Example 4.3 A circuit has a 2 Ω resistor, a 2 mH inductor and a 0.15 μF capacitor connected in series with a power supply that generates a square wave emf $\varepsilon(t)$ with a 1 V amplitude and a period of 0.3 ms. What is the current in the circuit?
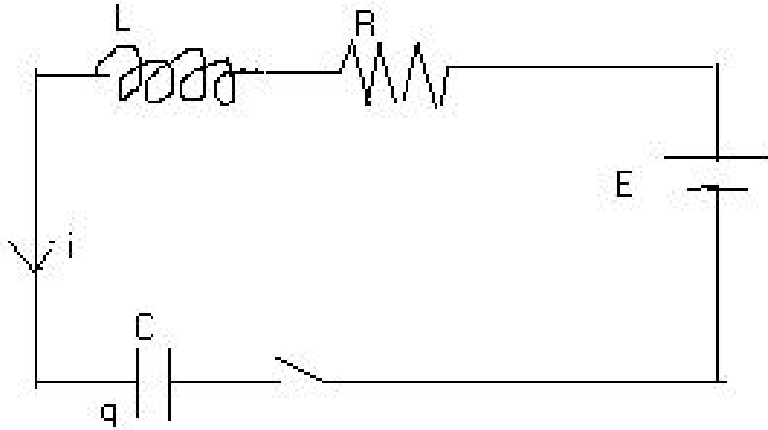

Figure 4.7: A series LRC circuit with a periodic applied EMF

The circuit is shown in Figure 4.7. Kirchhoff's rules applied to the circuit give the differential equation

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = \varepsilon(t)$$

where $I = \frac{dQ}{dt}$. Write the equation in standard form:

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = \frac{\varepsilon}{L}$$

The coefficients are:

$$2\alpha \equiv \frac{R}{L} = \frac{2}{2\ \text{mH}} = 1 \times 10^3\ \text{s}^{-1}$$

and

$$\omega_0^2 \equiv \frac{1}{LC} = \frac{1}{(2\ \text{mH})\,(0.15\ \mu\text{F})} = \frac{1}{3} \times 10^{10}\ \text{s}^{-2}$$

Then:

$$\omega_0 = \frac{10^5}{\sqrt{3}}\ \text{rad/s} = 5.77 \times 10^4\ \text{rad/s}$$

We have already found the series for the square wave emf ε (equation **??**)

$$\varepsilon(t) = \varepsilon_0 \left(\frac{1}{2} + \frac{1}{\pi} \sum_{n=-\infty,\ n\ \text{odd}}^{+\infty} \frac{i}{n} \exp\left(i\frac{2n\pi t}{\tau} \right) \right)$$

1

where τ is the period of 0.3 ms and $\mathcal{E}_0 = 1$ V. Next write the solution as a Fourier series:

$$Q(t) = \sum_{n=-\infty}^{\infty} q_n \exp\left(i\frac{2n\pi t}{\tau}\right)$$

(Note: it is much easier to use the series in exponential form here, because the equation has both first and second derivatives. When we differentiate the series, each term in the equation contains simple multiples of the original exponential terms. In contrast, the odd order derivatives mix the sines and cosines : $\frac{d}{dx}\sin x = \cos x$. Instead of one equation for each coefficient c_n, we would have two equations to solve simultaneously for the coefficients of the sines and cosines.)

Now substitute the series for $\mathcal{E}(t)$ and $Q(t)$ into the differential equation:

$$\sum_{n=-\infty}^{\infty} -\left(\frac{2n\pi}{\tau}\right)^2 q_n e^{i2n\pi t/\tau} + 2\alpha \sum_{n=-\infty}^{\infty} i\frac{2n\pi}{\tau} q_n e^{i2n\pi t/\tau} + \omega_0^2 \sum_{n=-\infty}^{\infty} q_n e^{i2n\pi t/\tau}$$

$$= \frac{\mathcal{E}_0}{L}\left(\frac{1}{2} + \frac{1}{\pi}\sum_{n=-\infty,\ n\ \text{odd}}^{+\infty} \frac{i}{n} e^{i2n\pi t/\tau}\right)$$

Next we make use of the orthogonality of the exponentials by multiplying the whole equation by $\exp\left(-i\frac{2m\pi t}{\tau}\right)$ and integrating over one period. Only the terms with $n = m$ survive. Since this is true for any integer m, we can equate the coefficients of each exponential separately.

The constant $(n = 0)$ term gives

$$\omega_0^2 q_0 = \frac{\mathcal{E}_0}{2L} \Rightarrow q_0 = \frac{\mathcal{E}_0 C}{2}$$

The other terms are given by:q

$$q_n\left(-\left(\frac{2n\pi}{\tau}\right)^2 + \frac{4\alpha i n\pi}{\tau} + \omega_0^2\right) = \begin{array}{ll} \frac{\mathcal{E}_0}{\pi L}\frac{i}{n} & \text{if} \quad n \text{ is odd} \\ 0 & \text{if} \quad n \text{ is even} \end{array}$$

$$q_n = \frac{\mathcal{E}_0\tau^2}{\pi L}\frac{i}{n}\frac{1}{\left(-(2n\pi)^2 + 4\alpha\tau i n\pi + \omega_0^2\tau^2\right)} \qquad n \text{ odd}$$

$$= \frac{\mathcal{E}_0\tau^2}{\pi L}\frac{i}{n}\frac{\left(-(2n\pi)^2 - 4\alpha\tau i n\pi + \omega_0^2\tau^2\right)}{\left[\omega_0^2\tau^2 - (2n\pi)^2\right]^2 + (4\alpha\tau n\pi)^2}$$

$$= \frac{\mathcal{E}_0\tau^2}{n\pi L}\frac{4\alpha\tau n\pi + \left[\omega_0^2\tau^2 - (2n\pi)^2\right]i}{\left[\omega_0^2\tau^2 - (2n\pi)^2\right]^2 + (4\alpha\tau n\pi)^2}$$

$$= \frac{\mathcal{E}_0\tau^2}{\pi L}\frac{4\alpha\tau\pi + \left[\omega_0^2\tau^2 - (2n\pi)^2\right]i/n}{\left[\omega_0^2\tau^2 - (2n\pi)^2\right]^2 + (4\alpha\tau n\pi)^2}$$

2

Notice that the real part of q_n is even in n while the imaginary part is odd. This is exactly what we expect if the resulting series is to be real. The real parts combine to give cosines while the imaginary parts combine to give sines:

$$Q\left(t\right) = \frac{\mathcal{E}_0 C}{2} + \frac{\mathcal{E}_0 \tau^2}{\pi L} \sum_{n=-\infty,\,\neq 0,\,\mathrm{odd}}^{\infty} \frac{4\alpha\tau\pi + \left[\omega_0^2\tau^2 - (2n\pi)^2\right]i/n}{\left[\omega_0^2\tau^2 - (2n\pi)^2\right]^2 + (4\alpha\tau n\pi)^2} \exp\left(i\frac{2n\pi t}{\tau}\right)$$

$$= \frac{\mathcal{E}_0 C}{2} + \frac{2\mathcal{E}_0\tau^2}{\pi L} \sum_{n=1}^{\infty} \frac{4\alpha\tau\pi \cos\frac{2n\pi t}{\tau} - \frac{1}{n}\left[\omega_0^2\tau^2 - (2n\pi)^2\right]\sin\frac{2n\pi t}{\tau}}{\left[\omega_0^2\tau^2 - (2n\pi)^2\right]^2 + (4\alpha\tau n\pi)^2}$$

Finally we can differentiate to get I. Then:

$$I\left(t\right) = -\frac{4\mathcal{E}_0\tau}{L} \sum_{n=1,\,\mathrm{odd}}^{\infty} \frac{4n\alpha\tau\pi\sin\frac{2n\pi t}{\tau} + \left[\omega_0^2\tau^2 - (2n\pi)^2\right]\cos\frac{2n\pi t}{\tau}}{\left[\omega_0^2\tau^2 - (2n\pi)^2\right]^2 + (4\alpha\tau n\pi)^2}$$

The constant in front of the sum is:

$$\frac{4\mathcal{E}_0\tau}{L} = \frac{4\left(1\text{ V}\right)\left(0.1\text{ ms}\right)}{\left(2\text{ mH}\right)} = 0.2\,\frac{\text{V}\cdot\text{s}}{\text{H}} = 0.2\text{ A}$$

Thus:

$$I\left(t\right) = -\left(0.2\text{ A}\right) \sum_{n=1,\,\mathrm{odd}}^{\infty} \frac{4n\alpha\tau\pi\sin\frac{2n\pi t}{\tau} + \left[\omega_0^2\tau^2 - (2n\pi)^2\right]\cos\frac{2n\pi t}{\tau}}{\left[\omega_0^2\tau^2 - (2n\pi)^2\right]^2 + (4\alpha\tau n\pi)^2}$$

Notice that if the natural frequency ω_0 of the circuit times the period τ of the emf is very close to $2n\pi$ for some integer n, then the current will be very large: there is a resonance at that frequency. In our example, $\omega_0\tau/\pi = \left(\frac{10^5}{\sqrt{3}}\text{ s}^{-1}\right)\left(0.3\times 10^{-3}\text{ s}\right)/\pi = 5.513\,4$ rad, which is close to $6{=}2{\times}3$. Thw solution is shown in Figure 4.8. The current is dominated by the $n=3$ term, which has three times the frequency of the square wave emf.
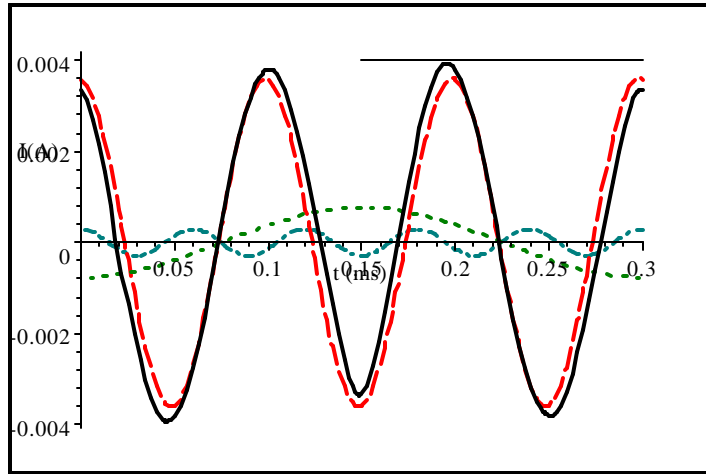
The heavy solid line is the complete solution. The green dotted line is the $n = 1$ term, red dashed $n = 3$, and blue-green dot-dash is the $n - 5$ term. The emf is shown as a thin line for reference. The vertical scale does not apply to this term.