}{\cosh \quad \tau}$$

For short times, $\tau \ll 1$, we have

$$v_x = \quad c\tau$$

and the speed increases linearly in time, with v_y remaining constant. But for τ large, we find tanh $\tau \to 1$, and so $v_x \to c$ and $v_y \to 0$. The latter result is initially surprising, since there is no force in the $y-$direction. But since the particle's γ is continuously increasing, and the momentum component stays fixed, the velocity component must decrease.

To express the results in terms of coordinate time, we integrate again. The $0-$component of the 4-velocity is

$$\gamma c = \frac{d}{d\tau}(ct) = \gamma_0 c \cosh \quad \tau$$

Thus

$$t = \frac{\gamma_0}{} \sinh \quad \tau$$

where we set the integration constant to zero so that $t = 0$ when $\tau = 0$. Then for $\tau \ll 1$, , $t = \gamma_0 \tau$, but for $\tau \gg 1$,

$$t = \frac{\gamma_0}{2} e^{\quad \tau}$$

and

$$\begin{aligned} v_x &= c\frac{t/\gamma_0}{\sqrt{1 + (\quad t/\gamma_0)^2}} \\ v_y &= \frac{v_0}{\sqrt{1 + (\quad t/\gamma_0)^2}} \end{aligned}$$

The graph shows β_x (black) and β_y (red) versus t/γ_0, with $\beta_y = 0.5$ at $t = 0$.